ALGEBRAIC THEORIES

A CATEGORICAL INTRODUCTION TO GENERAL ALGEBRA

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With a Preface by F.W. Lawvere

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Preface by F.W. Lawvere

Chapter 1

Algebraic theories and algebraic categories

In this chapter we introduce algebraic categories and study basic concepts, such that limits and colimits of algebras, and representable algebras.

1.1 Definition. An *algebraic theory* is a small category $\mathcal{T}$ with finite products. An *algebra* for the theory $\mathcal{T}$ is a functor $A\colon \mathcal{T} \to Set$ preserving finite products. We denote by $Alg\,\mathcal{T}$ the category of algebras of $\mathcal{T}$. Morphisms, called *homomorphisms*, are the natural transformations. That is, $Alg\,\mathcal{T}$ is a full subcategory of the functor category $Set^{\mathcal{T}}$.

1.2 Definition. A category is *algebraic* if it is equivalent to $Alg\,\mathcal{T}$ for some algebraic theory $\mathcal{T}$.

We will see in Chapter 10 that this corresponds well with varieties, i.e., equationally specified categories of (many-sorted, finitary) algebras.

1.3 Example.

1. Sets: The simplest algebraic category is the category of sets itself. An algebraic theory $\mathcal{T}_1$ for Set can be described as the full subcategory of Set^{op} whose objects are the natural numbers (compare with 7.12). In fact, since $n = 1 \times \ldots \times 1$ in Set^{op}, the category $\mathcal{T}_1$ has finite products. And every algebra $A\colon \mathcal{T}_1 \to Set$ is determined, up to isomorphism, by the set $A(1)$, since $A(n) \simeq A(1) \times \ldots \times A(1)$. More precisely, we have an equivalence functor

   $$E\colon Alg\,\mathcal{T}_1 \to Set\,, \quad A \mapsto A(1)\,.$$

   The category Set has other algebraic theories – we describe them all in Chapter 12.

2. Abelian groups: We can describe a theory $\mathcal{T}_{ab}$ for the category Ab of abelian groups as the full subcategory of Ab^{op} whose objects are of the

form $\mathbb{Z}^n$, for n a natural number (compare with 7.12). Alternatively, we denote by $\mathcal{T}_{ab}$ the category having natural numbers as objects, and morphisms from n to k are matrices of integers with n columns and k rows. Composition of $P\colon m \to n$ and $Q\colon n \to k$ is given by matrix multiplication $Q \cdot P\colon m \to k$, and identity morphisms are the unit matrices. If $n = 0$ or $k = 0$, the only $n \times k$ matrix is the empty one $[\,]$. $\mathcal{T}_{ab}$ has finite products. For example, 2 is the product 1×1 with projections $[1, 0]\colon 2 \to 1$ and $[0, 1]\colon 2 \to 1$. (In fact, given one-row matrices $P, Q\colon n \to 1$, there exists a unique two-row matrix $R\colon n \to 2$ such that $[1, 0]\cdot R = P$ and $[0, 1]\cdot R = Q$: the matrix with rows P and Q.) We will show in Chapter 7 that Ab is equivalent to $Alg\,\mathcal{T}_{ab}$. Here is a direct argument. Every abelian group G defines an algebra $\hat{G}\colon \mathcal{T}_{ab} \to Set$ whose object function is $\hat{G}(n) = G^n$. For every morphism $P\colon n \to k$ we define $\hat{G}(P)\colon G^n \to G^k$ by matrix multiplication

$$\hat{G}(P)\colon \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \mapsto P \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}.$$

The function $G \mapsto \hat{G}$ extends to a functor $\widehat{(-)}\colon Ab \to Alg\,\mathcal{T}_{ab}$ in a rather obvious way: given a group homomorphism $h\colon G_1 \to G_2$, then $\hat{h}\colon \hat{G}_1 \to \hat{G}_2$ is the natural transformation whose components are $h^n\colon G_1^n \to G_2^n$. It is obvious that $\widehat{(-)}$ is a well defined, full and faithful functor. To prove that it is an equivalence functor, we need, for every algebra $A\colon \mathcal{T}_{ab} \to Set$, to present an abelian group G with $A \simeq \hat{G}$. The underlying set of G is $A(1)$. The binary group operation is obtained from the morphism $[1, 1]\colon 2 \to 1$ in $\mathcal{T}_{ab}$ by $A[1, 1]\colon G^2 \to G$, the neutral element is $A[\,]\colon 1 \to G$ for the morphism $[\,]\colon 0 \to 1$ of $\mathcal{T}_{ab}$, and the inverse is given by $A[-1]\colon G \to G$. It is not difficult to check that the axioms of abelian group are fulfilled. For example, the axiom $x + 0 = x$ follows from the fact that A preserves the composition of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}\colon 1 \to 2$ with $[1, 1]\colon 2 \to 1$. Clearly, $A \simeq \hat{G}$ (consider the canonical isomorphism $A(n) = A(1 \times \ldots \times 1) \simeq A(1) \times \ldots \times A(1) = G^n = \hat{G}(n)$).

3. R-modules: Abelian groups are precisely $\mathbb{Z}$-modules. It is easy to generalize the previous example replacing abelian groups with R-modules, for R any unitary ring (in particular, R can be a field). Algebraic categories of the form *R-Mod* are treated in greater detail in Chapter 16.

4. Set-valued functors: If $\mathcal{C}$ is a small category, the functor category $Set^{\mathcal{C}}$ is algebraic. An algebraic theory $\mathcal{T}_{\mathcal{C}}$ of $Set^{\mathcal{C}}$ is a *free completion of $\mathcal{C}$ under finite products*. This can be described as the category whose objects are all finite families

$$(C_i)_{i \in I}\,,\ I \text{ finite}$$

of objects of $\mathcal{C}$ and whose morphisms from $(C_i)_{i\in I}$ to $(C'_j)_{j\in J}$ are pairs (a, α) where $a\colon J \to I$ is a function and $\alpha = (\alpha_j)_{j\in J}$ is a family of morphisms $\alpha_j\colon C_{a(j)} \to C'_j$ of $\mathcal{C}$. The composition of $(a,\alpha)\colon (C_i)_{i\in I} \to (C'_j)_{j\in J}$ and $(b,\beta)\colon (C'_j)_{j\in J} \to (C''_k)_{k\in K}$ is given by the function $a\cdot b\colon K \to I$ and the family $\beta_k \cdot \alpha_{b(k)}\colon C_{a(b(k))} \to C''_k$. It is easy to verify that the embedding

$$U\colon \mathcal{C} \to \mathcal{T}_{\mathcal{C}}\ , \quad C \mapsto (C)$$

is indeed a free completion of $\mathcal{C}$ under finite products. Consequently, the categories $\mathit{Set}^{\mathcal{C}}$ and $\mathit{Alg}\,\mathcal{T}_{\mathcal{C}}$ are equivalent: the functor $E\colon \mathit{Set}^{\mathcal{C}} \to \mathit{Alg}\,\mathcal{T}_{\mathcal{C}}$ which to every $H\colon \mathcal{C} \to \mathit{Set}$ assigns the unique algebra $H'\colon \mathcal{T}_{\mathcal{C}} \to \mathit{Set}$ with $H = H'\cdot U$ is an equivalence of categories.

5. Many-sorted sets: The category of S-sorted sets and S-sorted functions is simply the product category Set^{S}. This is example 4. above, where $\mathcal{C}$ is the discrete category with object set S. Here $\mathcal{T}_{\mathcal{C}}$ is the category of all finite families $(c_i)_{i\in I}$ in S, morphisms are $a\colon (c_i)_{i\in I} \to (c'_j)_{j\in J}$ where $a\colon J \to I$ is a function with $c_{a(j)} = c'_j$ for every j. If S has just one element, this is the theory $\mathcal{T}_1$ described in 1. above.

6. Graphs: We denote by Gr the category of directed graphs G with multiple edges: they are given by a set G_v of vertices, a set G_e of edges, and two functions $G_e \to G_v$ determining the target and the source of every edge. This is a special case of 4. with the obvious category $\mathcal{C}\colon\ e \rightrightarrows v$ (identity morphisms are not depicted).

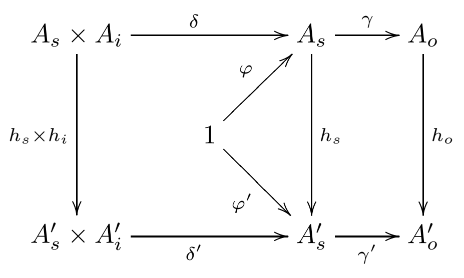
7. Sequential automata (Aut) : A sequential (deterministic) automaton A is given by a set A_s of states, a set A_i of input symbols, a set A_o of output symbols, and by three functions

   $\delta\colon A_s \times A_i \to A_s$ (next-state function)
   $\gamma\colon A_s \to A_o$ (output)
   $\varphi\colon 1 \to A_s$ (initial state)

   Given two sequential automata A and $A' = (A'_s, A'_i, A'_o, \delta', \gamma', \varphi')$, a morphism (simulation) is given by a triple of functions

$$h_s\colon A_s \to A'_s\ ,\quad h_i\colon A_i \to A'_i\ ,\quad h_o\colon A_o \to A'_o$$

   such that the diagram



   commutes. An algebraic theory for automata is described in 9.12.2. below.

1.4 Remark. The category of algebras is quite rich. Firstly, every object t of an algebraic theory $\mathcal{T}$ yields the algebra $Y(t)$ representable by t :

$$Y(t) = \mathcal{T}(t, -) \colon \mathcal{T} \to Set\,.$$

This, together with the Yoneda transformations, yields a full and faithful functor

$$Y_{\mathcal{T}} \colon \mathcal{T}^{op} \to Alg\,\mathcal{T}\ .$$

Secondly, $Alg\,\mathcal{T}$ is, as we observe below, closed under limits in the functor category $Set^{\mathcal{T}}$ which yields further examples.

1.5 Example. For abelian groups, we have

$$Y \colon \mathcal{T}_{ab}^{op} \to Alg\,\mathcal{T}_{ab} \simeq Ab$$

where, for $n \in \mathbb{N}$, $Y(n) = \mathbb{Z}^n$ is a free abelian group on n generators.

Other algebras can be obtained e.g. by the formation of limits and colimits. We will now show that limits always exist and are built up at the level of sets. Also colimits always exist but they are seldom built up at the level of sets. We will study colimits in Chapters 2 and 4.

1.6 Proposition. *For every algebraic theory $\mathcal{T}$, the category $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under limits.*

Proof. Limits are formed objectwise in $Set^{\mathcal{T}}$. Since limits and finite products commute, given a diagram in $Set^{\mathcal{T}}$ whose objects are functors preserving finite products, then a limit of that diagram also preserves finite products. $\square$

1.7 Corollary. *Every algebraic category is complete.*

1.8 Remark.

1. The previous proposition means that limits of algebras are formed at the level of underlying sets. For example, a product of two graphs has both the vertex set given by the cartesian product of the vertex sets, and the set of edges given by the cartesian product of the edge sets.

2. Monomorphisms in the category $Alg\,\mathcal{T}$ are precisely the homomorphisms that are componentwise monomorphisms (i.e., injective functions) in *Set.* In fact, this is true in $Set^{\mathcal{T}}$, and $Alg\,\mathcal{T}$ is closed under monomorphisms (being closed under limits) in $Set^{\mathcal{T}}$.

3. Recall that a *kernel pair* of a morphism $f \colon A \to B$ is a pair $k_1, k_2 \colon K \rightrightarrows A$ forming a pullback of f and f. In every algebraic category kernel pairs exist and are formed componentwise (in *Set*).

Chapter 2

Sifted and filtered colimits

Colimits in algebraic categories are, in general, not formed objectwise. In this chapter, we study the important case of sifted colimits, which are always formed objectwise. Prominent examples of sifted colimits are filtered colimits and reflexive coequalizers (see Chapter 3).

2.1 Definition.

1. A small category $\mathcal{D}$ is called *sifted* if finite product in *Set* commute with colimits over $\mathcal{D}$. Colimits of diagrams over sifted categories are called *sifted colimits.*

2. A small category $\mathcal{D}$ is called *filtered* if finite limits in *Set* commute with colimits over $\mathcal{D}$. Colimits of diagrams over filtered categories are called *filtered colimits.*

2.2 Remark.

1. Explicitly, a small category $\mathcal{D}$ is sifted if, given a diagram $D\colon \mathcal{D}\times\mathcal{J} \to Set$ where $\mathcal{J}$ is a finite discrete category, then the canonical map

$$\underset{\mathcal{D}}{colim} \left(\prod_{\mathcal{J}} D(d,j)\right) \to \prod_{\mathcal{J}} (\underset{\mathcal{D}}{colim}\ D(d,j))$$

   is an isomorphism. $\mathcal{D}$ is filtered if it satisfies the same condition, but with respect to every finite category $\mathcal{J}$ (replace $\prod_{\mathcal{J}}$ by $lim_{\mathcal{J}}$ in the previous formula).

2. The more usual definition of filtered category $\mathcal{D}$ is to say that every finite subcategory of $\mathcal{D}$ has a compatible cocone in $\mathcal{D}$. And a well-known result states that this implies the property of Definition 2.1.2: if $\mathcal{D}$ is a small filtered category, then finite limits commute with $\mathcal{D}$-limits in *Set*. See e.g. Theorem 2.13.6 in [Borceux 1]. The converse implication is trivial: if finite limits commute with $\mathcal{D}$-colimits in *Set* and $\mathcal{J}$ is a finite subcategory of $\mathcal{D}$, we want to show that a compatible cocone with

codomain some $d \in obj\mathcal{D}$ exists. The diagram $\mathcal{D}(-, d)\colon \mathcal{J}^{op} \to Set$ has as a limit the set of all such compatible cocones. Thus, we have to show that $\lim_{\mathcal{J}^{op}} \mathcal{D}(j, d) \neq \emptyset$ for some d, for which it is certainly sufficient to prove $colim_{\mathcal{D}}(lim_{\mathcal{J}^{op}}\mathcal{D}(j, d)) \neq \emptyset$. By assumption, the latter set is isomorphic to a limit of $colim_{\mathcal{D}}\mathcal{D}(j, d) \simeq 1$, thus, the above colimit is 1.

2.3 Proposition. *For every algebraic theory $\mathcal{T}$, the category $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under sifted colimits.*

Proof. Since sifted colimits and finite products commute in *Set*, they do so in $Set^{\mathcal{T}}$ (where they are computed objectwise). It follows that a sifted colimit in $Set^{\mathcal{T}}$ of functors preserving finite products also preserves finite products. □

2.4 Corollary. *In every algebraic category:*

1. *Sifted colimits commute with finite products;*

2. *Filtered colimits commute with finite limits.*

In order to provide useful characterization of sifted categories, let us look more carefully at the condition expressed in 2.2.

2.5 Remark.

1. A small category $\mathcal{D}$ is sifted iff it is nonempty and binary products in *Set* commute with colimits over $\mathcal{D}$. The latter condition means that, given diagrams $D, D'\colon \mathcal{D} \to Set$, then the canonical morphism

$$\underset{\mathcal{D}}{colim}\ (Dd \times D'd) \to (\ \underset{\mathcal{D}}{colim}\ Dd) \times (\ \underset{\mathcal{D}}{colim}\ D'd)$$

is an isomorphism. The fact that $\mathcal{D}$ is nonempty is equivalent to the condition of 2.2 for $\mathcal{J} = \emptyset$.

2. Consider the functor

$$D \times D'\colon \mathcal{D} \times \mathcal{D} \to Set\,, \quad (d, d') \mapsto Dd \times D'd'\,.$$

Since, for any set X, the functor $X \times -\colon Set \to Set$ preserves colimits, the canonical morphism

$$\underset{\mathcal{D}\times\mathcal{D}}{colim}\ (D \times D') \to (\ \underset{\mathcal{D}}{colim}\ Dd) \times (\ \underset{\mathcal{D}}{colim}\ D'd)$$

is an isomorphism. Therefore, the condition that binary products in *Set* commute with colimits over $\mathcal{D}$ can be restated saying that the canonical morphism

$$\underset{\mathcal{D}}{colim}\ ((D \times D') \cdot \Delta) \to \ \underset{\mathcal{D}\times\mathcal{D}}{colim}\ (D \times D')$$

is an isomorphism, where $\Delta\colon \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is the diagonal functor.

To obtain a characterization of sifted categories, it remains to recall some facts about final functors.

2.6 Definition.

1. A finctor $F\colon \mathcal{D}' \to \mathcal{D}$ is called *final* if, for every diagram $D\colon \mathcal{D} \to \mathcal{A}$ such that *colim* D exists in $\mathcal{A}$, then *colim* $D \cdot F$ exists and the canonical morphism *colim* $D \cdot F \to$ *colim* D is an isomorphism.

2. A category $\mathcal{A}$ is called *connected* if it is nonempty and for every pair of objects X and X' in $\mathcal{A}$ there exists a zig-zag of morphisms between X and X'
$$X \to X_1 \leftarrow X_2 \to \ldots \to X_n \leftarrow X'\,.$$

2.7 Example. A category with an initial object is connected.

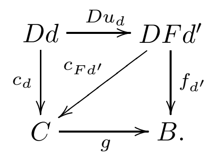
2.8 Lemma. *The following conditions on a functor $F\colon \mathcal{D}' \to \mathcal{D}$ are equivalent:*

1. *F is final;*

2. *F satisfies the finality condition with respect to all representable functors $\mathcal{D}(d,-)\colon \mathcal{D} \to Set$, $d \in obj\,\mathcal{D}$;*

3. *For every object d of $\mathcal{D}$, the slice category $d \downarrow F$ of all arrows $d \to Fd'$, $d' \in obj\,\mathcal{D}'$, is connected.*

Proof. 1 $\Rightarrow$ 2 is trivial and 2 $\Rightarrow$ 3 follows from the usual description of colimits in *Set* and the fact that since the diagram $\mathcal{D}(d,-)$ has colimit 1, so does the diagram $\mathcal{D}(d,F-) = \mathcal{D}(d,-)\cdot F$.
To prove 3 $\Rightarrow$ 1 let $D\colon \mathcal{D} \to \mathcal{A}$ be a diagram with a colimit $c_d\colon Dd \to C$ ($d \in obj\,\mathcal{D}$). We show that $D \cdot F$ has $c_{Fd'}\colon DFd' \to C$, ($d' \in obj\,\mathcal{D}'$) as a colimit. To prove the universal property of that cocone, let $f_{d'}\colon DFd' \to B$ be a cocone of $D \cdot F$. For every object d of $\mathcal{D}$, choose a morphism $u_d\colon d \to Fd'$ for some $d' \in obj\,\mathcal{D}'$, and put $g_d = f_{d'} \cdot D(u_d)\colon Dd \to B$. Since $d \downarrow F$ is connected, it is easy to verify that g_d does not depend on the choice of d' and u_d, and that these morphisms form a cocone of D. The unique factorization morphism $g\colon C \to B$ with $g \cdot c_d = g_d$ is a factorization as desired:



To show that g is unique use the fact that since $(c_d)_{d \in obj\,\mathcal{D}}$ is collectively epimorphic, so is $(c_{Fd'})_{d' \in obj\,\mathcal{D}'}$. ◻

2.9 Proposition. *A small category $\mathcal{D}$ is sifted if and only if it is nonempty and the diagonal functor $\Delta\colon \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is final.*

Proof. If Δ is final, then for every pair of diagrams $D, D'\colon \mathcal{D} \to Set$ the canonical morphism *colim* $((D \times D') \cdot \Delta) \to$ *colim* $(D \times D')$ is an isomorphism. Following 2.5.2, $\mathcal{D}$ is sifted.

