THE UNITARY THEORY OF EINSTEIN-MAYER

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In the following pages, I propose to discuss, in a geometrically intuitive fashion, the unitary theory of gravitation and electricity that Einstein, in collaboration with Mayer, published in 1931 in the *Sitzungsberichte* of the Berlin Academy ([1]).

The geometric basis that one may give to the theory is contained in the following three axioms:

AXIOM A. – *Spacetime is a four-dimensional manifold V_4 in a five-dimensional space with a Euclidean connection E_5.*

AXIOM B. – *The manifold V_4 is totally geodesic.*

AXIOM C. - *The geometry that is induced in V_4 by its presence in the interior of E_5 is Riemannian* (*zero torsion*).

To these purely geometrical axioms we add various other axioms that permit us to identify certain geometrical quantities in the space with certain physical quantities. The simplest and most important of them is the one that identifies the torsion of the space E_5 with the electromagnetic field (Axiom D). Another one permits us to recover the classical equations of motion of a particle in both a gravitational and electromagnetic field. Finally, the field equations include both the internal Riemannian curvature of V_4 and that of E_5.

I. – GEOMETRIC BASIS OF THE THEORY.

1. Notion of a space with Euclidean connection. – An n-dimensional space with a Euclidean connection, which is referred to an arbitrary $x^1, x^2, \ldots, x^n$, is defined by:

1. The *fundamental tensor* g_{ij}, which gives the square of the distance between two infinitely close points $g_{ij}\, dx^i\, dx^j$ or the square of the length of a vector $g_{ij}\, X^i\, Y^j$;

2. A system of n^3 quantities Γ^k_{ij} that permit us to define what we mean by the elementary geometric variation (or *absolute differential*) of a vector whose origin is

([1]) French translation by Maurice SOLOVINE in: EINSTEIN: *Théorie de la relativité,* Paris, Hermann, 1933, 73-98.

subjected to a given infintesimal displacement dx^i. The contravariant or covariant components of this differential are:

$$DX^i = dX^i + X^k \Gamma^i_{kh}\, dx^h,$$

and

$$DX_i = dX_i - X_k \Gamma^k_{ih}\, dx^h,$$

respectively.

The Γ^k_{ij} are not arbitrary; they must satisfy the condition that the length of a moving vector whose absolute differential is null remains constant. This translates into the relations:

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ijk} + \Gamma_{jik} = g_{jm}\Gamma^m_{ik} + g_{im}\Gamma^m_{jk}. \tag{1}$$

These relations express that the absolute differential of the tensor g_{ij} is null.

2. Other than the fundamental tensor g_{ij} , the space two important tensors, which embody the intrinsic geometric properties of the space:

1. The *torsion tensor* $S^k_{ij} = -S^k_{ji}$, which is defined by:

$$S^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}. \tag{2}$$

2. The *curvature tensor* R^k_{ijh} , which is defined by:

$$R^k_{ijh} = \frac{\partial \Gamma^k_{ij}}{\partial x^h} - \frac{\partial \Gamma^k_{ih}}{\partial x^j} + \Gamma^m_{ij}\Gamma^k_{mh} - \Gamma^m_{ih}\Gamma^k_{mj}. \tag{3}$$

If one considers an oriented surface element in space, which is defined, for example, by a parallelogram that is constructed from two infinitesimal vectors dx^i, δx^i of a certain order, then one may associate this element with:

1. A *torsion vector*, which represents an infinitesimal translation whose k^{th} components is:

$$\Omega^k = S^k_{ij}\, dx^i\, \delta x^j.$$

2. An infinitesimal *rotation* $\Omega_i^{\ k}$, which is defined by a two-index tensor (with $\Omega_{ik} = -\Omega_{ki}$):

$$\Omega_i^{\ k} = R^k_{ijh}\, dx^j \delta x^h.$$

The space is called *Riemannian* if it has null torsion; the symbols Γ^k_{ij} and Γ_{ikj} are then the well-known Christoffel symbols (of the second and first types). A Riemannian space is completely determined by the given of its fundamental tensor g_{ij}; a space with a Euclidean connection is completely determined by its fundamental tensor g_{ij} and its torsion tensor S_{ij}^{k}.

