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Euler's Inequality Revisited

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A quadrilateral ABCD is called a *double circle quadrilateral* if it has a circumcircle and an inscribed circle. In a previous article (see reference 1), we proved Euler's inequality for a double circle quadrilateral, $R \geq r\sqrt{2}$, where R and r denote the radii of the circumcircle and inscribed circle respectively. Here we shall find two expressions, involving the angles of such a quadrilateral, which lie between $r\sqrt{2}/R$ and 1. We denote the lengths of the sides AB, BC, CD, and DA by a, b, c, and d respectively, put $s = \frac{1}{2}(a+b+c+d)$, and denote the area of ABCD by Δ.

Theorem 1 *Let* ABCD *be a double circle quadrilateral. Then we obtain*

$$\frac{r\sqrt{2}}{R} \leq \frac{1}{2}\left(\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{B}}{2} + \sin\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2} + \sin\frac{\mathrm{C}}{2}\cos\frac{\mathrm{D}}{2} + \sin\frac{\mathrm{D}}{2}\cos\frac{\mathrm{A}}{2}\right) \leq 1.$$

In order to prove Theorem 1, we first prove the following lemma.

Lemma 1 *In a double circle quadrilateral* ABCD*, we have*

$$\sin\mathrm{A}\sin\mathrm{B} = \frac{r^2 + r\sqrt{r^2+4R^2}}{2R^2}.$$

Proof By reference 1, we have

$$\sin\mathrm{B} = \frac{2\Delta}{ab+cd}, \qquad \sin\mathrm{A} = \frac{2\Delta}{ad+bc}, \qquad \Delta = rs,$$
$$R = \frac{1}{4\Delta}\sqrt{(ab+cd)(ac+bd)(ad+bc)}.$$

Hence,

$$\frac{\sqrt{1+\sin\mathrm{A}\sin\mathrm{B}}}{\sin\mathrm{A}\sin\mathrm{B}} = \frac{\sqrt{(ab+cd)(ad+bc)}}{4\Delta^2}\sqrt{4\Delta^2+(ab+cd)(ad+bc)}.$$

By reference 1, we obtain

$$a+c = b+d = s, \qquad \sqrt{abcd} = \Delta, \qquad R \geq r\sqrt{2},$$

so that

$$\begin{aligned}\sqrt{4\Delta^2+(ab+cd)(ad+bc)} &= \sqrt{4abcd + (a^2bd + c^2bd) + (acb^2 + acd^2)}\\ &= \sqrt{bd(a^2+2ac+c^2) + ac(b^2+2bd+d^2)}\\ &= \sqrt{s^2bd + s^2ac}\\ &= s\sqrt{ac+bd}.\end{aligned}$$