Conversely, given two objects d and d' in $\mathcal{D}$, the representable functor $(\mathcal{D} \times \mathcal{D})((d, d'), -)$ is nothing but $D \times D' \colon \mathcal{D} \times \mathcal{D} \to \mathit{Set}$, with $D = \mathcal{D}(d, -)$ and $D' = \mathcal{D}(d', -)$, therefore 2.5.2 and 2.8.2 imply that Δ is final. $\square$

2.10 Example. Every small category with finite coproducts is sifted. In fact, it contains an initial object, and the slice category $(A, B) \downarrow \Delta$ is connected, having the coproduct of A and B as an initial object.

2.11 Remark. An analogy of 2.9 holds for filtered categories: a small category $\mathcal{D}$ is filtered iff for every finite category $\mathcal{J}$ the diagonal functor $\Delta \colon \mathcal{D} \to \mathcal{D}^{\mathcal{J}}$ is final. A proof can be found in [Gabriel-Ulmer].

Chapter 3

Reflexive coequalizers

An important case of sifted colimits are reflexive coequalizers, i.e., coequalizers of parallel pairs of split epimorphisms having a joint splitting. More formally:

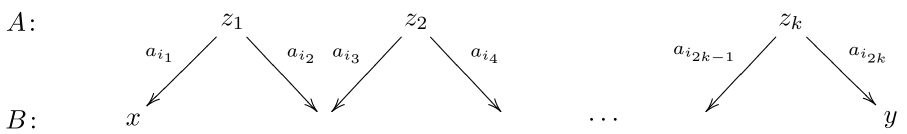
3.1 Example. Consider the category $\mathcal{D}$ given by the morphisms

$$A \;\substack{\xrightarrow{\;a_1\;} \\ \xleftarrow{\;\;d\;\;} \\ \xrightarrow[\;a_2\;]{}}\; B$$

(identity morphisms are not depicted) composed freely modulo $a_1 \cdot d = \mathrm{id}_B = a_2 \cdot d$. This category is sifted: it is an easy exercise to check that the categories $(A, A) \downarrow \Delta$, $(A, B) \downarrow \Delta$, $(B, B) \downarrow \Delta$ are connected. Colimits of diagrams $D \colon \mathcal{D} \to \mathcal{A}$ are called *reflexive coequalizers*, they are coequalizers of pairs having a common section.
Another method to prove that $\mathcal{D}$ is a sifted category is to prove directly that in *Set* reflexive coequalizers commute with binary products. In fact, suppose that

$$A \underset{a_2}{\overset{a_1}{\rightrightarrows}} B \xrightarrow{\;c\;} C \quad \text{and} \quad A' \underset{a'_2}{\overset{a'_1}{\rightrightarrows}} B' \xrightarrow{\;c'\;} C'$$

are reflexive coequalizers in *Set*. We can assume, without loss of generality, that c is the canonical function on the quotient $C = B/\sim$ modulo the equivalence relation described as follows: two elements $x, y \in B$ are equivalent iff there exists a zig-zag



where $i_1, i_2, \ldots, i_{2k}$ are 1 or 2 (and the values change from neighbor to neighbor). Thus, the zig-zag is determined by its length, k, and by its type, given by i_1(= 1 or 2). Now if the pair a_1, a_2 is reflexive, we can fix the type to be $i_1 = 1$, and we can increase the length to any number $k+1, k+2, \ldots$. Analogously, we can assume $C' = B'/\sim'$ where $\sim'$ is the equivalence relation given by zig-zags of a'_1 and a'_2. Now we form the pair

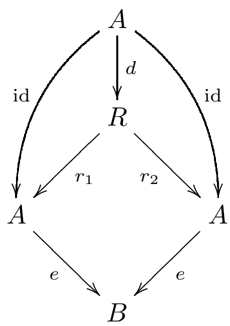
$$A \times A' \underset{a_2 \times a'_2}{\overset{a_1 \times a'_1}{\rightrightarrows}} B \times B'$$

and obtain a coequalizer given by the zig-zag equivalence $\approx$ on $B \times B'$. Now given any pair $(x, x') \approx (y, y')$ in $B \times B'$, we obviously have zig-zags both for $x \sim y$ and for $x' \sim y'$ (projection of the zig-zag for the given pair). But also the other way round: whenever $x \sim y$ and $x' \sim' y'$, then we choose the two zig-zags so that they both have type 1 and have the same lengths. They create an obvious zig-zag for $(x, x') \approx (y, y')$. From this it follows that the map

$$A \times A' \underset{a_2 \times a'_2}{\overset{a_1 \times a'_1}{\rightrightarrows}} B \times B' \xrightarrow{\ c \times c'\ } (B/\sim) \times (B'/\sim')$$

is a coequalizer, as required.

3.2 Example. Recall that a morphism $e\colon A \to B$ is called a *regular epimorphism* if it is a coequalizer of a parallel pair. Every regular epimorphism is a reflexive coequalizer (in a category with kernel pair). In fact, if r_1, r_2 is a kernel pair of $e\colon A \to B$



then e is a coequalizer of r_1, r_2. And since $e \cdot \mathrm{id} = e \cdot \mathrm{id}$, there exists a unique d with $r_1 \cdot d = \mathrm{id} = r_2 \cdot d$.

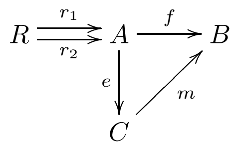
3.3 Corollary. *For every algebraic theory $\mathcal{T}$, the category $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under reflexive coequalizers and regular epimorphisms. Therefore regular epimorphisms in $Alg\,\mathcal{T}$ are precisely the homomorphisms which are componentwise epimorphisms (i.e., surjective functions) in Set.*

Proof. The first part of the statement follows from 2.3, 3.1 and 3.2. The second one follows, since it is true in $Set^{\mathcal{T}}$. ☐

In particular, every algebraic category is *co-wellpowered* with respect to regular epimorphisms. This means that, for a fixed object A, the regular epimorphisms with domain A constitute a small set. This is true in *Set* and therefore, by 3.3, in every algebraic category.

3.4 Corollary. *Every algebraic category has* regular factorizations*, i.e., every morphism is a composite of a regular epimorphism followed by a monomorphism.*

Proof. The category $Set^{\mathcal{T}}$ has regular factorizations: given a morphism $f\colon A \to B$, form a kernel pair $r_1, r_2\colon R \rightrightarrows A$ and its coequalizer $e\colon A \to C$. The factorizing morphism m



is a monomorphism. Since $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under kernel pairs and their coequalizers, it inherits the regular factorization from $Set^{\mathcal{T}}$. □

3.5 Example. In *Ab* we know that:

1. Coproducts are not formed at the level of sets. In fact, $A + B = A \times B$ for all abelian groups A, B.

2. Reflexive coequalizers are formed at the level of sets, but general coequalizers are not. Consider e.g. the pair $x \mapsto 2x$ and $x \mapsto 0$ of endomorphisms of $\mathbb{Z}$ whose coequalizer in *Ab* is finite and in *Set* is infinite.

3.6 Remark. We provided a simple characterization of monomorphisms (1.8) and regular epimorphisms (3.3) in algebraic categories. There does not seem to be a simple characterization of the dual concepts (epimorphisms and regular monomorphisms). In fact, there exist algebraic categories with non-surjective epimorphisms and with non-regular monomorphisms, as we show in the following example.

3.7 Example. In the category of semigroups, consider the multiplicative semigroups $\mathbb{Z}$ of integers and $\mathbb{Q}$ of rationals. The embedding $i\colon \mathbb{Z} \to \mathbb{Q}$ is both a monomorphism (because it is injective) and an epimorphism. In fact, consider homomorphisms $h, k\colon \mathbb{Q} \to A$ such that $h \cdot i = k \cdot i$. That is, $h(n) = k(n)$ for every integer n. To prove $h = k$, it is sufficient to verify $h(1/m) = k(1/m)$ for all integers $m \neq 0$: this follows from $h(m) \cdot h(1/m) = k(m) \cdot k(1/m) = 1$ (since $h(1) = k(1) = 1$). Consequently, i is not a regular epimorphism, nor a regular monomorphism.

3.8 Remark. Recall that in a finitely complete category $\mathcal{A}$ *relations* on an object A can be understood as subobjects of $A \times A$. These are represented by a monomorphism $r\colon R \to A \times A$ or by a parallel pair $r_1, r_2\colon R \rightrightarrows A$ of morphisms that are jointly monic. The following definitions formalize the corresponding concepts for relations in *Set*.

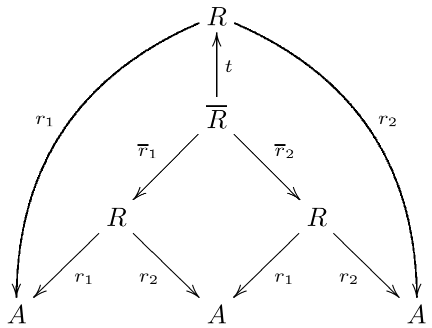
3.9 Definition. A relation $r_1, r_2\colon R \rightrightarrows A$ in a category $\mathcal{A}$ is called

1. *reflexive* if the pair r_1, r_2 is reflexive ($r_1 \cdot d = \mathrm{id} = r_2 \cdot d$ for some $d\colon A \to R$),

2. *symmetric* if there exists $s\colon R \to R$ with $r_1 = r_2 \cdot s$ and $r_2 = r_1 \cdot s$

and

3. *transitive* provided that for a pullback $\overline{R}$ of r_1 and r_2 there exists a morphism $t\colon \overline{R} \to R$ such that the diagram



commutes.

4. An *equivalence relation* is a relation which is reflexive, symmetric and transitive.

3.10 Remark.

1. An equivalence relation in *Set* is precisely an equivalence relation in the usual sense.

2. Given a parallel pair $r_1, r_2\colon R \rightrightarrows A$ of morphisms and an object X, we can define a relation $\underset{R}{\sim}$ in the hom-set $\mathcal{A}(X, A)$ as follows: $f \underset{R}{\sim} g$ when there exists a morphism $H\colon X \to R$ such that $r_1 \cdot H = f$ and $r_2 \cdot H = g$. It is easy to check that $r_1, r_2\colon R \to A$ is an equivalence relation in $\mathcal{A}$ iff $\underset{R}{\sim}$ is an equivalence relation in *Set* for all X in $\mathcal{A}$.

3. Kernel pairs (1.8.3) are equivalence relations.

4. A category is said to have *effective equivalence relations* provided that every equivalence relation is a kernel pair, i.e., is a pullback of some morphism with itself. It is easy to see that *Set* has this property, but e.g. the category of posets does not (take an arbitrary poset B and an equivalence relation R on the underlying set of B equipped with the discrete ordering, then the two projections $R \rightrightarrows B$ form an equivalence relation which is seldom a kernel pair).

3.11 Definition. A finitely complete category with coequalizers of kernel pairs is called *exact* if it has effective equivalence relations, and if its regular epimorphisms are stable under pullbacks. That is, in every pullback

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow[v]{} & D \end{array}$$

if v is a regular epimorphism, then so is u.

3.12 Corollary. *Every algebraic category is exact.*

Proof. Since equivalence relations are reflexive pairs, the exactness of $Alg\,\mathcal{T}$ follows from that (obvious) of $Set^{\mathcal{T}}$ using 1.8 and 3.3. □

Let us end this chapter quoting some more exactness properties of algebraic categories.

3.13 Corollary. *In every algebraic category:*

1. *Regular epimorphisms are stable under products. This means that, given regular epimorphisms $e_i \colon A_i \to B_i$ $(i \in I)$, then*

$$\prod_{i \in I} e_i \colon \prod_{i \in I} A_i \to \prod_{i \in I} B_i$$

   is a regular epimorphism.

2. *Filtered colimits distribute over products. This means that, given a set of filtered diagrams $D_i \colon \mathcal{D}_i \to \mathcal{A}$, $(i \in I)$, we can form the diagram*

$$D \colon \prod_{i \in I} \mathcal{D}_i \to \mathcal{A}\,, \quad (d_i) \mapsto \prod_{i \in I} D_i(d_i)$$

   and the canonical morphism

$$\underset{\prod \mathcal{D}_i}{colim}\; D \to \prod_{i \in I} (\underset{\mathcal{D}_i}{colim}\; D_i)$$

   is an isomorphism.

3. *Sifted colimits distribute over finite products. (As in 2., but with the $\mathcal{D}_i$ sifted categories and I a finite set.)*

Proof. Each of the three statements is easy to prove in *Set*, and then in $Set^{\mathcal{T}}$, where limits and colimits are formed objectwise. Following 1.6, 2.3 and 3.3, the statements hold in $Alg\,\mathcal{T}$ for every algebraic theory $\mathcal{T}$. □

3.14 Remark.

1. In 3.13.2 and 3.13.3 we implicitly assume the existence of colimits in an algebraic category. This is not a restriction, because every algebraic category is cocomplete, as we prove in Chapter 4.

2. In *Set*, all colimits distribute over finite products. This is not true in algebraic categories in general. It is easy to construct a counterexample using the empty diagram in the category of abelian groups.

Chapter 4

Algebraic categories as free cocompletions

In this chapter we prove that every algebraic category has colimits. In fact, the category $Alg\,\mathcal{T}$ is a free *FC-conservative* cocompletion of $\mathcal{T}^{op}$ under colimits, where FC-conservative means: preserving finite coproducts.

For the existence of colimits, since we already know that $Alg\,\mathcal{T}$ has sifted colimits and, in particular, reflexive coequalizers (see 2.1, 2.3 and 3.3), all we need to establish are finite coproducts. Indeed, coproducts then exist because they are filtered colimits of finite coproducts. And coproducts and reflexive coequalizers construct all colimits: the classical theorem that coproducts and coequalizers construct colimits (see e.g. [MacLane]) only uses reflexive coequalizers. Here is the first step: finite coproducts of representable algebras, including an initial object of $Alg\,\mathcal{T}$.

4.1 Lemma. *For every algebraic theory $\mathcal{T}$, the Yoneda embedding (1.4)*

$$Y_{\mathcal{T}} \colon \mathcal{T}^{op} \to Alg\,\mathcal{T}$$

preserves finite coproducts.

Proof. If 1 is a terminal object of $\mathcal{T}$ then $\mathcal{T}(1,-)$ is a initial object of $Alg\,\mathcal{T}$: for every algebra A we know that $A(1)$ is a terminal object, thus there is a unique morphism $\mathcal{T}(1,-) \to A$.
Given two objects t_1, t_2 in $\mathcal{T}$ then $\mathcal{T}(t_1 \times t_2, -)$ is a coproduct of $\mathcal{T}(t_1,-)$ and $\mathcal{T}(t_2,-)$ since for every algebra A the morphisms $\mathcal{T}(t_1 \times t_2, -) \to A$ correspond to elements of $A(t_1 \times t_2) = A(t_1) \times A(t_2)$. □

4.2 Remark. Recall that every functor $A \colon \mathcal{C}^{op} \to Set$ is in a canonical way a colimit of representable functors. In fact, consider the Yoneda embedding

$$Y \colon \mathcal{C} \to Set^{\mathcal{C}^{op}}$$

and the slice category $El\,A = Y \downarrow A$ of "elements of A". Then A is the colimit of

$$El\,A \xrightarrow{\Phi_A} \mathcal{C} \xrightarrow{Y} Set^{\mathcal{C}^{op}}$$

where Φ_A is the canonical projection which to every element of the set $A(X)$ assigns the object X.

4.3 Lemma. *Given an algebraic theory $\mathcal{T}$, for every functor A in $Set^{\mathcal{T}}$ the following conditions are equivalent:*

1. *A is an algebra,*
2. *$(El\,A)^{op}$ is a sifted category*

and

3. *A is a sifted colimit of representable algebras.*

Proof. $2 \Rightarrow 3$: This follows from 4.2.
$3 \Rightarrow 1$: Representable functors are in $Alg\,\mathcal{T}$ (1.4) and $Alg\,\mathcal{T}$ is closed in $Set^{\mathcal{T}}$ under sifted colimits (2.3).
$1 \Rightarrow 2$: Assume that $A\colon \mathcal{T} \to Set$ preserves finite products. We have to prove that $(El\,A)^{op}$ is sifted. Following 2.10, it suffices to prove that $El\,A$ have finite products. This is obvious: for example, the product of $(X \in \mathcal{T}, x \in A(X))$ and $(Z \in \mathcal{T}, z \in A(Z))$ is nothing but $(X \times Z, (x,z) \in A(X) \times A(Z) = A(X \times Z))$. ☐

4.4 Lemma.

1. *If two functors $F\colon \mathcal{D} \to \mathcal{C}$, $G\colon \mathcal{B} \to \mathcal{A}$ are final, then the product functor $F \times G\colon \mathcal{D} \times \mathcal{B} \to \mathcal{C} \times \mathcal{A}$ is final.*
2. *A product of two sifted categories is sifted.*

Proof. 1: This follows from 2.8.3 because, for any object (d,b) in $\mathcal{D} \times \mathcal{B}$, the slice category $(d,b) \downarrow F \times G$ is nothing but the product category $(d \downarrow F) \times (b \downarrow G)$, and the product of two connected categories is connected.
2: Obvious from 1. and 2.9. ☐

4.5 Theorem. *Every algebraic category is cocomplete.*

Proof. As explained at the beginning of this chapter, we only need to establish finite coproducts $A + B$ in $Alg\,\mathcal{T}$. Express A as a sifted colimit of representable algebras (4.3)

$$A = colim\,[Y \cdot \Phi_A\colon El\,A \to \mathcal{T}^{op} \to Set^{\mathcal{T}}]$$

and analogously for B. The category

$$\mathcal{D} = (El\,A)^{op} \times (El\,B)^{op}$$

is sifted by 4.4 and we have two colimits in $Alg\,\mathcal{T}$ over $\mathcal{D}$:

$$A = colim\,Y \cdot \Phi_A \cdot P_1 \quad \text{and} \quad B = colim\,Y \cdot \Phi_B \cdot P_2$$

for the projections P_1, P_2 of $\mathcal{D}$. The diagram $D\colon \mathcal{D} \to Alg\,\mathcal{T}$ assigning to every pair (x,z) a coproduct of the representable algebras (see 4.1)

$$(x,z) \mapsto Y \cdot \Phi_A(x) + Y \cdot \Phi_B(z) \quad \text{(in } Alg\,\mathcal{T})$$

is sifted, thus it has a colimit in $Alg\,\mathcal{T}$. Since colimits over $\mathcal{D}$ always commute with finite coproducts, we get

$$colim\, D = \underset{(x,z)}{colim}\ Y \cdot \Phi_A(x) + \underset{(x,z)}{colim}\ Y \cdot \Phi_B(z) = A + B.$$

□

4.6 Remark. We are ready to characterize algebraic categories as free conservative cocompletions. Before we define this and prove the result, let us recall the general concept of a free cocompletion of a category $\mathcal{C}$: this is, roughly speaking, a cocomplete category $\mathcal{A}$ in which $\mathcal{C}$ is a full subcategory such that every functor from $\mathcal{C}$ to a cocomplete category has an *essentially unique* extension (that is, unique up to natural isomorphism) to a colimit-preserving functor with domain $\mathcal{A}$. In the following definition we say this more precisely. Also, for a given class $\mathbb{D}$ of small categories we define a free cocompletion with respect to $\mathbb{D}$, meaning that all colimits considered are $\mathbb{D}$-colimits, i.e., colimits of diagrams with domains that are elements of $\mathbb{D}$.