3. Totally geodesic manifolds. – Consider a four-dimensional manifold V_4 in a space with Euclidean connection E_5. We assume, with no loss of generality, that V_4 is defined by the equation $x^5 = 0$, in such a way that the unitary vector **v** that is normal to V_4 has only one covariant component different from zero, namely, v_5. We denote the indices 1, 2, 3, 4 by the *Latin* letters $i, j, k, \dots$

If we consider a curve traced out in V_4 with the curvilinear abscissa s, and if we denote the unitary tangent vector to this curve by **t** then the absolute derivative $\frac{D\mathbf{t}}{ds}$ represents the *curvature vector.* The normal component $\mathbf{v}\frac{D\mathbf{t}}{ds}$ of that vector is the *normal curvature* $\frac{1}{R}$ of the curve. It is obtained immediately from the formula:

$$\frac{1}{R} = v_5 \frac{Dt^5}{ds},$$

or, since $t^5 = \frac{dx^5}{ds} = 0$:

$$\frac{1}{R} = v_5 t^k \Gamma_{kh}^{s} \frac{dx^h}{ds} = v_5 \Gamma_{kh}^{s} \frac{dx^k}{ds}\frac{dx^h}{ds}.$$

The manifold is called *totally geodesic* if the normal curvature of any curve that is traced out in the manifold is null, or, what amounts to the same thing, if any geodesic curve in V_4 is geodesic in the space.

Therefore, Axiom B *translates analytically into the relations:*

(I) $$\Gamma_{ij}^{s} + \Gamma_{ji}^{s} = 0 \qquad \text{(Axiom B)},$$

that the Γ *symbols for* E_5 *must satisfy at any point of spacetime* V_4.

4. Induced connection in a manifold V_4 **of** E_5. – The fact that the manifold V_4 (whose equation is $x^5 = 0$), is embedded in a space E_5 with a Euclidean connection permits us to introduce an *induced connection* in V_4. This Euclidean connection is defined by:

1. The fundamental tensor g_{ij} (i, j = 1, 2, 3, 4) which has the same components as the fundamental tensor of the ambient space at each point of V_4.

2. A system of symbols $\overline{\Gamma}_{ijk}$ that satisfy the condition that the absolute differential $\overline{D}\mathbf{X}$ of a vector **X** of V_4 be the tangential component of the absolute

differential $D\mathbf{X}$ of the same vector, as calculated with the Euclidean connection of the ambient space. From this, it results that for any vector of V_4 one has:

$$\overline{D}X_i = DX_i = dX_i - X_k \Gamma^k_{ih} dx^h - X_5 \Gamma^5_{ih} dx^h = dX_i - X^k \Gamma_{ikh} dx^h;$$

there is no reason to include the term $X^5\ \Gamma_{i5h}\ dx^h$ in the last expression of this triple equality, since the vector $\mathbf{X}$ belongs to V_4, its X^5 component is null. It then results that *the symbols of the induced connection* $\overline{\Gamma}_{ijk}$ *are the same as those of the connection in the ambient space.*

One may now go further by assuming, with no loss of generality, that the lines x^i = const. (i = 1, 2, 3, 4) are normal to V_4 at every point of V_4. Under these conditions, one has:

$$g_{i5} = 0$$

at every point of V_4; moreover, the $\overline{g}^{ij}$ of the induced connection are equal to the g^{ij} of the connection of the ambient space, and, as a result, *the symbols of the induced connection* $\overline{\Gamma}^k_{ij}$ *are the same as those of the Euclidean connection of the ambient space.*

Finally, we add that the contravariant components of the unitary normal vector $\mathbf{v}$ must all be zero except for v^5.

5. *Axiom* C now translates into relations that express that the induced connection on V_4 has zero torsion, namely:

(II) $$\Gamma^k_{ij} - \Gamma^k_{ji} = 0 \qquad \text{(Axiom C)},$$

It does not necessarily result from this that the ambient space E_5, is Riemannian, and this is what permits us to introduce the electromagnetic field in a geometric form.

We add that, from (I), the equations:

$$g_{i5} = 0, \qquad v^i = v_i = 0,$$

give:

$$\Gamma_{i5j} + \Gamma_{5ij} = 0,$$

so:

(4) $$\Gamma^i_{5j} = -g^{55}\Gamma^{i5}_j \qquad \text{or} \qquad v^5\Gamma^i_{5j} = -v_5\Gamma^{i5}_j.$$

Finally, since the absolute differential $D\mathbf{v}$ of a unitary vector is normal to that vector, i.e., has a null fifth component, one has ([2]):

(5) $$\frac{\partial v^5}{\partial x^i} + v^5\Gamma^5_{5i} = 0, \qquad \frac{\partial v_5}{\partial x^i} - v_5\Gamma^5_{5i} = 0.$$

([2]) With no loss of generality, it is possible to assume that $v_5 = 1$, and, as a result $v^5 = 1$; however, this will not simplify the calculations appreciably.