4.7 Definition. Let $\mathbb{D}$ be a class of small categories. By a *free cocompletion of a category* $\mathcal{C}$ *with respect to* $\mathbb{D}$ is meant a functor $H\colon \mathcal{C} \to \mathcal{A}$ such that

1. $\mathcal{A}$ is a category with $\mathbb{D}$-colimits

and

2. for every functor $F\colon \mathcal{C} \to \mathcal{B}$, where $\mathcal{B}$ is a category with $\mathbb{D}$-colimits, there exists an essentially unique functor $F^*\colon \mathcal{A} \to \mathcal{B}$ preserving $\mathbb{D}$-colimits with F naturally isomorphic to $F^* \cdot H$.

If $\mathbb{D}$ consists of all small categories, $F\colon \mathcal{C} \to \mathcal{A}$ is called a *free cocompletion* of $\mathcal{C}$.

4.8 Proposition. *For every small category* $\mathcal{C}$, *the Yoneda embedding*

$$Y\colon \mathcal{C} \to Set^{\mathcal{C}^{op}}$$

is a free cocompletion of $\mathcal{C}$.

Proof. Let $F\colon \mathcal{C} \to \mathcal{B}$ a functor, where $\mathcal{B}$ has colimits. Since $F^*\colon Set^{\mathcal{C}^{op}} \to \mathcal{B}$ should extend F and preserve colimits, we are forced to define it on objects (using the notation of 4.2) by

$$F^*(A) = \underset{El\,A}{colim}\ (F \cdot \Phi_A)\,.$$

The definition on morphisms $h\colon A_1 \to A_2$ is also obvious: h induces a functor

$$El\,h\colon El\,A_1 \to El\,A_2\ , \quad (X \in \mathcal{C}, x \in A_1(X)) \mapsto (X, h_X(x) \in A_2(X))\,.$$

By the universal property of the colimit, $El\,h$ induces a morphism

$$h' \colon colim\,(F \cdot \Phi_{A_1}) \to colim\,(F \cdot \Phi_{A_2})$$

and we are forced to define $F^*(h) = h'$.
Conversely, the above rule $A \mapsto colim\,(F \cdot \Phi_A)$ defines a functor $F^* \colon Set^{\mathcal{C}^{op}} \to \mathcal{B}$ which fulfils $F^* \cdot Y = F$ because, for $A = Y(X) = \mathcal{C}(-, X)$, a colimit of $F \cdot \Phi_A = \mathcal{B}(-FX)$ is FX. It remains to prove that F^* preserves colimits: this follows from the fact that F^* is left adjoint to the functor

$$\mathcal{B}(F-, -) \colon \mathcal{B} \to Set^{\mathcal{C}^{op}}.$$

The natural bijection

$$\mathcal{B}(F^*A, B) \simeq \mathrm{Nat}(A, \mathcal{B}(F-, B))$$

immediately follows from the definition of F^*A and the Yoneda lemma. □

4.9 Remark. A famous classical example is a free cocompletion with respect to filtered colimits. For a small category $\mathcal{C}$ let $Ind\mathcal{C}$ be the category of all filtered colimts of representable functors in $Set^{\mathcal{C}^{op}}$. The codomain restriction of the Yoneda embedding

$$Y_{Ind} : \mathcal{C} \to Ind\mathcal{C}$$

is a free cocompletion of $\mathcal{C}$ under filtered colimits. We do not prove this result here, but we proceed analogously with sifted colimits, where we present a proof.

4.10 Notation. Let $\mathcal{C}$ be a small category. We denote by $Sind\mathcal{C}$ the full subcategory of $Set^{\mathcal{C}^{op}}$ of those functors $A \colon \mathcal{C}^{op} \to Set$ which are sifted colimits of representable functors. We write

$$Y_{Sind} : \mathcal{C} \to Sind\mathcal{C}$$

for the codomain restriction of the Yoneda embedding. Following 4.3, if $\mathcal{C}$ has finite coproducts then $Sind\mathcal{C} = Alg\,(\mathcal{C}^{op})$.

4.11 Definition. Let $\mathcal{C}$ be a small category with finite coproducts. A functor $H \colon \mathcal{C} \to \mathcal{A}$ is a *free FC-conservative cocompletion* of $\mathcal{C}$ if:

1. $\mathcal{A}$ has colimits and H preserves finite coproducts

and

2. for every functor $F \colon \mathcal{C} \to \mathcal{B}$ preserving finite coproducts, where $\mathcal{B}$ is a cocomplete category, there exists an essentially unique functor $F^* \colon \mathcal{A} \to \mathcal{B}$ preserving colimits with F naturally isomorphic to $F^* \cdot H$.

4.12 Remark. Following 4.3, if $\mathcal{C}$ has finite coproducts then $Sind\mathcal{C} = Alg\,(\mathcal{C}^{op})$.

4.13 Proposition. *Let $\mathcal{C}$ be a small category with finite coproducts. The Yoneda embedding*

$$Y_{Sind} : \mathcal{C} \to Sind\mathcal{C}$$

is a free FC-conservative cocompletion of $\mathcal{C}$.

Proof. Following 4.5, $Sind\mathcal{C} = Alg(\mathcal{C}^{op})$ is cocomplete. Moreover, by 4.1, Y_{Sind} preserves finite coproducts. To prove the universal property of Y_{Sind} we can follow step by step the proof of 4.8. Indeed, $F^*\colon Sind\mathcal{C} \to \mathcal{B}$ still has a right adjoint. In fact, since $F\colon \mathcal{C} \to \mathcal{B}$ preserves finite coproducts, $\mathcal{B}(F-,-)\colon \mathcal{B} \to Set^{\mathcal{C}^{op}}$ factors through $Alg(\mathcal{C}^{op})$. □

4.14 Corollary. *A category is algebraic if and only if it is a free FC-conservative cocompletion of a small category with finite coproducts.*

Proof. Every category $Alg\mathcal{T}$ is, by 4.3, equal to $Sind(\mathcal{T}^{op})$.
Conversely, if a category $\mathcal{A}$ is a free FC-conservative cocompletion of a small category $\mathcal{C}$ with finite coproducts, then (since free cocompletions are unique up to equivalence of categories) $\mathcal{A}$ is equivalent to $Sind\mathcal{C} = Alg(\mathcal{C}^{op})$. □

4.15 Remark. Algebraic categories are also precisely the free cocompletions with respect to sifted colimits of small categories with finite coproducts. This is a bit technical to prove, we thus only present a (simple) proof of a slightly less canonical result: in the following proposition we assume that the category $\mathcal{B}$ has all (not necessarily sifted) colimits.

4.16 Proposition. *For every small category $\mathcal{C}$ with finite coproducts, given a functor $F\colon \mathcal{C} \to \mathcal{B}$ where $\mathcal{B}$ is cocomplete, there exists an essentially unique functor $F^*\colon Sind\mathcal{C} \to \mathcal{B}$ preserving sifted colimits and such that F is naturally isomorphic to $F^* \cdot Y_{Sind}$.*

Proof. The requested functor is the restriction of the functor $F^*\colon Set^{\mathcal{C}^{op}} \to \mathcal{B}$ of 4.8. Since the latter preserves colimits (it has a right adjoint) and $Sind\mathcal{C}$ is closed in $Set^{\mathcal{C}^{op}}$ under sifted colimits (see 2.3), the former preserves sifted colimits. The essential uniqueness of $F^*\colon Sind\mathcal{C} \to \mathcal{B}$ follows from 4.3 (as in the proof of 4.8). □

4.17 Remark.

1. Lemma 4.3 can be adapted to filtered colimits: if $\mathcal{C}$ is a finitely complete small category, then a functor $A\colon \mathcal{C} \to Set$ preserves finite limits iff $El\,A$ is a filtered category iff A is a filtered colimit of representable functors. The proof is similar to that of 4.3.

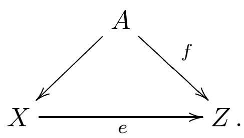
2. We mentioned in 4.9 that the Yoneda embedding $Y_{Ind}\colon \mathcal{C} \to Ind\mathcal{C}$ is a free cocompletion of $\mathcal{C}$ under filtered colimits. More is true: if $\mathcal{C}$ is finitely cocomplete, then $Ind\mathcal{C}$ is cocomplete and $Y_{Ind}\colon \mathcal{C} \to Ind\mathcal{C}$ is a free cocompletion of $\mathcal{C}$ conservative with respect to finite colimits. Once again, the proof is similar to that of 4.13 and we omit it.

Chapter 5

Properties of algebras

In this chapter properties of algebras such as finite presentability, regular projectivity, etc., are studied. These will be later used for a characterization of algebraic categories.

5.1 Definition. An object A of a category $\mathcal{A}$ is called *regular projective* if its hom-functor $\mathcal{A}(A,-)\colon \mathcal{A} \to \mathit{Set}$ preserves regular epimorhisms. That is, for every regular epimorphism $e\colon X \to Z$ and every morphism $f\colon A \to Z$ there exists a commutative triangle



5.2 Example.

1. In *Set* all objects are regular projective.

2. In *Ab* all free abelian groups are projective. Conversely, a projective abelian group is free: express A as a regular quotient $e\colon X \to A$ of a free group X and apply the previous definition to $f = \mathrm{id}_A$. This proves that $X \simeq A \times \operatorname{Ker} e$, and then A is free.

3. A graph G is regular projective in *Gr* (see 1.3.4) iff its edges are pairwise disjoint. That is, both functions $G_e \to G_v$ are monomorphisms.

5.3 Definition. Let $\mathcal{A}$ be a category. An object A of $\mathcal{A}$ is:

1. *finitely presentable* if the hom-functor $\mathcal{A}(A,-)\colon \mathcal{A} \to \mathit{Set}$ preserves filtered colimits;

2. *projectively finitely presentable* if the hom-functor $\mathcal{A}(A,-)\colon \mathcal{A} \to \mathit{Set}$ preserves sifted colimits.

5.4 Remark. Any projectively finitely presentable object is finitely presentable (because filtered colimits are sifted) and regular projective (because regular epimorphisms are coequalizers of reflexive pairs). We will show in 5.14 that in an algebraic category also the converse implication holds. In fact, the converse implication holds in any cocomplete exact category, see 15.2.

5.5 Example.

1. In *Set* projectively finitely presentable means finite.
2. An S-sorted set, i.e., an object of the product category Set^S, is (projectively) finitely presentable iff it is a *finite* S*-sorted set*, i.e., a collection $X = (X_s)_{s \in S}$ with $\coprod_{s \in S} X_s$ finite.
3. An abelian group A is finitely presentable in the above sense in the category Ab iff it is finitely presentable in the usual algebraic sense: A can be presented by finitely many generators and finitely many equations. This is easily seen from the fact that every abelian group is a filtered colimit of abelian groups that are finitely presentable (in the algebraic sense). An abelian group is projectively finitely presentable iff it is a free abelian group on finitely many generators, see 5.13.
4. In $Set^{\mathcal{T}}$ every representable functor is projectively finitely presentable. If $\mathcal{T}$ is an algebraic theory and $Y \colon \mathcal{T}^{op} \to Alg\,\mathcal{T}$ is the Yoneda embedding, then all algebras $Y(t)$ are projectively finitely presentable. This follows from $Alg\,\mathcal{T}$ being closed under sifted colimits in $Set^{\mathcal{T}}$.
5. In a poset considered as a category the (projectively) finitely presentable objects are precisely the *compact* elements x, i.e., such that for every directed join $y = \bigvee_{i \in I} y_i$, from $x \leq y$ it follows that $x \leq y_i$ for some $i \in I$.
6. A graph is finitely presentable in Gr iff it has finitely many vertices and finitely many edges. In fact, it is easy to see that for each such graph G the hom-functor $Gr(G, -)$ preserves filtered colimits. Conversely, if G is finitely presentable, use the fact that G is a filtered colimit of all its subgraphs on finitely many vertices and finitely many edges. A graph is projectively finitely presentable iff it has finitely many vertices and finitely many pairwise disjoint edges.

5.6 Remark. In categories $Set^{\mathcal{C}}$ the representable objects have a stronger property: their hom-functors preserve *all* colimits. We call objects whose hom-functors preserve all colimits *absolutely presentable*. In algebraic categories, these are typically rare. For example no abelian group A is absolutely presentable in Ab : for the initial object 1, the object $Ab(A, 1)$ is never intial in *Set*. However, the categories $Set^{\mathcal{C}}$ are an exception: every object there is a colimit of absolutely presentable objects.

Recall that an object B is a *retract* of an object A if there are morphisms $f \colon B \to A$ and $g \colon A \to B$ such that $g \cdot f = \mathrm{id}_B$.

5.7 Lemma. *Every retract of a regular projective object is regular projective. Every retract of a (projectively) finitely presentable object also has that property.*

Proof. If $f\colon B \to A$ and $g\colon A \to B$ are such that $g \cdot f = \mathrm{id}_B$, then

$$\alpha = \mathcal{A}(g,-)\colon G = \mathcal{A}(B,-) \to F = \mathcal{A}(A,-) \text{ and } \beta = \mathcal{A}(f,-)\colon F \to G$$

are such that $\beta \cdot \alpha = \mathrm{id}_G$. Therefore,

$$F \underset{\mathrm{id}_F}{\overset{\alpha\cdot\beta}{\rightrightarrows}} F \xrightarrow{\beta} G$$

is a coequalizer. By interchange of colimits, G preserves every colimit preserved by F. □

5.8 Remark. Absolutely presentable functors in $\mathit{Set}^{\mathcal{C}}$ are precisely the retracts of the representable functors.

5.9 Lemma.

1. *Projectively finitely presentable objects are closed under finite coproducts.*
2. *Finitely presentable objects are closed under finite colimits.*

Proof. Let us prove the first statement (the second one is similar). Consider a finite family $(A_i)_{i\in I}$ of projectively finitely presentable objects. Since $\mathcal{A}(\coprod_I A_i,-) \simeq \prod_I \mathcal{A}(A_i,-)$, the claim follows from the fact that a finite product of functors $\mathcal{A} \to \mathit{Set}$ preserving sifted colimits also preserves them. □

5.10 Lemma. *Regular projective objects are closed under coproducts.*

Proof. Let $(A_i)_{i\in I}$ be a family of regular projective objects and $e\colon X \to Z$ a regular epimorphism. The claim follows from the formula $\mathcal{A}(\coprod_I A_i, e) \simeq \prod_I \mathcal{A}(A_i, e)$ and the fact that in *Set* regular epimorphisms are stable under products (3.13). □

5.11 Corollary. *Let $\mathcal{T}$ be an algebraic theory. In the category $\mathit{Alg}\,\mathcal{T}$:*

1. *The projectively finitely presentable objects are precisely the retracts of representable algebras.*
2. *The regular projective objects are precisely the retract of coproducts of representable algebras.*

Proof. 1: Following 5.5 and 5.7, a retract of a representable algebra is projectively finitely presentable. Conversely, following 4.3, we can express a projectively finitely presentable algebra A as a sifted colimit of representable algebras. Since $\mathit{Alg}\,\mathcal{T}(A,-)$ preserves this colimit, it follows that id_A factors through some of the colimit morphism $e\colon Y(t) \to A$. Thus e is a split epimorphism and A is a retract of $Y(t)$.
2: Following 5.5, 5.7 and 5.10, a retract of a coproduct of representable algebras

is regular projective. Conversely, following 4.3, we can express any algebra A as a colimit of representable algebras. Since $Alg\,\mathcal{T}$ is cocomplete, this implies that A is a regular quotient of a coproduct of representable algebras

$$e\colon \coprod \mathcal{T}(t,-) \to A\,.$$

Therefore, if A is regular projective, it is a retract of $\coprod \mathcal{T}(t,-)$. □

5.12 Corollary. *Every algebraic category has enough regular projective objects, i.e., for every algebra A there is a regular projective algebra P and a regular epimorphism $e\colon P \to A$.*

Proof. As in the proof of 5.11.2, we have a regular epimorphism

$$e\colon \coprod \mathcal{T}(t,-) \to A.$$

Following 5.5 and 5.10, $\coprod \mathcal{T}(t,-)$ is a regular projective object. □

5.13 Example. In *Ab* projectively finitely presentable objects are precisely the finitely generated free abelian groups. In fact, these are precisely the representable algebras, and every retract of a free abelian group is also free (se 5.2.2).

5.14 Corollary. *In an algebraic category, an algebra is projectively finitely presentable if and only if it is finitely presentable and regular projective.*

Proof. One implication holds in any category, see 5.4. Conversely, if P is a regular projective object in $Alg\,\mathcal{T}$, following 5.11.2 P is a retract of a coproduct of representable algebras. Since every coproduct is a filtered colimit of its finite subcoproducts, if P is also finitely presentable, then it is a retract of a finite coproduct of representable algebras. Following 5.5, 5.7 and 5.9.1, P is projectively finitely presentable. □

5.15 Proposition. *In every algebraic category the finitely presentable algebras are precisely the coequalizers of reflexive pairs of homomorphisms between representable algebras.*

Proof. Consider an algebra A. Following 4.3, A is a (sifted) colimit of representable algebras. Thus, A is a filtered colimit of finite colimits of representable algebras, that is, A is a filtered colimit of coequalizers of morphisms between finite coproducts of representable algebras. If A is finitely presentable, then it is a retract of a coequalizer of morphisms between finite coproducts of representable algebras. Following 4.1, A is a retract of a coequalizer of a reflexive pair of morphisms between representable algebras, say

$$Y(s) \underset{k}{\overset{h}{\rightrightarrows}} Y(t) \xrightarrow{\ e\ } Q \underset{u}{\overset{s}{\rightleftarrows}} A$$

where $s,t \in \mathcal{T}$, Y is the Yoneda embedding $\mathcal{T}^{op} \to Alg\,\mathcal{T}$, e is a coequalizer of h and k, and $s \cdot u = \mathrm{id}_A$. Since $Y(t)$ is regular projective (5.4 and 5.5) and e

is a regular epimorphism, there is an homomorphism $g\colon Y(t) \to Y(t)$ such that $e \cdot g = u \cdot s \cdot e$. Let us prove that the diagram

$$Y(s) \overset{h,\; k,\; g\cdot h}{\rightrightarrows} Y(t) \xrightarrow{\quad s\cdot e \quad} A$$

is a multiple coequalizer. Firstly, $s\cdot e\cdot k = s\cdot e\cdot h = s\cdot u\cdot s\cdot e\cdot h = s\cdot e\cdot g\cdot h$. Assume that $f\colon Y(t) \to X$ coequalizes $h, k, g \cdot h$. Then there is a unique homomorphism $v\colon Q \to X$ with $v\cdot e = f$. Hence $f\cdot h = f\cdot g\cdot h = v\cdot e\cdot g\cdot h = v\cdot u\cdot s\cdot e\cdot h$. Since h is an epimorphism (because the pair h, k is reflexive) we get $f = v \cdot u \cdot s \cdot e$. Now, from the usual description of finite colimits, we have that A is a reflexive coequalizer of a pair of homomorphisms between finite coproducts of representable algebras. By 4.1 again, we get the claim.
The converse implication is obvious: representable algebras are (projectively) finitely presentable (5.5), and finitely presentable objects are closed under finite colimits (5.9). □

Chapter 6

A characterization of algebraic categories

The aim of this chapter is to characterize algebraic categories among cocomplete categories with a generating set of objects.

6.1 Remark. Recall that an epimorphism $f\colon A \to B$ in a category $\mathcal{A}$ is called an *extremal epimorphism* if, for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ & {\scriptstyle u}\searrow \quad \nearrow{\scriptstyle m} & \\ & X & \end{array}$$

where m is a monomorphism, then m is an isomorphism. It is called a *strong epimorphism* if, for every commutative square

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow[\;m\;]{} & Z \end{array}$$

where m is a monomorphism, there is a (necessarily unique) morphism $g\colon B \to X$ such that $m \cdot g = v$ (and then $g \cdot f = u$).
Let us recall some elementary facts on extremal, strong and regular epimorphisms. More can be found is Chapter 14.

1. Any regular epimorphism is strong, any strong epimorphism is extremal. If the category $\mathcal{A}$ has finite limits, then any extremal epimorphism is strong (to check this, use that monomorphisms are stable under pullbacks).

2. If the category $\mathcal{A}$ has binary products, then the condition of being an epimorphism in the definition of strong epimorphism is redundant. The same holds for extremal epimorphisms if the category $\mathcal{A}$ has equalizers.

3. If a composite $f \cdot g$ is a strong epimorphism, then f is a strong epimorphism. The same holds for extremal epimorphisms.

4. If f is a monomorphism and an extremal epimorphism, then it is an isomorphism.

6.2 Definition. Consider a set of objects $\mathcal{G}$ in a category $\mathcal{A}$. Consider also the functor

$$Y_{\mathcal{G}} \colon \mathcal{A} \to Set^{\mathcal{G}}\ , \quad Y_{\mathcal{G}}(A)(G) = \mathcal{A}(G, A).$$

1. $\mathcal{G}$ is a *generator* if $Y_{\mathcal{G}}$ is faithful.

2. $\mathcal{G}$ is a *strong generator* if $Y_{\mathcal{G}}$ is faithful and conservative (i.e., it reflects isomorphisms).

6.3 Remark.

1. $\mathcal{G}$ is a generator iff two morphisms $x, y \colon A \rightrightarrows B$ are equal whenever $x \cdot g = y \cdot g$ for all $g \colon G \to A$, G varying in $\mathcal{G}$.

2. If $\mathcal{A}$ has coproducts, then $\mathcal{G}$ is a generator iff every object of $\mathcal{A}$ is a quotient of a coproduct of objects in $\mathcal{G}$. That is, an epimorphism

   $$a \colon \coprod_{i \in I} G_i \to A$$

   exists with all G_i in $\mathcal{G}$. Another equivalent formulation uses the canonical morphism

   $$u_A \colon \coprod_{g \in \mathcal{G} \downarrow A} G \to A$$

   such that $u_A \cdot \rho_g = g$ for all coproduct injections

   $$\rho_g \colon G \to \coprod_{\mathcal{G} \downarrow A} G.$$

   $\mathcal{G}$ is a generator iff for every object A the canonical morphism u_A is an epimorphism.

3. The following proposition suggests that "strong" generator should more properly be called "extremal", the present terminology has just historical reasons.

6.4 Proposition. *Let $\mathcal{A}$ be a category with coproducts and $\mathcal{G}$ a set of objects of $\mathcal{A}$. The following conditions are equivalent:*

1. *$\mathcal{G}$ is a strong generator;*

2. *For every object in $\mathcal{A}$, an extremal epimorphism*

   $$a \colon \coprod_{i \in I} G_i \to A$$

   exists with all G_i in $\mathcal{G}$;

3. *For every object A in $\mathcal{A}$, the canonical morphism*

$$u_A \colon \coprod_{g \in \mathcal{G} \downarrow A} G \to A$$

is an extremal epimorphism.

Proof. Let us check the equivalence between 1 and 3. We already know that the faithfulness of $Y_{\mathcal{G}}$ corresponds to the fact that u_A is an epimorphism for all A in $\mathcal{A}$. Consider now a morphism $f \colon B \to A$ such that $\mathcal{A}(G, f) \colon \mathcal{A}(G, B) \to \mathcal{A}(G, A)$ is bijective for all G in $\mathcal{G}$. Since for any $g \colon G \to A$ there is $g' \colon G \to B$ such that $g = f \cdot g'$, we get $f' \colon \coprod_{\mathcal{G} \downarrow A} G \to B$ such that $u_A = f \cdot f'$. This implies that f is an extremal epimorphism because, by assumption, u_A is an extremal epimorphism. It is also a monomorphism: given $x, y \colon X \rightrightarrows B$ such that $f \cdot x = f \cdot y$, we can assume that X is in $\mathcal{G}$ (because $\mathcal{G}$ is a generator), so that $\mathcal{A}(X, f)$ is injective and, therefore, $x = y$.
Conversely, if $u_A = m \cdot h$ with m a monomorphism, then, for all G in $\mathcal{G}$, $\mathcal{A}(G, m)$ is surjective (because $m \cdot h \cdot \rho_g = u_A \cdot \rho_g = g$) and injective (because m is a monomorphism). Since $Y_{\mathcal{G}}$ is conservative, this implies that m is an isomorphism. □

6.5 Corollary. *If $\mathcal{A}$ has coproducts and each object of $\mathcal{A}$ is a colimit of objects from $\mathcal{G}$, then $\mathcal{G}$ is a strong generator.*

Proof. This follows from 6.4, because any regular epimorphism is extremal □

6.6 Example.

1. Every nonempty set forms a (singleton) strong generator in *Set*.
2. The group of integers form a (singleton) strong generator in *Ab*.
3. Let $\mathcal{C}$ be a small category. Then $Set^{\mathcal{C}}$ has a strong generator formed by all representable functors. This follows from 4.2 and 6.5.
4. Let $\mathcal{T}$ be an algebraic theory. Then $Alg\,\mathcal{T}$ has a strong generator formed by all representable functors. This follows from 4.3 and 6.5.
5. In the category of posets and order-preserving functions the terminal (one-element) poset forms a singleton generator – but this generator is not strong. In contrast, a two-element chain is a strong (singleton) generator.

6.7 Lemma. *Let $\mathcal{A}$ be a cocomplete category. If $\mathcal{A}$ has a set $\mathcal{G}_{pfp}$ of projectively finitely presentable objects such that every object of $\mathcal{A}$ is a sifted colimit of objects of $\mathcal{G}_{pfp}$, then $\mathcal{A}$ has, up to isomorphism, only a set of projectively finitely presentable objects.*

Proof. Express an object A of $\mathcal{A}$ as a sifted colimit of objects from $\mathcal{G}_{pfp}$. If A is projectively finitely presentable, then it is a retract of an object from $\mathcal{G}_{pfp}$. Since each object from $\mathcal{A}$ has only a set of retracts (because each retract of an object B gives rise to an idempotent morphism $e \colon B \to B$, $e \cdot e = e$), our claim is proved. □

6.8 Theorem. *(Characterization of algebraic categories) The following conditions on a category $\mathcal{A}$ are equivalent:*

1. *$\mathcal{A}$ is algebraic;*
2. *$\mathcal{A}$ is cocomplete and has a set $\mathcal{G}_{pfp}$ of projectively finitely presentable objects such that every object of $\mathcal{A}$ is a sifted colimit of objects of $\mathcal{G}_{pfp}$;*
3. *$\mathcal{A}$ is cocomplete and has a strong generator consisting of projectively finitely presentable objects.*

Proof. $1 \Rightarrow 2$: Let $\mathcal{T}$ be an algebraic theory. Then $Alg\,\mathcal{T}$ is cocomplete (4.5), the representable algebras form a set of projectively finitely presentable objects (5.5), and every algebra is a sifted colimit of representable algebras (4.3).
$2 \Rightarrow 3$: Consider the family $\mathcal{A}_{pfp}$ of the projectively finitely presentable objects of $\mathcal{A}$. By 6.7, $\mathcal{A}_{pfp}$ is essentially a set. By 6.5, $\mathcal{A}_{pfp}$ is a strong generator.
$3 \Rightarrow 1$: Let $\mathcal{G}$ be a strong generator consisting of projectively finitely presentable objects. Since projectively finitely presentable objects are closed under finite coproducts (5.9), we can assume without loss of generality that $\mathcal{G}$ is closed under finite coproducts (if this is not the case, we can replace $\mathcal{G}$ by its closure in $\mathcal{A}$ under finite coproduccts, which still is a strong generator). We are going to prove that $\mathcal{A}$ is equivalent to $Alg\,(\mathcal{G}^{op})$, where $\mathcal{G}$ is seen as a full subcategory of $\mathcal{A}$.
(1) We prove first that $\mathcal{G}$ is *dense*, i.e., for every object K of $\mathcal{A}$ the canonical diagram of all arrows from $\mathcal{G}$

$$D_K \colon \mathcal{G} \downarrow K \to \mathcal{A}\,,\ \ (g\colon G \to K) \mapsto G \quad \text{for } G \in \mathcal{G}$$

has K as colimit, with $(g\colon G \to K)$ as colimit cocone. To prove this, form a colimit cocone of D_K:

$$(c_g\colon G \to K^*) \quad \text{for all } g\colon G \to K \text{ in } \mathcal{G} \downarrow K.$$

We have to prove that the unique factorizing morphism $\lambda\colon K^* \to K$ is an isomorphism. Consider the coproduct

$$\rho_g\colon G \to \coprod_{\mathcal{G}\downarrow K} G$$

and the morphisms

$$u\colon \coprod_{\mathcal{G}\downarrow K} G \to K \qquad v\colon \coprod_{\mathcal{G}\downarrow K} G \to K^*$$

such that $u \cdot \rho_g = g$ and $v \cdot \rho_g = c_g$ for all $g \in \mathcal{G} \downarrow K$. Since $\lambda \cdot v = u$ and u is an extremal epimorphism (because $\mathcal{G}$ is a strong generator), then λ is an extremal epimorphism. It remains to prove that λ is a monomorphism. Consider two morphisms $x, y\colon X \rightrightarrows K^*$ such that $\lambda \cdot x = \lambda \cdot y$, and let us prove that $x = y$. Since $\mathcal{G}$ is a (strong) generator, we can assume without loss of generality that

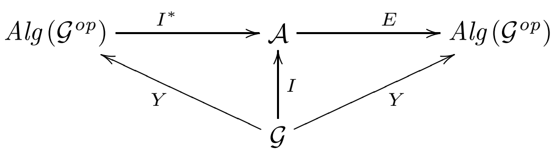
X is in $\mathcal{G}$. Since $\mathcal{G} \downarrow K$ is sifted (in fact, it has finite coproducts, because $\mathcal{G}$ has, and so it is sifted by 2.10) and X is projectively finitely presentable, we have

$$\mathcal{A}(X, K^*) \simeq \underset{\mathcal{G}\downarrow K}{colim}\ \mathcal{A}(X, G)\,.$$

Therefore, both x and y factor through some term of the colimit, i.e., there are $a\colon A \to K$ and $b\colon B \to K$ in $\mathcal{G} \downarrow K$ and $x'\colon X \to A$, $y'\colon X \to B$ such that $c_a \cdot x' = x$ and $c_b \cdot y' = y$. This gives rise to an object $(X, \lambda \cdot x) = (X, \lambda \cdot y)$ and two morphisms $x'\colon (X, \lambda \cdot x) \to (A, a)$ and $y'\colon (X, \lambda \cdot y) \to (B, b)$ in $\mathcal{G} \downarrow K$. Finally, $x = c_a \cdot x' = c_{\lambda \cdot x} = c_{\lambda \cdot y} = c_b \cdot y' = y$. (Note that this proves the implication $3 \Rightarrow 2$.)
(2) It follows from (1) that the functor

$$E\colon \mathcal{A} \to Alg\,(\mathcal{G}^{op})\,, \quad K \mapsto \mathcal{A}(-, K)\colon \mathcal{G}^{op} \to Set$$

is full and faithful. Indeed, given an homomorphism $\alpha\colon \mathcal{A}(-, K) \to \mathcal{A}(-, L)$, for every $g\colon G \to K$ in $\mathcal{G} \downarrow K$ we have a morphism $\alpha_G(g)\colon G \to L$. Those morphisms $\alpha_G(g)$ form a cocone on $\mathcal{G} \downarrow K$, so that there exists a unique morphism $\hat{\alpha}\colon K \to L$ such that $\hat{\alpha} \cdot g = \alpha_G(g)$ for all g in $\mathcal{G} \downarrow K$. It is easy to check that $E(\hat{\alpha}) = \alpha$ and that $\widehat{E(f)} = f$ for all $f\colon K \to L$ in $\mathcal{A}$.
(3) Let us prove now that $E\colon \mathcal{A} \to Alg\,(\mathcal{G}^{op})$ preserves sifted colimits. Consider a sifted diagram $D\colon \mathcal{D} \to \mathcal{A}$ with colimit $(h_d\colon Dd \to H)$. For any G in $\mathcal{G}$, we have that $(\mathcal{A}(G, h_d)\colon \mathcal{A}(G, Dd) \to \mathcal{A}(G, H))$ is a colimit of $\mathcal{A}(G, -) \cdot D$ in Set (because $\mathcal{G}$ is formed by projectively finitely presentable objects). This implies that $(E(h_d)\colon E(Dd) \to E(H))$ is a colimit of $E \cdot D$ in $Alg\,(\mathcal{G}^{op})$ (because sifted colimits are computed objectwise in $Alg\,(\mathcal{G}^{op})$, see 2.3).
(4) It follows from (3) that $E\colon \mathcal{A} \to Alg\,(\mathcal{G}^{op})$ is essentially surjective on objects (i.e., each object of $Alg\,(\mathcal{G}^{op})$ is isomorphic to an object of the form EA for some object A in $\mathcal{A}$). In fact, we have the following diagram, commutative up to natural isomorphism,



where I is the inclusion and I^* is its colimit preserving extension (4.13). Since $E \cdot I^* \cdot Y \simeq Y$ and $E \cdot I^*$ preserves sifted colimits, it follows from 4.16 that $E \cdot I^*$ is naturally isomorphic to the identity functor. □

6.9 Notation. For every algebraic category $\mathcal{A}$ we denote by $\mathcal{A}_{pfp}$ a full, small subcategory representing all projectively finitely presentable algebras (see 6.7).

6.10 Corollary. *For every algebraic category $\mathcal{A}$ the dual of $\mathcal{A}_{pfp}$ is an algebraic theory of $\mathcal{A}$: the Yoneda embedding*

$$Y_{pfp}\colon \mathcal{A} \to Alg\,(\mathcal{A}_{pfp}^{op})\,, \quad A \mapsto \mathcal{A}(-, A)\colon \mathcal{A}_{pfp}^{op} \to Set$$

is an equivalence functor.

Proof. Following 5.9.1, $\mathcal{A}^{op}_{pfp}$ is an algebraic theory. Since $\mathcal{A}$ satisfies condition 2 of **??** and $\mathcal{A}_{pfp}$ is a strong generator (see 6.5) the above proof demonstrates that Y_{pfp} is an equivalence functor. □

6.11 Corollary. *Two algebraic categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if the categories $\mathcal{A}_{pfp}$ and $\mathcal{B}_{pfp}$ are equivalent.*

Proof. This follows immediately from 6.10 and the fact that equivalence functors preserve projective finite presentability. □

6.12 Lemma. *Consider an adjunction*

$$\mathcal{A} \underset{I}{\overset{R}{\leftrightarrows}} \mathcal{B}$$

with R left adjoint to I.