I. – THE ELECTROMAGNETIC FIELD AND THE MOTION OF A CHARGED PARTICLE.

6. From Axioms B and C, which translate into formulas (I) and (II), the torsion vector of E_5 that is associated with a surface element for the spacetime V_4 is normal to V_4; it is in this torsion vector that we shall look for the electromagnetic field.

AXIOM D. – *The electromagnetic field at an arbitrary point of spacetime V_4 has components F_{ij}, that are defined by the quantities $\frac{1}{2}v^5 S_{ij}^5$ that define the mean of the torsion tensor, as measured on the normal to V_4.*

Relations (1) and (2) then give:

$$\text{(III)} \qquad \Gamma_{ij}^5 = v^5 F_{ij}, \qquad \Gamma_{5j}^i = v_5 F^i{}_j = v_5 F_j{}^i \qquad \text{(Axiom D).}$$

Axiom D may be stated in a more intuitive manner by considering what one may call the generalized flux of the magnetic field through a surface element of spacetime, namely:

$$\iint F_{ij} dx^i dx^j ,$$

in which the (non-indicated) summation is performed over *pairs* of the indices 1, 2, 3, 4. *This flux is equal to the normal component of the torsion vector that is associated with the surface element being considered,* namely:

$$\iint v_5 (\Gamma_{ij}^5 - \Gamma_{ij}^5) dx^i dx^j .$$

The integral, , which has a natural intrinsic significance, may be called the flux of the magnetic field, since, in special relativity, when it is taken over a surface element in space (*at constant time*), it gives:

$$\iint H_x dydz + H_y dzdx + H_z dxdy ,$$

if we agree to set:

$$F_{23} = H_x, \qquad F_{31} = H_y, \qquad F_{12} = H_z .$$

On the contrary, the F_{i4} define the electric field.

7. Motion of charged particle. – We shall arrive at the equations of motion of particle in a field of gravitation and electricity by stating with the following axiom:

AXIOM E. – *The space vector whose tangential component along V_4 represents the energy-momentum of a particle, and whose normal component V_4 has the charge of this particle for its measure, remains equipollent to itself when the particle is placed in a field of gravitation and electricity.*

Let **t** be the unitary vector that is tangent to the trajectory of the particle in spacetime V_4; moreover, let m_0 be the *rest mass* of the particle and e, its charge. The space vector under consideration in the statement of Axiom E is:

$$m_0\mathbf{t} + e\mathbf{v}.$$

One must therefore have:

$$dm_0\mathbf{t} + de\mathbf{v} + m_0\,D\mathbf{t} + eD\mathbf{v} = 0.$$

Now, since the manifold V_4 is totally geodesic, the two vectors $D\mathbf{t}$ and $D\mathbf{v}$ are both perpendicular to **t** and **v**. The preceding equation may be decomposed into three equations, namely:

$$dm_0 = 0, \qquad de = 0, \qquad m_0\,D\mathbf{t} + eD\mathbf{v} = 0.$$

The first two provide the conservation theorems for the rest mass and charge ([3]). As for the third, when one remarks that the contravariant components t^i of **t** are $\frac{dx^i}{ds}$, it provides the four equations:

$$m_0\left(d\frac{dx^i}{ds} + \Gamma^i_{kh}\frac{dx^k}{ds}\frac{dx^h}{ds}\right) + ev^5\Gamma^i_{5h}\frac{dx^h}{ds} = 0\,,$$

or, accounting for (III):

$$\text{(IV)} \qquad m_0\left(d\frac{dx^i}{ds} + \Gamma^i_{kh}\frac{dx^k}{ds}\frac{dx^h}{ds}\right) + eF_h{}^i\frac{dx^h}{ds} = 0\,;$$

these are the classical equations.