1. *If I is faithful and conservative, $\mathcal{B}$ has finite limits and $\mathcal{G}$ is a strong generator in $\mathcal{B}$, then $R(\mathcal{G})$ is a strong generator in $\mathcal{A}$.*

2. *If I preserves sifted colimits and X is projectively finitely presentable in $\mathcal{B}$, then RX is projectively finitely presentable in $\mathcal{A}$.*

Proof. 1: $R(\mathcal{G})$ is a generator because I is a faithful right adjoint. Next, consider a morphism $a\colon A \to A'$ in $\mathcal{A}$ such that every morphism $RG \to A'$, with $G \in \mathcal{G}$, uniquely factors through a. This implies, by adjunction, that every morphism $G \to IA'$ uniquely factors through Ia. Since $\mathcal{G}$ is a strong generator, Ia is an isomorphism. Since I is conservative, a is an isomorphism.
2: In fact, $\mathcal{A}(RX,-) \simeq \mathcal{B}(X,I-) = \mathcal{B}(X,-)\cdot I$, so that $\mathcal{A}(RX,-)$ is the composite of two functors preserving sifted colimits. □

6.13 Proposition. *Let $\mathcal{T}$ be an algebraic theory. Then $Alg\,\mathcal{T}$ is a reflective subcategory of $Set^{\mathcal{T}}$.*

Proof. We are going to construct a left adjoint R to the full inclusion $I\colon Alg\,\mathcal{T} \to Set^{\mathcal{T}}$. For $A \in Set^{\mathcal{T}}$, we define RA to be a colimit of the diagram

$$(El\,A)^{op} \xrightarrow{\ \Phi_A\ } \mathcal{T}^{op} \xrightarrow{\ Y\ } Alg\,\mathcal{T}$$

(this makes sense because, by 4.5, $Alg\,\mathcal{T}$ is cocomplete). Following 4.2, A is the colimit of $I\cdot Y\cdot\Phi_A$. The colimit cocone $(x\colon \mathcal{T}(X,-) \to A)_{(X,x)\in El\,A}$ gives rise to a cocone on $Y\cdot\Phi_A$, and therefore to a natural transformation $\eta_A\colon A \to I(RA)$. It is straightforward to check the requested universal property. □

6.14 Theorem. *A category is algebraic if and only if it is a reflective subcategory of $Set^{\mathcal{C}}$ closed under sifted colimits, for some small category $\mathcal{C}$.*

Proof. Let $\mathcal{T}$ be an algebraic theory. Following 6.13 and 2.3, $Alg\,\mathcal{T}$ is a relfective subcategory closed under sifted colimits of $Set^{\mathcal{T}}$.
Conversely, let $\mathcal{C}$ be a small category. By 1.3.4 $Set^{\mathcal{C}}$ is an algebraic category, so that it fulfils the conditions of 6.8.3. Following 6.12, those conditions are inherited by any reflective subcategory closed under sifted colimits of $Set^{\mathcal{C}}$. □

6.15 Remark. Once again, if we replace sifted colimits by filtered colimits, the main results of this chapter still hold. Let state them explicitly. The proofs are basically the same and can be omitted.

1. Analogously to $Sind(\mathcal{T}^{op}) = Alg\mathcal{T}$ for algebraic theories (see 4.3), for every small, finitely complete category $\mathcal{C}$, $Ind(\mathcal{C}^{op})$ of 4.9 is equivalent to $Lex\mathcal{C}$, where $Lex\mathcal{C}$ denotes the full subcategory of $Set^{\mathcal{C}}$ on all functors preserving finite limits.

2. A category $\mathcal{A}$ is called *locally finitely presentable* if it is cocomplete and has a set $\mathcal{G}_{fp}$ of finitely presentable objects such that every object of $\mathcal{A}$ is a filtered colimit of objects of $\mathcal{G}_{fp}$.

3. A locally finitely presentable category $\mathcal{A}$ has, up to isomorphism, only a set of finitely presentable objects.

4. The following conditions on a category $\mathcal{A}$ are equivalent:
   (a) $\mathcal{A}$ is equivalent to $Lex\mathcal{C}$ for some small category $\mathcal{C}$ with finite limits;
   (b) $\mathcal{A}$ is locally finitely presentable;
   (c) $\mathcal{A}$ is cocomplete and has a strong generator consisting of finitely presentable objects.

5. If $\mathcal{A}$ is a locally finitely presentable category and $\mathcal{A}_{fp}$ is a small full subcategory representing the finitely presentable objects, then $\mathcal{A} \simeq Lex(\mathcal{A}_{fp}^{op}) \simeq Ind(\mathcal{A}_{fp})$.

6. For every small category $\mathcal{C}$ with finite limits, $Lex\mathcal{C}$ is reflective and closed under filtered colimits in $Set^{\mathcal{C}}$.

7. A category is locally finitely presentable iff it is a reflective subcategory of $Set^{\mathcal{C}}$ closed under filtered colimits , for some small category $\mathcal{C}$.

6.16 Corollary. *Every algebraic category is locally finitely presentable.*

6.17 Example. The category of posets is an example of a locally finitely presentable category which is not algebraic.

6.18 Corollary. *Let $\mathcal{C}$ be a small category with finite colimits. Then $Ind\mathcal{C}$ is a reflective subcategory of $Sind\mathcal{C}$. Equivalently, $Lex(\mathcal{C}^{op})$ is a reflective subcategory of $Alg(\mathcal{C}^{op})$.*

Proof. By 4.3, $Sind\mathcal{C} = Alg(\mathcal{C}^{op})$ and, analogously, $Ind\mathcal{C} = Lex(\mathcal{C}^{op})$. Consider the full inclusions

$$\begin{array}{ccc} Lex(\mathcal{C}^{op}) & \xrightarrow{\;I_1\;} & Set^{\mathcal{C}^{op}} \\ & {\scriptstyle I_2}\searrow \quad \nearrow{\scriptstyle I_3} & \\ & Alg(\mathcal{C}^{op}) & \end{array}$$

By 6.15.6, I_1 has a left adjoint, say R. Since I_3 is full and faithful, $R \cdot I_3$ is left adjoint to I_2. $\square$

6.19 Remark. Let us finish this chapter by quoting another characterization theorem, similar to 6.8.
The following conditions on a category $\mathcal{A}$ are equivalent:

1. $\mathcal{A}$ is equivalent to $Set^{\mathcal{C}}$ for some small category $\mathcal{C}$;
2. $\mathcal{A}$ is cocomplete and has a set of absolutely presentable objects $\mathcal{G}_{ap}$ such that every object of $\mathcal{A}$ is a colimit of objects of $\mathcal{G}_{ap}$;
3. $\mathcal{A}$ is cocomplete and has a strong generator consisting of absolutely presentable objects.

Moreover, if $\mathcal{A} = Set^{\mathcal{C}}$ and $\mathcal{A}_{ap}$ is its full subcategory of absolutely presentable objects (which, by 5.8, are precisely the retracts of representable functors), then $\mathcal{A} \simeq Set^{\mathcal{A}_{ap}^{op}}$ (compare with 7.8).

Chapter 7

Idempotent completion

We have studied algebraic categories as individual categories so far. It turns out that there is a natural concept of morphism between algebraic categories, which we call an algebraic functor in Chapter 8, so that we obtain a 2-category of all algebraic categories and a duality of this 2-category with the 2-category of algebraic theories.

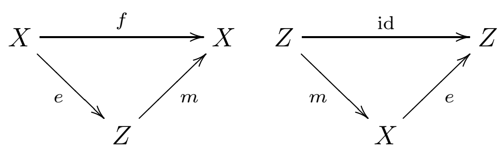
The crucial idea is to start with morphisms of algebraic theories and to work with idempotent complete algebraic theories. To prepare the duality theorem between idempotent complete algebraic theories and algebraic categories, we study in this chapter the idempotent completion of a category.

7.1 Definition.

1. Given an idempotent morphism

$$f\colon X \to X\ , \quad f \cdot f = f$$

in a category $\mathcal{C}$, by a *splitting* of f is meant a factorization $f = m \cdot e$ such that $e \cdot m$ is the identity morphism



2. A category $\mathcal{C}$ is called *idempotent-complete* provided that every idempotent in $\mathcal{C}$ has a splitting.

7.2 Remark.

1. In the situation of 7.1, observe that Z is a retract of X.

2. A splitting of an idempotent f is unique up to isomorphism:

   (a) for every isomorphism $i\colon Z \to \bar{Z}$ the morphisms $\bar{e} = i \cdot e$ and $\bar{m} = m \cdot i^{-1}$ form a splitting of f;

   (b) for every splitting $f = \bar{m} \cdot \bar{e}$, $\bar{e} \cdot \bar{m} = \mathrm{id}$, there exists a unique isomorphism i such that $i \cdot e = \bar{e}$ and $m \cdot i^{-1} = \bar{m}$ (just put $i = \bar{e} \cdot m$ and $i^{-1} = e \cdot \bar{m}$).

3. If an idempotent f has a factorization $f = m \cdot e$ with m a monomorphism and e an epimorphism, then $f = m \cdot e$ is a splitting of f. Indeed, $m \cdot e \cdot m \cdot e = f \cdot f = f = m \cdot e = m \cdot \mathrm{id} \cdot e$, so that $e \cdot m = \mathrm{id}$.

4. To be idempotent-complete is a self-dual notion: $\mathcal{C}$ is idempotent complete iff $\mathcal{C}^{op}$ is so.

7.3 Example.

1. Every category which has either equalizers or coequalizers is idempotent-complete. In fact, form an equalizer m of the idempotent $f\colon X \to X$ and $\mathrm{id}\colon X \to X$

$$Z \xrightarrow{\ m\ } X \underset{\mathrm{id}}{\overset{f}{\rightrightarrows}} X\,.$$

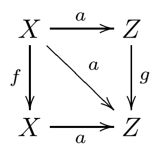
   Since $f \cdot f = \mathrm{id} \cdot f$, f factors as $f = m \cdot e$ for some $e\colon X \to Z$. Now, $m \cdot e \cdot m = f \cdot m = m$ and m is a monomorphism, so that $e \cdot m = \mathrm{id}$.

2. A full subcategory $\mathcal{D}$ of an idempotent-complete category $\mathcal{C}$ is idempotent complete iff $\mathcal{D}$ is closed in $\mathcal{C}$ under retracts.

7.4 Definition. Let $\mathcal{C}$ be a category. The *category $Ic\mathcal{C}$ of idempotents* of $\mathcal{C}$ is defined as follows:

1. an object of $Ic\mathcal{C}$ is an idempotent $f\colon X \to X$ of $\mathcal{C}$;

2. an arrow from $f\colon X \to X$ to $g\colon Z \to Z$ is an arrow $a\colon X \to Z$ in $\mathcal{C}$ such that $a = g \cdot a \cdot f$ (or, equivalently, such that the diagram



   commutes);

3. the identity of $f\colon X \to X$ is f itself;

4. composition is as in $\mathcal{C}$.

There is a (full and faithful) functor $E\colon \mathcal{C} \to Ic\mathcal{C}$ defined by $E(X) = \mathrm{id}_X$ and $E(a) = a$.

7.5 Proposition. *The functor $E\colon \mathcal{C} \to Ic\mathcal{C}$ is an* idempotent completion *of $\mathcal{C}$. This means that*

1. *$Ic\mathcal{C}$ is idempotent-complete, and*

2. *for any idempotent-complete category $\mathcal{B}$ and for any functor $F\colon \mathcal{C} \to \mathcal{B}$ there is an essentially unique functor $F^*\colon Ic\mathcal{C} \to \mathcal{B}$ such that $F^* \cdot E \simeq F$.*

Proof. 1: Let $a\colon (f\colon X \to X) \to (f\colon X \to X)$ be an idempotent in $Ic\mathcal{C}$. A splitting of a is given by

$$(f\colon X \to X) \xrightarrow{\ a\ } (a\colon X \to X) \xrightarrow{\ a\ } (f\colon X \to X)\ .$$

2: Each object $(f\colon X \to X)$ of $Ic\mathcal{C}$ is a splitting of the idempotent $f\colon EX \to EX$. Thus, we are forced to define $F^*(f\colon X \to X)$ as the (essentially unique) splitting of $Ff\colon FX \to FX$. ☐

7.6 Remark.

1. A category $\mathcal{C}$ is idempotent-complete iff the functor $E\colon \mathcal{C} \to Ic\mathcal{C}$ is an equivalence.

2. Clearly, $Ic\,(\mathcal{C}^{op}) \simeq (Ic\mathcal{C})^{op}$.

7.7 Remark. Let $\mathcal{C}$ be a small category. The idempotent completion $E\colon \mathcal{C} \to Ic\mathcal{C}$ of 7.4 can be equivalently described as the codomain restriction of the Yoneda embedding

$$Y_{Ic}\colon \mathcal{C} \to Ic\mathcal{C}$$

to the full subcategory of $Set^{\mathcal{C}^{op}}$ of those functors which are retracts of representable functors. Indeed, if

$$R \underset{m}{\overset{e}{\leftrightarrows}} \mathcal{C}(-, X)$$

is a retract, by the Yoneda lemma we get an idempotent $m \cdot e\colon X \to X$ in $\mathcal{C}$. The argument for morphisms is analogous.

7.8 Corollary. *For two small categories $\mathcal{C}$ and $\mathcal{D}$ the corresponding functor categories $Set^{\mathcal{C}}$ and $Set^{\mathcal{D}}$ are equivalent if and only if $\mathcal{C}$ and $\mathcal{D}$ have a common idempotent completion:*

$$Set^{\mathcal{C}} \simeq Set^{\mathcal{D}} \quad \text{iff} \quad Ic\mathcal{C} \simeq Ic\mathcal{D}\,.$$

Proof. The universal property of $E\colon \mathcal{C} \to Ic\mathcal{C}$ clearly implies $Set^{\mathcal{C}} \simeq Set^{Ic\mathcal{C}}$. Conversely, if $Set^{\mathcal{C}} \simeq Set^{\mathcal{D}}$, then the subcategories of absolutely presentable objects are equivalent. By 5.8 and 7.7, this means that $Ic\,(\mathcal{C}^{op}) \simeq Ic\,(\mathcal{D}^{op})$. By 7.6.2, we have $Ic\mathcal{C} \simeq Ic\mathcal{D}$. ☐

An algebraic category $\mathcal{A}$ can be described, up to equivalence, as $Alg\mathcal{T}$ for several, non equivalent algebraic theories $\mathcal{T}$ (Chapter 12 is devoted to study this fact in great detail). Nevertheless, among all the possible algebraic theories describing a given algebraic category, one is special:

7.9 Proposition. *For every algebraic category $\mathcal{A}$ there is an idempotent-complete algebraic theory $\mathcal{T}$ such that $\mathcal{A} \simeq Alg\,\mathcal{T}$.*

Proof. In fact, we can choose $\mathcal{T} = \mathcal{A}_{pfp}^{op}$, the dual of the full subcategory of projectively finitely presentable objects. Following 6.10, $\mathcal{A} \simeq Alg\,(\mathcal{A}_{pfp}^{op})$. By 5.11.1 and 7.3.2, $\mathcal{A}_{pfp}$ is idempotent-complete, and then $\mathcal{A}_{pfp}^{op}$ also is idempotent-complete. □

In fact, more is true: $\mathcal{A}_{pfp}^{op}$ is, up to equivalence, the unique idempotent-complete algebraic theory $\mathcal{T}$ such that $\mathcal{A} \simeq Alg\,\mathcal{T}$. More precisely:

7.10 Corollary.

1. *Let $\mathcal{A}$ be an algebraic category and $\mathcal{T}$ an algebraic theory. $\mathcal{A} \simeq Alg\,\mathcal{T}$ iff $\mathcal{A}_{pfp}^{op} \simeq Ic\,\mathcal{T}$.*

2. *Let $\mathcal{T}_1, \mathcal{T}_2$ be algebraic theories. $Alg\,\mathcal{T}_1 \simeq Alg\,\mathcal{T}_2$ iff $Set^{\mathcal{T}_1} \simeq Set^{\mathcal{T}_2}$.*

Proof. 1: Following 6.9, $\mathcal{A} \simeq Alg\,\mathcal{T}$ iff $\mathcal{A}_{pfp} \simeq (Alg\,\mathcal{T})_{pfp}$. Following 5.11.1 and 7.7, $(Alg\,\mathcal{T})_{pfp} \simeq Ic\,(\mathcal{T}^{op})$. We conclude by 7.6.2.
2: This follows from 1. and 7.6. □

7.11 Definition. The (essentially unique) idempotent-complete algebraic theory of an algebraic category is called its *canonical theory.*

7.12 Example.

1. Following 5.5.1, the canonical theory of the category *Set* is precisely the theory $\mathcal{T}_1$ described in 1.3.1.

2. Following 5.13, the canonical theory of the category *Ab* is precisely the theory $\mathcal{T}_{ab}$ described in 1.3.2.

Chapter 8

Algebraic functors

The notion of morphism of algebraic theories is obvious. The notion of morphism of algebraic categories is more subtle.

8.1 Definition. Let $\mathcal{T}_1, \mathcal{T}_2$ be algebraic theories. A functor $H\colon \mathcal{T}_1 \to \mathcal{T}_2$ is called a *morphism of algebraic theories* if it preserves finite products.

8.2 Proposition. *Let $H\colon \mathcal{T}_1 \to \mathcal{T}_2$ be a morphism of algebraic theories. H induces an adjunction*

$$Alg\,\mathcal{T}_2 \underset{Alg\,H}{\overset{H^*}{\leftrightarrows}} Alg\,\mathcal{T}_1 \ , \quad H^* \dashv Alg\,H,$$

where $Alg\,H$ is given by $(-)\cdot H$. Moreover, $Alg\,H$ preserves sifted colimits, and H^* is the essentially unique functor which preserves colimits and makes the diagram

$$\begin{array}{ccc} \mathcal{T}_1^{op} & \xrightarrow{\ Y\ } & Alg\,\mathcal{T}_1 \\ {\scriptstyle H^{op}}\downarrow & & \downarrow{\scriptstyle H^*} \\ \mathcal{T}_2^{op} & \xrightarrow[\ Y\ ]{} & Alg\,\mathcal{T}_2 \end{array}$$

commutative up to natural isomorphism.

Proof. To get the adjunction $H^* \dashv Alg\,H$ and to prove the last part of the claim, it suffices to apply 4.13 to the finite coproduct preserving functor $Y \cdot H^*$. Following the proof of 4.13, the right adjoint $Alg\,H$ is given by $\mathrm{Nat}(Y(H-), -)$. By the Yoneda lemma, this is nothing but composition with H:

$$Alg\,H\colon Alg\,\mathcal{T}_2 \to Alg\,\mathcal{T}_1 \ , \quad (A\colon \mathcal{T}_2 \to Set) \mapsto (A \cdot H\colon \mathcal{T}_1 \to \mathcal{T}_2 \to Set)\,.$$

This immediately implies that $Alg\,H$ preserves sifted colimits, because they are calculated objectwise in $Alg\,\mathcal{T}_1$ and $Alg\,\mathcal{T}_2$ (see 2.3). □

8.3 Example. The previous proposition generalizes the fact that, for every algebraic theory $\mathcal{T}$, the category $Alg\,\mathcal{T}$ is reflective in $Set^{\mathcal{T}}$ (6.13). Indeed, consider the free completion under finite products $\mathcal{T} \to \mathcal{T}_{\Pi}$ described in 1.3.4. Since $\mathcal{T}$ has finite products, the identity functor $\mathcal{T} \to \mathcal{T}$ extends to a morphism of algebraic theories $H\colon \mathcal{T}_{\Pi} \to \mathcal{T}$. Up to the equivalence $Alg\,\mathcal{T}_{\Pi} \simeq Set^{\mathcal{T}}$, the inclusion $Alg\,\mathcal{T} \to Set^{\mathcal{T}}$ is nothing but the functor $Alg\,H$ induced by H.

8.4 Definition. A functor between two categories with sifted colimits is called *algebraic* provided it is a right adjoint and preserves sifted colimits.

8.5 Example.

1. The forgetful functor $Ab \to Set$ is algebraic.
2. Given an algebra A in an algebraic category $\mathcal{A}$, then $\mathcal{A}(A, -)\colon \mathcal{A} \to Set$ is algebraic iff A is projectively finitely presentable.
3. A constant functor with value A between algebraic categories is algebraic iff A is a terminal object.

8.6 Remark.

1. We know from 8.2 that every morphism of theories induces an algebraic functor between the corresponding algebraic categories. If, moreover, the algebraic theories are canonical (7.11), then the algebraic functors are "precisely" those induced by morphisms of theories, as we prove below. This will motivate us to define "morphisms of algebraic categories" as the algebraic functors.
2. Does the above mean that there is a duality between the category of all algebraic categories and that of all idempotent-complete algebraic theories? This is "almost" true, see the duality theorem 8.14 and 8.15, but a more subtle formulation is needed: just look at the simplest of all algebraic categories, *Set*, and the simplest of its endomorphisms, the identity functor Id_{Set}. It is easy to find a proper class of functors naturally isomorphic to Id_{Set} – and each of them is algebraic. However, in the category of all theories no such phenomenon occurs. We thus need to work with morphisms of algebraic categories "up to natural isomorphism".
3. We have to move from categories to 2-categories. The reader does not need to know much about 2-categories. Let us recall that a 2-category $\mathbb{A}$ has objects and, instead of hom-sets $\mathbb{A}(A, B)$, it has hom-categories $\mathbb{A}(A, B)$ – the objects of $\mathbb{A}(A, B)$ are called 1-cells and the morphisms 2-cells. A prototype of a 2-category is the 2-category *Cat* of all small categories: 1-cells are functors and 2-cells are natural transformations. We essentially work just with this 2-category and its sub-2-categories.
4. Let us recall the concept of a 2-functor $F\colon \mathbb{A} \to \mathbb{B}$: it assigns objects FA of $\mathbb{B}$ to objects A of $\mathbb{A}$; for every pair A, A' of objects of $\mathbb{A}$, it defines a functor $F_{A,A'}\colon \mathbb{A}(A, A') \to \mathbb{B}(FA, FA')$ and fulfils some canonical requirements about compositions and identities.

5. The category of small categories and all functors considered up to natural isomorphism is denoted $\pi_0(Cat)$. More generally, if $\mathbb{A}$ is a 2-category, we denote by $\pi_0(\mathbb{A})$ the category whose objects are those of $\mathbb{A}$, and whose morphisms are 2-isomorphism classes of 1-cells of $\mathbb{A}$.

6. A 2-functor $F\colon \mathbb{A} \to \mathbb{B}$ is a *biequivalence* provided that

   - for every pair A, A' of objects of $\mathbb{A}$, the functor $F_{A,A'}$ is an equivalence of categories, and
   - every object of $\mathbb{B}$ is isomorphic, in $\pi_0(\mathbb{B})$, to FA for some object A of $\mathbb{A}$. Clearly, every biequivalence $\mathbb{A} \to \mathbb{B}$ induces an equivalence functor $\pi_0(\mathbb{A}) \to \pi_0(\mathbb{B})$.

7. For every 2-category $\mathbb{A}$ we denote by $\mathbb{A}^{op}$ the 2-category in which the direction of 1-cells is reversed (and the direction of 2-cells remains non-reversed).

8.7 Notation. We define the 2-category Th of theories to have

objects: all algebraic theories,

1-cells: all morphisms of algebraic theories (i.e., functors preserving finite products),

2-cells: all natural transformations.

This is a full sub-2-category of Cat, i.e., composition of 1-cells and 2-cells are defined in Th as the usual composition of functors and natural transformations, respectively.

8.8 Notation. We will use also a smaller 2-category Th_{ic} whose objects are all idempotent-complete algebraic theories (and 1-cells and 2-cells are as in Th).

8.9 Notation. We define the 2-category ALG of algebraic categories to have

objects: all algebraic categories,

1-cells: all algebraic functors,

2-cells: all natural transformations.

Once again, compositions are the usual compositions of functors and natural transformations.

8.10 Remark. We need to be a little careful about fondations here: there is, as remarked above, a proper class of 1-cells in $ALG(Set, Set)$, for example. However, if we consider the 1-cells up to natural isomorphism, all problems disappear: this is one consequence of the duality theorem below. Ignoring the foundational considerations, we consider ALG as a sub-2-category of the 2-category of all categories. (Whereas the latter is highly non foundational, the duality we prove below tells us that ALG is essentially just the dual of Th_{ic}.)

8.11 Definition. We denote by

$$Alg\colon Th^{op} \to ALG$$

the 2-functor assigning to every algebraic theory $\mathcal{T}$ the category $Alg\,\mathcal{T}$, to every 1-cell $H\colon \mathcal{T}_1 \to \mathcal{T}_2$ the above functor

$$Alg\,H\colon Alg\,\mathcal{T}_2 \to Alg\,\mathcal{T}_1\ , \quad A \mapsto A \cdot H$$

and to every 2-cell $\alpha\colon H \to K\colon \mathcal{T}_1 \rightrightarrows \mathcal{T}_2$ the natural transformation

$$Alg\,\alpha\colon Alg\,H \to Alg\,K$$

whose component at a $\mathcal{T}_2$-algebra A is $A \cdot \alpha\colon A \cdot H \to A \cdot K$.

8.12 Remark. The 2-functor *Alg* is well-defined by 8.2: for every morphism of theories H, the functor $Alg\,H$ is algebraic. The fact that for every natural transformation α we get a natural transformation $Alg\,\alpha$ is easy to verify. We leave to the rader the routine verification that *Alg* is indeed a 2-functor.

The following version of Yoneda lemma is needed in the proof of the duality theorem 8.14.

8.13 Lemma. *Consider two functors between small categories* $H, K\colon \mathcal{C}_1 \rightrightarrows \mathcal{C}_2$ *and the induced functors* $-\cdot H, -\cdot K\colon Set^{\mathcal{C}_2} \rightrightarrows Set^{\mathcal{C}_1}$*. Then every natural transformation* $-\cdot H \to -\cdot K$ *is induced by a unique natural transformation* $H \to K$*. That is, the map*

$$\mathrm{Nat}(H,K) \to \mathrm{Nat}(-\cdot H, -\cdot K)\ , \quad \alpha \mapsto -\cdot\alpha$$

is bijective.

Proof. The proof follows the same lines that the proof of Yoneda lemma. Let just indicate how to construct the inverse map. Let $\lambda\colon -\cdot H \to -\cdot K$ be a natural transformation, and consider an object X in $\mathcal{C}_1$. Since $\mathcal{C}_2(HX,-) \in Set^{\mathcal{C}_2}$, we have a map $\lambda_{\mathcal{C}_2(HX,-)}(X)\colon \mathcal{C}_2(HX,HX) \to \mathcal{C}_2(HX,KX)$ and we put $\alpha(X) = \lambda_{\mathcal{C}_2(HX,-)}(X)(\mathrm{id}_{HX})\colon HX \to KX$. The family $(\alpha(X))_{X\in\mathcal{C}_1}$ is the natural transformation $H \to K$ we are looking for. ☐

8.14 Theorem. *(Duality of Algebraic Categories and Theories) The restriction of the 2-functor Alg to idempotent-complete algebraic theories*

$$Alg\colon Th_{ic}^{op} \to ALG$$

is a biequivalence.

Proof. Following 7.9, every algebraic category $\mathcal{A}$ is equivalent to $Alg\,\mathcal{T}$ for an idempotent-complete algebraic theory $\mathcal{T}$, just take $\mathcal{T} = \mathcal{A}_{pfp}^{op}$.
We have to prove that, for $\mathcal{T}_1, \mathcal{T}_2$ two idempotent-complete algebraic theories, the functor

$$Alg_{\mathcal{T}_1,\mathcal{T}_2}\colon Th_{ic}(\mathcal{T}_1,\mathcal{T}_2) \to ALG(Alg\,\mathcal{T}_2, Alg\,\mathcal{T}_1)$$

is an equivalence of categories.
(1) Consider an algebraic functor $G\colon Alg\,\mathcal{T}_2 \to Alg\,\mathcal{T}_1$ and let L be its left adjoint. We are going to prove that L restricts to a functor F which preserves finite coproducts and makes the diagram

$$\begin{array}{ccc} \mathcal{T}_1^{op} & \xrightarrow{\;Y\;} & Alg\,\mathcal{T}_1 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle L} \\ \mathcal{T}_2^{op} & \xrightarrow[\;Y\;]{} & Alg\,\mathcal{T}_2 \end{array}$$

commutative, up to natural isomorphism. Consider an object X in $\mathcal{T}_1$; by adjunction, we have a natural isomorphism

$$\hom(L(Y(X)), -) \simeq \hom(Y(X), G-)\,.$$

Since $Y(X)$ is projectively finitely presentable (5.4) and G preserves sifted colimits, the above natural isomorphism says that $L(Y(X))$ is projectively finitely presentable. By 5.11, $L(Y(X))$ is a retract of a representable algebra and, since $\mathcal{T}_2$ is idempotent-complete, $L(Y(X))$ is itself a representable algebra (see 7.3.2). This means that there is an essentially unique object in $\mathcal{T}_2$, say FX, such that $L(Y(X)) \simeq Y(FX)$. In this way, we get a map on objects $F\colon obj\,\mathcal{T}_1 \to obj\,\mathcal{T}_2$ which, by the Yoneda lemma, extends to a functor $F\colon \mathcal{T}_1^{op} \to \mathcal{T}_2^{op}$ making the previous diagram commutative up to natural isomorphism. Clearly, F preserves finite coproducts, because Y preserves (4.1) and reflects finite coproducts and L preserves them. It remains to prove that $G \simeq Alg\,H$, where $H = F^{op}\colon \mathcal{T}_1 \to \mathcal{T}_2$, or, equivalently, that $L \simeq H^*$. But this follows from the commutativity of the above diagram and the last part of 8.2. This proves that $Alg_{\mathcal{T}_1,\mathcal{T}_2}$ is essentially surjective.
(2) $Alg_{\mathcal{T}_1,\mathcal{T}_2}$ is full and faithful: this is precisely Lemma 8.13. □

8.15 Corollary. *There is an equivalence of categories* $\pi_0(Th_{ic})^{op} \simeq \pi_0(ALG)$.

8.16 Corollary. *A functor between algebraic categories*

$$G\colon \mathcal{A}_2 \to \mathcal{A}_1$$

is algebraic if and only if it is induced by a morphism of theories. That is: there exists a morphism of algebraic theories $H\colon \mathcal{T}_1 \to \mathcal{T}_2$ *and two equivalence functors* $E_1\colon Alg\,\mathcal{T}_1 \to \mathcal{A}_1$, $E_2\colon Alg\,\mathcal{T}_2 \to \mathcal{A}_2$ *such that*

$$\begin{array}{ccc} Alg\,\mathcal{T}_1 & \xrightarrow{\;E_1\;} & \mathcal{A}_1 \\ {\scriptstyle Alg\,H}\uparrow & & \uparrow{\scriptstyle G} \\ Alg\,\mathcal{T}_2 & \xrightarrow[\;E_2\;]{} & \mathcal{A}_2 \end{array}$$

commutes up to natural isomorphism.

Proof. Since the condition to be algebraic is stable under composition with an equivalence and under natural isomorphism, one implication follows from 8.2 and the other one from 8.14. □

8.17 Remark. The 2-functor

$$Alg\colon Th_{ic}^{op} \to ALG$$

is a kind of representable 2-functor: forgetting the size condition (an algebraic theory is by definition a small category) we have $Alg\,\mathcal{T} = Th(\mathcal{T}, Set)$ for every algebraic theory $\mathcal{T}$. In fact, Set is a dualizing object, in the sense that also the converse 2-functor

$$ALG^{op} \to Th_{ic}\,, \quad \mathcal{A} \mapsto \mathcal{A}_{pfp}^{op}$$

is representable by Set : there is an equivalence of categories

$$\mathcal{A}_{pfp}^{op} \simeq ALG(\mathcal{A}, Set)\,.$$

Indeed, if $A \in \mathcal{A}_{pfp}^{op}$, then $G = \mathcal{A}(A, -)\colon \mathcal{A} \to Set$ preserves sifted colimits and has a left adjoint

$$L\colon Set \to \mathcal{A}\,, \quad L(S) = \coprod_S A\,.$$

Conversely, if $G\colon \mathcal{A} \to Set$ has a left adjoint L, then $G \simeq \mathcal{A}(L(*), -)$, and $L(*)$ is projectively finitely presentable provided that G preserves sifted colimits.

Let us come back to algebraic functors between algebraic categories (8.4). We wish to characterize them in terms of exactness properties. For this we will use Freyd's representation theorem. For the reader's convenience, we recall the statement here; a proof can be found e.g. in [MacLane], Section V.6.

8.18 Lemma. *A functor $F\colon \mathcal{B} \to Set$, with $\mathcal{B}$ complete, is representable if and only if it preserves limits and satisfies the Solution Set Condition: there is a set $\mathcal{G}$ of objects of $\mathcal{B}$ such that for any object B of $\mathcal{B}$ and any element $b \in FB$, there are $X \in \mathcal{G}$, $x \in FX$ and $f\colon X \to B$ such that $(Ff)(x) = b$.*

8.19 Theorem. *For every functor G between algebraic categories, the following conditions are equivalent:*

1. *G is algebraic,*
2. *G preserves limits and sifted colimits,*

and

3. *G preserves limits, filtered colimits and regular epimorphisms.*

Proof. 1 ⇒ 2 is obvious and 2 ⇒ 3 follows from the fact that: a) filtered implies sifted, and b) every regular epimorphism is a reflexive coequalizer (of its kernel pair).
To prove 3 ⇒ 1, let $G\colon \mathcal{A} \to \mathcal{B}$ be a functor preserving limits, filtered colimits

and regular epimorphisms for given algebraic categories $\mathcal{A}$ and $\mathcal{B}$.
(1) G has a left adjoint. That is, for every object A of $\mathcal{A}$ we are to prove that the functor

$$\mathcal{A}(A, G-)\colon \mathcal{B} \to Set$$

is representable.
(1a) Assume first that A is projectively finitely presentable. Since G preserves limits, it remains to prove that $\mathcal{A}(A, G-)$ satisfies the Solution Set Condition of Lemma 8.18. Recall that $\mathcal{B}$ is locally finitely presentable (6.16). Consider the set $\mathcal{B}_{fp}$ of finitely presentable objects of $\mathcal{B}$ (see 6.15), and take

$$\mathcal{G} = \{GX \mid X \in \mathcal{B}_{fp}\}\,.$$

Every object B of $\mathcal{B}$ is a filtereded colimit of objects from $\mathcal{B}_{fp}$ (6.15). Write $(\sigma_X\colon X \to B)$ for the colimit cocone. Since G preserves filtered colimits, $(G(\sigma_X)\colon GX \to GB)$ is still a colimit cocone. Since $\mathcal{A}(A, -)$ preserves sifted colimits, every $b\colon A \to GB$ factors through one of the $G(\sigma_X)$, that is, there are $\sigma_X\colon X \to B$ and $\varphi\colon A \to GX$ such that

$$\begin{array}{ccc} & & GX \\ & \overset{\varphi}{\nearrow} & \big\downarrow {\scriptstyle G(\sigma_X)} \\ A & \xrightarrow[b]{} & GB \end{array}$$

commutes. The Solution Set Condition is satisfied.
(1b) If A is an arbitrary object of $\mathcal{A}$, we know that A is a sifted colimit of projectively finitely presentable objects (6.8), say $A = colim\, A_i$. From (1a), we know that $\mathcal{A}(A_i, G-)$ is representable, say $\mathcal{A}(A_i, G-) \simeq \mathcal{B}(B_i, -)$. Finally, we have the following natural isomorphisms

$$\mathcal{A}(A, G-) = \mathcal{A}(colim\, A_i, G-) \simeq \lim \mathcal{A}(A_i, G-) \simeq$$

$$\simeq \lim \mathcal{B}(B_i, -) \simeq \mathcal{B}(colim\, B_i, -)$$

so that $\mathcal{A}(A, G-)$ is representable.
(2) G is algebraic. To prove this, it is by 8.2 sufficient to verify that G is induced by a theory morphism between the canonical algebraic theories. That is, there exists a functor

$$H\colon \mathcal{B}^{op}_{pfp} \to \mathcal{A}^{op}_{pfp}$$

preserving finite products and such that the square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad G \quad} & \mathcal{B} \\ {\scriptstyle Y^{\mathcal{A}}_{pfp}} \big\downarrow & & \big\downarrow {\scriptstyle Y^{\mathcal{B}}_{pfp}} \\ Alg\,(\mathcal{A}^{op}_{pfp}) & \xrightarrow[Alg\,H]{} & Alg\,(\mathcal{B}^{op}_{pfp}) \end{array}$$

commutes up to natural isomorphism – see 6.10 for the equivalence functors $Y^{\mathcal{A}}_{pfp}$ and $Y^{\mathcal{B}}_{pfp}$. Let $F\colon \mathcal{B} \to \mathcal{A}$ be a left adjoint of G. Then F preserves finite presentability (because G preserves filtered colimits) and regular projectivity (because G preserves regular epimorphisms): in fact, for all B in $\mathcal{B}$,

$$\mathcal{A}(FB,-) \simeq \mathcal{B}(B,G-) = \mathcal{B}(B,-)\cdot G\,.$$

By 5.14, F maps $\mathcal{B}_{pfp}$ into $\mathcal{A}_{pfp}$. Let $H\colon \mathcal{B}_{pfp}^{op} \to \mathcal{A}_{pfp}^{op}$ denote the dual functor of the domain-codomain restriction of F. In order to prove that the square above commutes up to natural isomorphism, it is sufficient to prove that the square below commutes up to natural isomorphism

$$\begin{array}{ccc} \mathcal{A} & \xleftarrow{\;F\;} & \mathcal{B} \\ {\scriptstyle Y^{\mathcal{A}}_{pfp}}\downarrow & & \downarrow{\scriptstyle Y^{\mathcal{B}}_{pfp}} \\ Alg\,(\mathcal{A}^{op}_{pfp}) & \xleftarrow[H^*]{} & Alg\,(\mathcal{B}^{op}_{pfp}) \end{array}$$

This follows immediately from 8.2 and the fact that H^{op} is the restriction of F. ☐

8.20 Remark. We end this chapter by mentioning the Gabriel-Ulmer duality for locally finitely presentable categories. The proof is similar to that of 8.14. Write *LEX* for the 2-category of small categories with finite limits, finite limit preserving functors, and natural transformations. Write *LFP* for the 2-category of locally finitely presentable categories, right adjoint preserving filtered colimits, and natural transformations. The 2-functor

$$Lex\colon LEX^{op} \to LFP\;, \quad \mathcal{C}_1 \overset{H}{\underset{K}{\rightrightarrows}}\!\!\Downarrow\alpha\; \mathcal{C}_2 \;\mapsto\; Lex\mathcal{C}_1 \overset{-\cdot H}{\underset{-\cdot K}{\leftleftarrows}}\!\!\Downarrow -\cdot\alpha\; Lex\mathcal{C}_2$$

(6.15) is a biequivalence. The converse 2-functor associates to a locally finitely presentable category $\mathcal{A}$ the small and finitely complete category $\mathcal{A}^{op}_{fp}$.