8. If we assume, as in special relativity, that:

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2, \qquad (x^1 = x,\ x^2 = y,\ x^3 = z,\ x^4 = t),$$

and if we set:

$$F_{23} = H_x, \qquad F_{31} = H_y, \qquad F_{12} = H_z,$$
$$F_{41} = E_x, \qquad F_{42} = E_y, \qquad F_{43} = E_z,$$

then equations (IV) become, upon remarking that the mass m, in the ordinary sense of the word, is equal to $m_0\frac{dt}{ds}$:

([3]) In their memoir, Einstein and Mayer considered the vector that had components on V_4 that were equal to the unitary vector **t**, and a normal component equal to the ratio $\frac{e}{m}$; the constancy of this ratio, which is derived from the hypothesis that this vector remains equipollent to itself, seems to be interpreted, due to its significance, as the statement that this ratio has the same value *for all electrons.*

$$\begin{aligned}
d\left(m\frac{dx}{dt}\right) &= e(-\mathrm{H}_z dy + \mathrm{H}_y dz + \mathrm{E}_x dt), \\
d\left(m\frac{dy}{dt}\right) &= e(-\mathrm{H}_x dz + \mathrm{H}_z dx + \mathrm{E}_y dt), \\
d\left(m\frac{dz}{dt}\right) &= e(-\mathrm{H}_y dx + \mathrm{H}_x dy + \mathrm{E}_z dt), \\
dm &= e(\ \mathrm{E}_z dx + \mathrm{E}_y dy + \mathrm{E}_z dz).
\end{aligned}$$

The generalized flux of the magnetic field in this case is therefore:

$$\iint H_x\, dydz + H_y\, dxdy - E_x\, dxdt - E_y\, dydt - E_z\, dzdt.$$

II. – THE SPATIAL RIEMANNIAN CURVATURE OF THE FIELD.

9. At every point of spacetime V_4, and for every surface element of V_4 at this point there is reason to consider the *intrinsic* Riemannian curvature of V_4 that is defined by the classical components $R_{i\ jh}^{\ k}$, and the Riemannian curvature of the ambient space E_5, which we represent by the components $\overline{R}_{5\ jh}^{\ k}$.

From (3), one has:

$$\overline{R}_{ijh}^{k} = R_{ijh}^{k} + \Gamma_{ij}^{5}\Gamma_{5h}^{k} - \Gamma_{ih}^{5}\Gamma_{5j}^{k},$$

or, on account of (III):

$$\overline{R}_{ijh}^{k} = R_{ijh}^{k} + F_{ij}F_{h}^{\ k} - F_{ih}F_{j}^{\ k}; \tag{6}$$

from this, upon contracting the indices k and h, one deduces:

$$\overline{R}_{ij} = R_{ij} - F_{ik}F_{h}^{\ k} \qquad \text{or} \qquad \overline{R}_{i}^{\ j} = R_{i}^{\ j} - F_{ik}F^{\ jk}; \tag{7}$$

and then, by another contraction:

$$\overline{R} = R - F_{hk}F^{hk}. \tag{8}$$

Similarly, upon accounting for (3) and (III), one has:

$$\overline{R}_{5jh}^{k} = \frac{\partial(v_5 F_j^{\ k})}{\partial x^h} - \frac{\partial(v_5 F_h^{\ k})}{\partial x^j} + v_5(F_j^{\ m}\Gamma_{mh}^{k} - F_h^{\ m}\Gamma_{mj}^{k} + \Gamma_{5j}^{5}F_h^{\ k} - \Gamma_{5h}^{5}F_j^{\ k}).$$

If we take (5) into account, a simple calculation gives ([4]):

$$v^5\overline{R}_{5\ jh}^{\ k} = F_{j\ ;k}^{\ k} - F_{h\ ;j}^{\ k}. \tag{9}$$

([4]) Along with Einstein, we designate the covariant derivative *that exists on the manifold* V_4 by a semi-colon.

From this one deduces, upon contracting over the indices, k and h:

$$\text{(10)} \qquad v^5 \overline{R}_{5j} = F_j^{\ k}{}_{;k} \qquad \text{and} \qquad v^5 \overline{R}_5^{\ j} = F^{jk}{}_{;k}\,.$$

10. In order to prepare the field equations, we shall introduce the quantities:

$$\text{(11)} \qquad \begin{cases} R^*_{ij} = \overline{R}_{ij} - \tfrac{1}{2} g_{ij} \overline{R} - \tfrac{1}{4} g_{ij} F^{hk} F_{hk} \\ \qquad = R_{ij} - \tfrac{1}{2} g_{ij} R - F_{ik} F_j^{\ k} + \tfrac{1}{4} g_{ij} F^{hk} F_{hk} \\ R^*_{5i} = v^5 R_{5i} = F_i^{\ k}{}_{;k}\,. \end{cases}$$

The quantities satisfy the following identities:

$$\text{(12)} \qquad \begin{cases} R_i^{*k}{}_{;k} - F_{im} R_5^{*m} \equiv \tfrac{1}{2} F^{hk} (F_{ih;k} + F_{hk;i} + F_{ki;h}) \\ R_5^{*k}{}_{;k} \equiv 0. \end{cases}$$