Chapter 9

Many-sorted algebraic categories

Classical algebraic categories, such as groups, lattices, etc. are not only abstract categories: their objects are sets with a structure and their morphisms are functions preserving the structure. Thus, they are concrete categories over *Set*, which means that a "forgetful" functor into *Set* is given. In computer science one often considers S-sorted algebras, where S is a given nonempty set (of sorts) and algebras are not sets with operations, but rather S-indexed families of sets with operations of given sort. This means that the forgetful functor is into Set^S rather than into *Set*.

In this chapter we study S-sorted algebraic theories and S-sorted algebraic categories. To start, we need a general lemma on morphisms of theories.

9.1 Lemma. *Let $H\colon \mathcal{T}_1 \to \mathcal{T}_2$ be a functor between small categories. If H is essentially surjective on objects, then for every category $\mathcal{B}$ the induced functor*

$$- \cdot H\colon [\mathcal{T}_2, \mathcal{B}] \to [\mathcal{T}_1, \mathcal{B}]$$

is faithful and conservative.
In particular, if H is an essentially surjective morphism of algebraic theories, then

$$Alg\, H\colon Alg\,\mathcal{T}_2 \to Alg\,\mathcal{T}_1$$

is faithful and conservative.

Proof. Consider a natural transformation $\alpha\colon A \to B\colon \mathcal{T}_2 \rightrightarrows \mathcal{B}$ and assume that $\alpha \cdot H\colon A \cdot H \to B \cdot H\colon \mathcal{T}_1 \rightrightarrows \mathcal{B}$ is a natural isomorphism. For every object X in $\mathcal{T}_2$ there is an object X' in $\mathcal{T}_1$ and an isomorphism $x\colon HX' \to X$. By naturality of α, we have $\alpha(X) = B(x) \cdot \alpha(HX') \cdot A(x)^{-1}$. Since $\alpha(HX')$ is an isomorphism, this implies that $\alpha(X)$ is an isomorphism. The proof that $- \cdot H$ is faithful is similar.
The second part of the statement follows from the first part taking $\mathcal{B} = Set$ and using that $Alg\,\mathcal{T}$ is a full subcategory of $Set^{\mathcal{T}}$. □

9.2 Remark. Classical algebraic categories $\mathcal{A}$, e.g. Ab, have their forgetful functor to Set representable by an algebra A (in Ab choose the group of integers; more generally, A is a free algebra on one generator as used in general algebra). The algebraic theory $\mathcal{T}$ which is the full subcategory of $\mathcal{A}^{op}$ on all finite products of A is then a typical algebraic theory of the given algebraic category. In case of Ab this is precisely the theory $\mathcal{T}_{ab}$ of 1.3.2. Analogously for typical algebraic categories $\mathcal{A}$ of S-sorted algebras: if for every sort $s \in S$ we have a free algebra A_s of $\mathcal{A}$ on one generator of sort s, then the forgetful functor from $\mathcal{A}$ to Set^S has components representable by $(A_s)_{s \in S}$. Again, the full subcategory of $\mathcal{A}^{op}$ on all finite products of these algebras is a typical algebraic theory of $\mathcal{A}$. We now formalize S-sorted algebraic categories.

9.3 Definition. Let S be a set. An *S-sorted algebraic theory* is a pair

$$(\mathcal{T}, \sigma)$$

where $\mathcal{T}$ is an algebraic theory and $\sigma \colon S \to obj\,\mathcal{T}$ is a map such that every object of $\mathcal{T}$ is a finite product of objects in $\sigma(S)$.
If S is a one-element set, S-sorted algebraic theories are called *one-sorted*

9.4 Remark.

1. A one-sorted theory consists of an algebraic theory $\mathcal{T}$ and a chosen object T such that all objects of $\mathcal{T}$ are powers T^n ($n \in \mathbb{N}$). One often works with the equivalent algebraic theory $\mathcal{T}_0$ whose objects are the natural numbers, with
$$\mathcal{T}_0(n, k) = \mathcal{T}(T^n, T^k)$$
for all $n, k \in \mathbb{N}$. The chosen object of $\mathcal{T}_0$ is, then, 1.

2. For every one-sorted theory $\mathcal{T}$, the category $Alg\,\mathcal{T}$ is equipped with a faithful functor
$$U = \mathcal{T}(T, -) \colon Alg\,\mathcal{T} \to Set$$
whose left adjoint $F \colon Set \to Alg\,\mathcal{T}$ is given by copowers of T. In particular, $T = F(1)$ is now a free algebra on one generator, if $Alg\,\mathcal{T}$ is viewed as a concrete category over Set. Observe that Examples 1.3.1 and 1.3.2 are of this type: in Set we chose 1, and in Ab we chose $\mathbb{Z}$ as free algebra on one generator.

3. Every algebra A of an algebraic category $\mathcal{A}$ (or, more generally, every object of a category with finite coproducts) defines an algebraic theory $\mathcal{T}(A)$ whose objects are natural numbers and whose morphisms from n to k are the morphisms of $\mathcal{A}$ from $kA = A + \ldots + A$ (k summands) to nA. More precisely, $\mathcal{T}(A)$ is equivalent to the full subcategory of $\mathcal{A}^{op}$ on all finite copowers of A under the equivalence functor $n \mapsto nA$. The corresponding category of $\mathcal{T}(A)$-algebras can be equivalent to $\mathcal{A}$, as we have seen in the example $\mathcal{A} = Ab$ and $A = \mathbb{Z}$. In fact, if $\mathcal{A} = Alg\,\mathcal{T}$ for a one-sorted algebraic theory and A is a free algebra on one generator, then

$\mathcal{A}$ is always equivalent to $Alg\,\mathcal{T}(A)$. These theories $\mathcal{T}(A)$ are often seen as the "natural" algebraic theories in classical algebra (e.g. for $\mathcal{A}$ = groups, lattices, monoids, etc.).

4. Recall the theory $\mathcal{T}_1$ of *Set* in Example 1.3.1, which is a free completion of a terminal category under finite products. Every one-sorted algebraic theory $(\mathcal{T}, T)$ defines a unique functor $\sigma\colon \mathcal{T}_1 \to \mathcal{T}$ preserving finite products with $\sigma(1) = T$. More generally:

9.5 Remark. For an S-sorted algebraic theory $(\mathcal{T}, \sigma)$, we can view $\sigma\colon S \to obj\,\mathcal{T}$ as a functor $\sigma\colon S \to \mathcal{T}$ with S a discrete category. Such a functor extends uniquely to a functor (still denoted σ)

$$\sigma\colon \mathcal{T}_S \to \mathcal{T}$$

which preserves finite products, where $\mathcal{T}_S$ is the free completion of S under finite products described in 1.3.5. The condition on the map $\sigma\colon S \to obj\,\mathcal{T}$ correspond to the fact that the functor $\sigma\colon \mathcal{T}_S \to \mathcal{T}$ is essentially surjective on objects.

9.6 Proposition. *Let $(\mathcal{T}, \sigma)$ be an S-sorted algebraic theory. The canonical forgetful functor*

$$U_\sigma\colon Alg\,\mathcal{T} \to Set^S\,, \quad A \mapsto \langle A(\sigma(s))\rangle_{s\in S}$$

is algebraic, faithful, and conservative.
Therefore, U_σ preserves and reflects limits, sifted colimits, monomorphisms, and regular epimorphisms.

Proof. Following 8.2, the morphism $\sigma\colon \mathcal{T}_S \to \mathcal{T}$ induces an algebraic functor

$$Alg\,\sigma\colon Alg\,\mathcal{T} \to Alg\,\mathcal{T}_S\,.$$

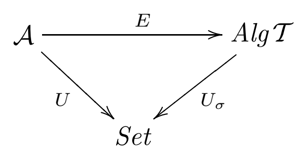
Since $\sigma\colon \mathcal{T}_S \to \mathcal{T}$ is essentially surjective, $Alg\,\sigma$ is faithful and conservative by 9.1. Since $Alg\,\mathcal{T}_S \simeq Set^S$ (1.3.5), the proof is completed by observing that 8.2 implies the concrete form $U_\sigma A = \langle A(\sigma(s))\rangle$ above. ☐

9.7 Definition.

1. By a *concrete category* over Set^S is meant a category $\mathcal{A}$ together with a faithful functor $U\colon \mathcal{A} \to Set^S$.

2. Given concrete categories $(\mathcal{A}, U)$ and $(\mathcal{A}', U')$ over Set^S, a *concrete functor* is a functor $F\colon \mathcal{A} \to \mathcal{A}'$ such that U is naturally isomorphic to $U' \cdot F$.

3. An *S-sorted algebraic category* is a concrete category $(\mathcal{A}, U)$ over Set^S for which there exists an S-sorted algebraic theory $(\mathcal{T}, \sigma)$ such that $Alg\,\mathcal{T}$ is *concretely equivalent* to $\mathcal{A}$, i.e., there exists an equivalence functor $E\colon \mathcal{A} \to Alg\,\mathcal{T}$ such that the diagram



commutes up to natural isomorphisms.

9.8 Example.

1. Ab is a one-sorted algebraic category with respect to the canonical forgetful functor $Ab \to Set$.

2. Gr is a two-sorted algebraic category when considered with the forgetful functor $U\colon Gr \to Set \times Set$ assigning to every graph G the pair (G_v, G_e).

9.9 Remark.

1. For an S-sorted algebraic theory $(\mathcal{T}, \sigma)$, we can describe explicitly a left adjoint $F_\sigma\colon Set^S \to Alg\,\mathcal{T}$ of U_σ : using 8.2 and the description of the equivalence $Alg\,\mathcal{T}_S \simeq Set^S$, we get:

$$F_\sigma(\langle X_s\rangle_{s\in S}) = \coprod_{s\in S}\left(\coprod_{X_s}\mathcal{T}(\sigma(s), -)\right).$$

This is a *free algebra* of the category $Alg\,\mathcal{T}$ on the object $\langle X_s\rangle_{s\in S}$. It is called *finitely generated* if the coproduct $\coprod_{s\in S} X_s$ is a finite set.

2. For $S = \{*\}$, an S-sorted algebraic theory $(\mathcal{T}, \sigma)$ is an algebraic theory $\mathcal{T}$ together with a choice of an object T of $\mathcal{T}$ such that every object of $\mathcal{T}$ is a finite power of T. In this case, we get:

- $U_\sigma\colon Alg\,\mathcal{T} \to Set\,, \quad U_\sigma(A) = A(T)$
- $F_\sigma\colon Set \to Alg\,\mathcal{T}\,, \quad F_\sigma(X) = \coprod_X \mathcal{T}(T, -)\,.$

In particular, a free algebra on one generator is represented by T : $F_\sigma(1) = \mathcal{T}(T, -)\,.$

9.10 Example.

1. Let $\mathcal{C}$ be a small category and consider its free completion under finite products $\Gamma\colon \mathcal{C} \to \mathcal{T}_\mathcal{C}$ described in 1.3.4. Precomposing Γ with the inclusion $obj\,\mathcal{C} \to \mathcal{C}$, we get an S-sorted algebraic theory

$$(\mathcal{T}_\mathcal{C}, S = obj\,\mathcal{C} \to \mathcal{C} \to \mathcal{T}_\mathcal{C})\,.$$

The induced functor $U_\sigma\colon Alg\,\mathcal{T}_\mathcal{C} \to Set^S$ is the equivalence $Alg\,\mathcal{T}_\mathcal{C} \simeq Set^\mathcal{C}$ described in 1.3.4, followed by the functor $Set^\mathcal{C} \to Set^S$ which forgets the action of $A\colon \mathcal{C} \to Set$ on the morphisms of $\mathcal{C}$.

2. As a special case of 1. above, consider a set S and the theory $\mathcal{T}_S$ described in 1.3.5. We get an S-sorted algebraic theory

$$(\mathcal{T}_S, S \to \mathcal{T}_S)\,.$$

The induced functor U_σ is the equivalence $Alg\,\mathcal{T}_S \simeq Set^S$ described in 1.3.5.

3. Every algebraic theory $\mathcal{T}$ can be seen as an S-sorted algebraic theory in a trivial way: put $S = obj\mathcal{T}$ and use simply the inclusion $S = obj\mathcal{T} \to \mathcal{T}$. The induced functor $U_\sigma \colon Alg\mathcal{T} \to Set^S$ forgets the action of an algebra $A \colon \mathcal{T} \to Set$ on the morphisms of $\mathcal{T}$. But this forgetful functor usually is not the typical one. (For example, Ab is one-sorted rather than many-sorted.)

4. The theory $\mathcal{T}_{ab}$ (1.3.2) together with the choice of the object $\mathbb{Z}$ is a one-sorted algebraic theory. Up to the equivalence $Alg\mathcal{T}_{ab} \simeq Ab$, the induced adjunction $F_\sigma \dashv U_\sigma$ is the usual adjunction $Ab \leftrightarrows Set$.

5. The theory $\mathcal{T}_{gr}$ of the category Gr of graphs described in 1.3.6 is a two-sorted algebraic theory. Recall that $\mathcal{T}_{gr}$ is a free completion under finite products of a category with two objects e and v. Observe that Gr cannot be described as the category of algebras for a one-sorted theory. In fact, a terminal object in Gr is the graph with one vertex and one edge, and it has a proper subobject given by the graph G with one vertex and no edge. Observe that G is neither terminal nor initial in Gr. This is in contrast with the following general fact on one-sorted algebraic theories.

9.11 Lemma. *In a one-sorted algebraic category a terminal algebra A has no nontrivial subobjects: for every subobject $m \colon B \to A$ either B is an initial algebra or a terminal one.*

Proof. Given a one-sorted algebraic theory $(\mathcal{T}, \sigma)$, since $U_\sigma \colon Alg\mathcal{T} \to Set$ preserves limits (9.6), $U_\sigma(B)$ is a subset of $U_\sigma(A) = 1$. If $U_\sigma(B) = 1$, then $U_\sigma(m)$ is an isomorphism and then m is an isomorphism (U_σ is conservative). If $U_\sigma(B) = \emptyset$, consider the unique monomorphism $a \colon F_\sigma(\emptyset) \to B$ and the induced map $U_\sigma(a) \colon U_\sigma(F_\sigma(\emptyset)) \to U_\sigma(B) = \emptyset$. Such a map necessarily is an isomorphism, and then a also is an isomorphism. □

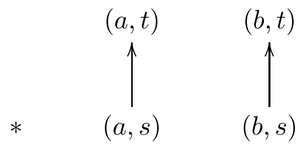
Let us stress that whereas in one-sorted algebraic categories we speak about free algebras on a set of generators, in S-sorted algebraic categories the generators must be assigned concrete sorts: here we form free algebras on S-sorted sets.

9.12 Example.

1. A free graph (1.3.6) on a two-sorted set $X = (X_v, X_e)$ is

$$F_\sigma(X) = (X_v + (X_e \times \{s,t\}), X_e)\,,$$

where the source function is $X_e \to X_e \times \{s,t\}\,,\ x \mapsto (x,s)$ and the target function is $X_e \to X_e \times \{s,t\}\,,\ x \mapsto (x,t)$. For example, if $X_v = \{*\}$ and $X_e = \{a,b\}$, we get



2. An algebraic theory of automata (see 1.3.7). We can describe a left adjoint of the forgetful functor

$$U\colon \mathit{Aut} \to \mathit{Set}^{\{s,i,o\}}$$

as follows: given a three-sorted set $X = (X_s, X_i, X_o)$ an automaton $A = FX$ has the given set of inputs: $A_i = X_i$. The states of A are freely generated from φ (the initial state) and the states of X_s by repeated inputs: $A_s = (X_s + \{\varphi\}) \times X^*$. Here the pair consisting of φ and the empty word is the initial state of A, and the next-state function is given by concatenation:

$$\delta\colon q \mid x_1 \dots x_n, x \mapsto q \mid x_1 \dots x_n x\,.$$

Finally, the outputs are $A_o = A_s + X_o$ with the left-hand copruduct injection as the output function. The full subcategory $\mathcal{T}$ of Aut^{op} on all finitely generated free automata is a three-sorted algebraic theory of Aut.

9.13 Example.

1. For one-sorted algebraic theories $(\mathcal{T}, \sigma)$ the free algebras $F_\sigma(n)$ on n generators, considered for $n = 0, 1, 2, \dots$ as a full subcategory of $(\mathit{Alg}\,\mathcal{T})^{op}$, form an algebraic theory of the category $\mathit{Alg}\,\mathcal{T}$. Such an algebraic theory is equivalent to the theory $\mathcal{T}_0$ described in 9.4.1: using 4.1 and Yoneda lemma, we have

$$\mathcal{T}_0(n,k) = \mathcal{T}(T^n, T^k) \simeq \mathit{Alg}\,\mathcal{T}(\mathcal{T}(T^k, -), \mathcal{T}(T^n, -)) \simeq$$

$$\simeq \mathit{Alg}\,\mathcal{T}\left(\coprod_k \mathcal{T}(T,-), \coprod_n \mathcal{T}(T,-)\right) = \mathit{Alg}\,\mathcal{T}(F_\sigma(k), F_\sigma(n))\,.$$

For $k = 1$, the homomorphisms from $F_\sigma(k)$ to $F_\sigma(n)$ precisely correspond to the elements of $U_\sigma(F_\sigma(n))$. For other k's we use the fact that $k = 1 \times \dots \times 1$ in $\mathcal{T}_0$, thus $\mathcal{T}_0(n,k)$ consists of k-tuples of elements of the free algebra on n generators:

$$\mathcal{T}_0(n,k) \simeq (U_\sigma(F_\sigma(n)))^k\,.$$

2. Analogously for S-sorted theories $(\mathcal{T}, \sigma)$: denote by S^* the set of all words in the alphabet S (i.e., all n-tuples for $n \in \mathbb{N}$ where the simplified notation $s_1 \dots s_n$ is used). $\mathit{Alg}\,\mathcal{T}$ has a theory $\mathcal{T}_0$ whose objects are the words of S^*. For every word $w = s_1 \dots s_n$, let $X_w = \{x_1, \dots, x_n\}$ be an S-sorted set of n elements, where s_i is the sort of the i-th element $(i = 1, \dots, n)$. The morphisms of $\mathcal{T}_0$ are

$$\mathcal{T}_0(w,v) = \mathit{Alg}\,\mathcal{T}(F_\sigma(X_v), F_\sigma(X_w))\,.$$

Using free algebras we can restate some facts from Chapter 5.