The last identity of (12) is easily verified. As for the first, one may prove it by first remarking that, from the classical Bianchi identities, one has:

$$\left(R_i^{\ k} \ \tfrac{1}{2} g_i^{\ k} R\right)_{;k} = 0\,.$$

What then remains is:

$$\begin{aligned} R_i^{*k}{}_{;k} &= F_{im} F^{km}{}_{;k} - F^{km} F_{im;k} + \tfrac{1}{2} F^{km} F_{km;i} \\ &= + F_{im} R_5^{*m} + \tfrac{1}{2} F^{km} (F_{km;i} - F_{im;k} + F_{ik;m})\,, \end{aligned}$$

which is precisely the formula we are trying to prove.

IV. – THE FIELD EQUATIONS.

11. We will obtain the field equations after we introduce two new axioms:

AXIOM F. – *The generalized flux of the magnetic field through the two-dimensional boundary of a three-dimensional region of spacetime is null.*

Since the generalized flux of the magnetic field is $\iint F_{ij} dx^i dx^j$, this axiom translates into the relations:

$$\text{(V)} \qquad F_{ij;k} + F_{jk;i} + F_{ik;j} = 0,$$

(first group of Maxwell equations).

AXIOM G. – *The matter tensor* T_{ij} (*energy-momentum*) *and the electricity tensor* T_{5i} (*charge and current density*) *are represented by* R^*_{ij} *and* R^*_{5i}, *up to a constant factor.*

This gives the equations:

$$R_{ij} - \tfrac{1}{2} g_{ij} R - F_{ik} F_j^{\ k} + g_{ij} F^{hk} F_{hk} = T_{ij}, \tag{VI}$$

$$F_{i\ ;k}^{\ k} = T_{5i}, \tag{VII}$$

in which the T_{ij} and T_{5i} in the right-hand sides have the *physical* significance that was described in the statement. The right-hand sides are null in vacuo.

In order to have the proportionality factor be a constant, we put ourselves in the special relativistic context. Equations (VII) have the left-hand sides:

$$-\frac{\partial H_z}{\partial y} + \frac{\partial H_y}{\partial z} - \frac{\partial E_x}{\partial t},$$
$$-\frac{\partial H_x}{\partial z} + \frac{\partial H_z}{\partial x} - \frac{\partial E_y}{\partial t},$$
$$-\frac{\partial H_y}{\partial x} + \frac{\partial H_x}{\partial y} - \frac{\partial E_z}{\partial t},$$
$$-\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z}.$$

The last quantity is equal to the charge density multiplied by 4π; the others give the current densities i_x, i_y, i_z, up to the same factor.

12. The identities, (12), give the following relations between the matter tensor and the electricity tensor:

$$\begin{cases} T_{i\ ;k}^{\ k} - F_{ik} T_5^{\ k} = 0, \\ T_{5\ ;k}^{\ k} = 0, \end{cases} \tag{VIII}$$

in which the latter expresses the theorem of the conservation of electricity. *In the continuum, these relations are equivalent to equations* (IV) *that gave us the motion of a particle, combined with the theorems of the constancy of the rest mass and the charge of that particle.*

13. In order to see this equivalence as neatly as possible, we place ourselves in the ideal case of diffuse matter without pressure. If we consider a three-dimensional volume element of spacetime, then the matter and the electricity that are contained in this element may be assimilated into a particle whose state is represented by a five-dimensional vector. The covariant components Π_i, Π_5 of that vector are given, up to a factor of 4π, by the formulas:

$$\begin{aligned}\Pi_i &= \sqrt{-g}\,(T_i^4 dx^1 dx^2 dx^3 - T_i^1 dx^2 dx^3 dx^4 \\ &\qquad - T_i^2 dx^3 dx^1 dx^4 - T_i^3 dx^1 dx^2 dx^4)\,, \\ \Pi_5 &= \sqrt{-g}\,(T_5^4 dx^1 dx^2 dx^3 - T_5^1 dx^2 dx^3 dx^4 \\ &\qquad - T_5^2 dx^3 dx^1 dx^4 - T_5^3 dx^1 dx^2 dx^4)\,.\end{aligned}$$

Here, we are dealing with some vectorial differential forms that are the covariant components of an infinitesimal vector that is attached to a three-dimensional spacetime element.

Therefore consider a four-dimensional domain $\mathcal{D}$ of spacetime, and the three-dimensional boundary, F, of that domain. A vector $\mathbf{\Pi}$ is thus attached to every element of the boundary. *Equations* (VIII) *express that the geometric sum of all of these vectors is null.*

In full rigor, in a non-euclidean space one may not speak of the geometric sum of vectors that do not have the same origin. However, if the domain $\mathcal{D}$ is infinitesimal, and if A is a point that is interior to $\mathcal{D}$ then one may parallel transport any vector that is attached to a point M of the boundary $\mathcal{F}$ to A by displacing its origin, for example, along the geodesic that joins the point M to the point A. Since the vectors now have the same origin one may add them, and *the principal part of the geometric sum does not depend on the chosen point A.*

Each component of the geometric sum is given by a triple integral taken over the boundary $\mathcal{F}$, which may, from the generalized Stokes's formula, be transformed into a quadruple integral taken over $\mathcal{D}$. If one denotes the exterior derivative of one of the forms $\mathbf{\Pi}_i$ by $\mathbf{\Pi}'$ – i.e., the quantity under the $\iiiint$ sign when one transforms the triple integral that was taken over $\mathcal{F}$ into a quadruple integral taken over $\mathcal{D}$ – then one proves that the covariant components of the desired geometric sum are the quadruple differential forms:

$$\begin{aligned}&\Pi'_i - \Gamma^k_{ih} dx^h \Pi_k - \Gamma^5_{ih} dx^h \Pi_5\,, \\ &\Pi'_5 - \Gamma^k_{5h} dx^h \Pi_k - \Gamma^5_{5h} dx^h \Pi_5\,.\end{aligned}$$

One therefore finds:

$$\begin{aligned}&-\sqrt{-g}\,(T_{i\ \ ;k}^{\ k} - F_{ik} T_5^k) dx^1 dx^2 dx^3 dx^4\,, \\ &-v^5 \sqrt{-g}\,(T_{5\ \ ;k}^{\ k} - F_h^{\ k} T_k^h) dx^1 dx^2 dx^3 dx^4\,.\end{aligned}$$

The first four of these quantities are null, from the first identities of (VIII); as for the last, it is also null, by virtue of the last identity of (VIII), and the remark that the sum, $F_h^{\ k} T_h^{\ k} = F^{hk} T_{kh}$, is null because of the antisymmetry of F^{kh}, combined with the symmetry of T_{kh}.

The theorem is thus proved: *The geometric sum of the vectors of the space E_5 that represent the states of the elements of the boundary $\mathcal{F}$ of the domain $\mathcal{D}$ is null.*

14. We shall now show that if Axiom F is not in contradiction with Axiom D then the preceding theorem is a necessary consequence of Axiom D. Indeed, consider the domain $\mathcal{D}$ an a particle whose worldline traverses the domain; suppose, to simplify, that it enters $\mathcal{D}$ at a point M on the boundary $\mathcal{F}$, and that it leaves at a point M'. From Axiom D, the five-dimensional vectors that represent the state of the particle at these two points are equipollent to each other, but when considered as being attached to two points of the oriented boundary, F, they must be regarded as having a null geometric sum. Since the vectors $\mathbf{\Pi}$ that are attached to the different points of $\mathcal{F}$ have a null geometric sum pairwise, their total geometric sum must be null. Q.E.D.

There is thus an equivalence between the equations of motion of a particle that are provided by Axiom D, *and the field equations that are provided by Axioms* E *and* F.

15. We may add an interesting remark that says nothing new. The last identity of (III) shows that the integral $\iiint v^5 \mathbf{\Pi}_5$ is null when taken over the three-dimensional boundary of a four-dimensional domain in spacetime. From a theorem of classical analysis, it results that this triply-extended integral over an arbitrary three-dimensional region is equal to a certain doubly-extended integral over the two-dimensional boundary of this region. This double integral is the following:

$$\begin{aligned}\iint \sqrt{-g}\,(&F^{23}dx^1dx^2 + F^{31}dx^2dx^4 \\ &+ F^{12}dx^3dx^4 + F^{14}dx^2dx^3 + F^{24}dx^3dx^1 + F^{34}dx^1dx^2);\end{aligned}$$

indeed, when transformed into a triple integral, it gives:

$$\iiint \sqrt{-g}\,(F^{4k}{}_{;k}\,dx^1dx^2dx^3 - F^{1k}{}_{;k}\,dx^2dx^3dx^4 \ldots).$$

The double integral is the *generalized flux of the electric field;* when this flux is taken over the boundary of a three-dimensional region it is equal to the quantity of electricity that is contained in that region.