9.14 Proposition. *Let $(\mathcal{T}, \sigma)$ be an S-sorted algebraic theory.*

1. *Free algebras are precisely the coproducts of representable algebras.*
2. *Every algebra is a regular quotient of a free algebra.*
3. *Regular projective algebras are precisely the retracts of free algebras.*

Proof. 1: Following 9.9.1, every free algebra is a corpoduct of representable algebras. Conversely, consider a coproduct of representable algebras

$$A = \coprod_{t \in \mathcal{T}} \left(\coprod_{X_t} \mathcal{T}(t, -) \right)$$

and, for every $t \in \mathcal{T}$, fix a decomposition $t = \sigma(s_{t,1}) \times \ldots \times \sigma(s_{t,n_t})$. Since the Yoneda embedding $Y : \mathcal{T}^{op} \to Alg\mathcal{T}$ preserves finite coproducts (4.1), we have

$$A \simeq \coprod_{s \in S} \left(\coprod_{X_s} \mathcal{T}(\sigma(s), -) \right) \simeq F_\sigma(\langle X_s \rangle_{s \in S})$$

where $X_s = \coprod_{t \in \mathcal{T}} (\coprod X_t)$ and $\coprod X_t$ consists of a copy of X_t for every occurence of s in the decomposition of t.
2: Following 4.3, every algebra is a regular quotient of a coproduct of representable algebras and then, by point 1, it is a regular quotient of a free algebra.
3: This follows from 1. and 5.11.2. ☐

9.15 Proposition. *Let $(\mathcal{T}, \sigma)$ be an S-sorted algebraic theory.*

1. *Finitely generated free algebras are precisely the representable algebras.*
2. *Projectively finitely presentable algebras are precisely the retracts of finitely generated free algebras.*
3. *Finitely presentable algebras are precisely the coequalizers of (reflexive) pairs of morphisms between finitely generated free algebras.*

Proof. 1: If $\langle X_s \rangle_{s \in S}$ is a finite S-set, then $X_s = \emptyset$ for all but a finite number $s_1, \ldots, s_n$ of sorts. Then

$$F_\sigma(\langle X_s \rangle_{s \in S}) = \left(\coprod_{X_{s_1}} \mathcal{T}(\sigma(s_1), -) \right) + \ldots + \left(\coprod_{X_{s_n}} \mathcal{T}(\sigma(s_n), -) \right) \simeq$$

$$\simeq \mathcal{T}(\sigma(s_1)^{X_{s_1}}, -) + \ldots + \mathcal{T}(\sigma(s_n)^{X_{s_n}}, -) \simeq \mathcal{T}(\sigma(s_1)^{X_{s_1}} \times \ldots \times \sigma(s_n)^{X_{s_n}}, -)$$

where we use that each X_{s_i} is finite and that the Yoneda embedding preserves finite coproducts. Conversely, consider a representable algebra $\mathcal{T}(t, -)$ and write $t = \sigma(s_1) \times \ldots \times \sigma(s_n)$. Then $\mathcal{T}(t, -) \simeq F_\sigma(\langle X_s \rangle_{s \in S})$, where X_s has as many elements as the occurences of s in the decomposition of t. Therefore, $\langle X_s \rangle_{s \in S}$ is a finite S-set.

2: This follows from 1. and 5.11.1.
3: This follows from 1. and 5.15. □

In Chapter 6 we characterized algebraic categories as abstract categories. We want now to characterize algebraic categories as concrete categories, that is categories equipped with a faithful functor into a power of *Set*. From 9.6, we know that an S-sorted algebraic theory $\sigma\colon S \to \mathcal{T}$ gives rise to a conservative and algebraic functor $U_\sigma\colon Alg\,\mathcal{T} \to Set^S$. Conversely, we have the following result (recalling 9.7).

9.16 Theorem. *(Characterization of Many-Sorted Algebraic Categories) Let $\mathcal{A}$ be a cocomplete category and $U\colon \mathcal{A} \to Set^S$ a conservative algebraic functor. Then $\mathcal{A}$ is an S-sorted algebraic category.*

Proof. We have the following situation

$$\mathcal{A} \underset{U}{\overset{F}{\rightleftarrows}} Set^S \simeq Alg\,\mathcal{T}_s \xleftarrow{\quad Y \quad} \mathcal{T}_S^{op}$$

where $F \dashv U$ and $\mathcal{T}_S$ is as in 1.3.5. Let $\mathcal{G}$ be the set of representable algebras in $Alg\,\mathcal{T}_S$. Following 5.5.4 and 6.6.4, $\mathcal{G}$ is a strong generator in $Alg\,\mathcal{T}_s$ and it is formed by projectively finitely presentable objects. Moreover, by 4.1, it is closed under finite coproducts. Therefore, $F(\mathcal{G})$ is closed in $\mathcal{A}$ under finite coproducts and, by 6.12.2, it is formed by projectively finitely presentable objects. Let us prove that $F(\mathcal{G})$ is a strong generator in $\mathcal{A}$. Consider an object A in $\mathcal{A}$. Since $\mathcal{G}$ is a strong generator (and Set^S has coproducts and finite limits), there is a strong epimorphism

$$a\colon \coprod_{i\in I} G_i \to U(A)$$

with the G_i's in $\mathcal{G}$ (see 6.1 and 6.4). Since a left adjoint preserves strong epimorphisms, we have a strong epimorphism

$$\textstyle\coprod_{i\in I} FG_i \simeq F\left(\coprod_{i\in I} G_i\right) \xrightarrow{\quad Fa \quad} F(UA) \ .$$

Since strong epimorphisms are stable under composition, it suffices to prove that the the counit $\epsilon_A\colon F(UA) \to A$ of the adjunction $F \dashv U$ is a strong epimorphism. Because of the triangular identities, we know that $U(\epsilon_A)$ is a split and therefore regular epimorphism. Since U preserves sifted colimits, it preserves in particular regular epimorphisms, and then it reflects them because it is conservative. Thus, ϵ_A is a regular and therefore strong epimorphism. Following the proof of 6.8 (implication 3 ⇒ 1), the functor

$$E\colon \mathcal{A} \to Alg\,(F(\mathcal{G})^{op}) \ , \quad K \mapsto \mathcal{A}(-,K)\colon F(\mathcal{G})^{op} \to Set$$

is an equivalence. Clearly, $F(\mathcal{G})^{op}$ is an S-sorted algebraic theory: the essentially surjective functor $\sigma\colon \mathcal{T}_S \to F(\mathcal{G})^{op}$ is the codomain restriction of $F\cdot Y$. It remains to check that $U_\sigma \cdot E \simeq U$: for an element s in S, we define $\hat{s} \in Set^S$ by

$$\hat{s}(t) = 1 \ \text{ if } \ t = s \ , \quad \hat{s}(t) = \emptyset \ \text{ if } \ t \neq s \, .$$

Following 9.9.1 and using the adjunction $F \dashv U$, we have

$$U_\sigma(EK) = \langle \mathcal{A}(F(\hat{s}), K) \rangle_{s \in S} \simeq \langle \mathit{Set}^S(\hat{s}, UK) \rangle_{s \in S} \simeq \langle (UK)_s \rangle_{s \in S} = UK\,.$$

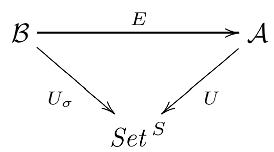
□

9.17 Corollary. *S-sorted algebraic categories are, up to concrete equivalence, precisely the cocomplete categories over Set^S whose forgetful functor is conservative and algebraic.*

More detailed: for a faithful functor

$$U\colon \mathcal{A} \to \mathit{Set}^S$$

with $\mathcal{A}$ cocomplete, there exists an S-sorted algebraic category $U_\sigma\colon \mathcal{B} \to \mathit{Set}^S$ and an equivalence functor



making the above diagram commutative up to natural isomorphism iff U is a conservative right adjoint preserving sifted colimits.

9.18 Remark. Let $U\colon \mathcal{A} \to \mathit{Set}^S$ be an S-sorted algebraic category. It follows from 9.16 that

1. the forgetful functor U has a left adjoint $F\colon \mathit{Set}^S \to \mathcal{A}$, and
2. an S-sorted algebraic theory $\mathcal{T}$ for $\mathcal{A}$ can be described as in 9.13.2:

$$\mathit{obj}\,\mathcal{T} = S^* \ \text{ and } \ \mathcal{T}(w,v) = \mathcal{A}(F(X_v), (FX_w))\,.$$

It is not difficult to adapt the duality between idempotent-complete algebraic theories and algebraic categories (8.14) to a duality between S-sorted algebraic theories and S-sorted algebraic categories.

9.19 Definition. Let S be a set.

1. The 2-category ALG_S is defined as follows:
   - objects are S-sorted algebraic categories $(\mathcal{A}, U)$;
   - 1-cells are concrete functors (see 9.7.1);
   - 2-cells are natural transformations.
2. The 2-category Th_S is defined as follows:
   - objects are S-sorted algebraic theories $(\mathcal{T}, \sigma)$;

- a 1-cell $H\colon (\mathcal{T}_1, \sigma_1) \to (\mathcal{T}_2, \sigma_2)$ is a functor $H\colon \mathcal{T}_1 \to \mathcal{T}_2$ such that $\sigma_2 \cdot H \simeq \sigma_1$;
- 2-cells are natural transformations.

9.20 Remark.

1. Every concrete functor between S-sorted algebraic categories is faithful, conservative, and algebraic (use 8.19).
2. The condition $\sigma_1 \cdot H \simeq \sigma_2$ in 9.19 implies that H is essentially surjective on objects and preserves finite products.

9.21 Definition. We denote by

$$Alg_S\colon Th_S^{op} \to ALG_S$$

the 2-functor defined by

$$(\mathcal{T}_1, \sigma_1) \underset{H_2}{\overset{H_1}{\rightrightarrows}} (\mathcal{T}_2, \sigma_2) \;\; (\downarrow \alpha) \quad \mapsto \quad (Alg\,\mathcal{T}_1, Alg\,\sigma_1) \underset{Alg\,H_2}{\overset{Alg\,H_1}{\leftleftarrows}} (Alg\,\mathcal{T}_2, Alg\,\sigma_2) \;\; (\downarrow Alg\,\alpha)$$

where $Alg\colon Th^{op} \to ALG$ is the 2-functor defined in 8.10, and $Alg\,\sigma$ is an abuse of notation for

$$Alg\,\mathcal{T} \xrightarrow{Alg\,\sigma} Alg\,\mathcal{T}_S \simeq Set^S \; .$$

9.22 Proposition. *The 2-functor*

$$Alg_S\colon Th_S^{op} \to ALG_S$$

is a biequivalence.

Proof. Following 9.6, Alg_S is well-defined. Following 9.16, every object of ALG_S is equivalent to $Alg_S(\mathcal{T}, \sigma)$ for some object $(\mathcal{T}, \sigma)$ of Th_S. The proof that Alg_s is locally full and faithful is the same as in Theorem 8.14. It remains to prove that Alg_S is locally essentially surjective: consider a concrete functor

$$\begin{array}{ccc} Alg\,\mathcal{T}_1 & \xrightarrow{G} & Alg\,\mathcal{T}_2 \\ & {\scriptstyle Alg\,\sigma_1}\searrow \quad \simeq \quad \swarrow{\scriptstyle Alg\,\sigma_2} & \\ & Set^S & \end{array}$$

and the left adjoints $F \dashv G$, $\sigma_i^* \dashv Alg\,\sigma_i$. We are going to prove that $F \cdot Y$ factors (up to natural isomorphism) through $\mathcal{T}_2$:

$$\begin{array}{ccc} \mathcal{T}_1^{op} & \xrightarrow{Y} & Alg\,\mathcal{T}_1 \\ {\scriptstyle H^{op}}\downarrow & \simeq & \downarrow{\scriptstyle F} \\ \mathcal{T}_2^{op} & \xrightarrow[Y]{} & Alg\,\mathcal{T}_2 \end{array}$$

Indeed, for every object $T \in \mathcal{T}_1$, there is an object $t \in \mathcal{T}_S$ and an isomorphism $T \simeq \sigma_1(t)$. Therefore

$$F(Y(T)) \simeq F(Y(\sigma_1(t))) \simeq F(\sigma_1^*(Y_S(t))) \simeq \sigma_2^*(Y_S(t)) \simeq Y(\sigma_2(t))$$

where we write Y_S for the Yoneda embedding Y composed with the equivalence of 1.3.5:

$$\mathcal{T}_S^{op} \xrightarrow{\quad Y \quad} Alg\,\mathcal{T}_S \simeq Set^S \ .$$

In this way, we get a map on objects

$$H\colon obj\,\mathcal{T}_1 \to obj\,\mathcal{T}_2\ , \quad T \mapsto \sigma_2(t)$$

which extends to a functor (because Y is full and faithful). Such a functor provides the needed morphism $H\colon (\mathcal{T}_1, \sigma_1) \to (\mathcal{T}_2, \sigma_2)$ of S-sorted algebraic theories. Indeed, following 8.2, $H^* \simeq F$, and then $Alg\,H \simeq G$. □

9.23 Remark. The 2-categories Th_S and ALG_S defined in 9.19 can be slightly modified to obtain full sub-2-categories Th'_S and ALG'_S of comma 2-categories: take as 1-cells in Th'_S not just functors $H\colon \mathcal{T}_1 \to \mathcal{T}_2$ such that $\sigma_2 \cdot H \simeq \sigma_1$, but pairs (H, φ), with φ a specified natural isomorphism $\varphi\colon \sigma_2 \cdot H \to \sigma_1$; 2-cells are then natural transformations $\alpha\colon H_1 \to H_2$ compatible with φ_1 and φ_2. One has to modify in the same way 1-cells and 2-cells in ALG_S. We still have a duality

$$Alg_S\colon (Th'_S)^{op} \to ALG'_S$$

as in 9.22, just observe that α satisfies the compatibility condition to be a 2-cell in Th'_S iff $Alg\,\alpha$ satisfies the compatibility condition to be a 2-cell in ALG'_S.

Chapter 10

Birkhoff's variety theorem

So far we have not treated one of the central concepts of algebra: equations. If $\mathcal{T}$ is an algebraic theory, an equation in $\mathcal{T}$ is simply a parallel pair $u, v\colon s \rightrightarrows t$ of morphisms in $\mathcal{T}$. An algebra $A\colon \mathcal{T} \to \mathit{Set}$ satisfies the equation "$u = v$" if $A(u) = A(v)$. Observe that if A satisfies the equation "$u = v$", then it satisfies also all the equations of the form "$u \cdot x = v \cdot x$" and "$y \cdot u = y \cdot v$" for $x\colon s' \to t$ and $y\colon t \to t'$ in $\mathcal{T}$. Moreover, if the equation "$u_i = v_i$", for $u_i, v_i\colon s \to t_i$, $i = 1, \ldots, n$, are satisfied by A, then A satisfies also "$\langle u \rangle = \langle v \rangle$", where $\langle u \rangle$ and $\langle v \rangle$ are the extensions of the families u_i, v_i to the product $t_1 \times \ldots \times t_n$. For this reason, we will state the definition of equationally specified subcategory using congruences as well as equations (see 10.6 and 10.7).

We need a preliminary result which refines Lemma 9.1.

10.1 Lemma. *Let $H\colon \mathcal{T}_1 \to \mathcal{T}_2$ be a functor between small categories. If H is full and essentially surjective, then for every category $\mathcal{B}$ the induced functor*

$$- \cdot H\colon [\mathcal{T}_2, \mathcal{B}] \to [\mathcal{T}_1, \mathcal{B}]$$

is full and faithful.
In particular, if H is a full and essentially surjective morphism of algebraic theories, then

$$\mathit{Alg}\, H\colon \mathit{Alg}\,\mathcal{T}_2 \to \mathit{Alg}\,\mathcal{T}_1$$

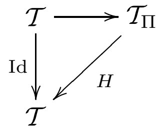
is full and faithful.

Proof. Consider two functors $A, B\colon \mathcal{T}_2 \rightrightarrows \mathit{Set}$ and a natural transformation $\beta\colon A \cdot H \to B \cdot H$. If X is an object in $\mathcal{T}_2$, we can chose an object Z in $\mathcal{T}_1$ and an isomorphism $x\colon HZ \to X$. We can now define $\alpha(X)$ by the following composition

$$AX \xrightarrow{A(x^{-1})} A(HX) \xrightarrow{\beta(X)} B(HX) \xrightarrow{B(x)} BX \ .$$

Using that H is full, it is straightforward to check the naturality of $\alpha(X)$ with respect to X, and that $\alpha(X)$ does not depend on the choice of Z and x. This implies in particular that $\alpha \cdot H = \beta$. □

10.2 Remark. The previous lemma cannot be inverted. Indeed, consider an algebraic theory $\mathcal{T}$, its free completion under finite products $\mathcal{T} \to \mathcal{T}_\Pi$ (1.3.4) and the extension $H\colon \mathcal{T}_\Pi \to \mathcal{T}$ of the identity functor on $\mathcal{T}$



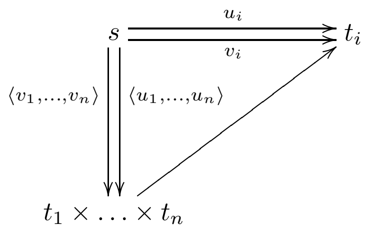
H is a morphism of algebraic theories, but it is not full. Nevertheless, the induced functor $Alg\,H$ is nothing but the full inclusion $Alg\,\mathcal{T} \to Alg\,\mathcal{T}_\Pi \simeq Set^{\mathcal{T}}$.

10.3 Definition. Let $\mathcal{T}$ be an algebraic theory. A *congruence* on $\mathcal{T}$ is a collection $\sim$ of equivalence relations $\underset{s,t}{\sim}$ on hom-sets $\mathcal{T}(s,t)$, where (s,t) ranges over $obj\,\mathcal{T} \times obj\,\mathcal{T}$, such that

1. if $u \underset{s,t}{\sim} v$ and $x \underset{r,s}{\sim} y$, then $u \cdot x \underset{r,t}{\sim} v \cdot y$

$$r \underset{y}{\overset{x}{\rightrightarrows}} s \underset{v}{\overset{u}{\rightrightarrows}} t$$

2. if $u_i \underset{s,t_i}{\sim} v_i$ for $i = 1, \ldots, n$, then $\langle u_1, \ldots, u_n \rangle \underset{s,t}{\sim} \langle v_1, \ldots, v_n \rangle$, where $t = t_1 \times \ldots \times t_n$



10.4 Example. Consider a category $\mathcal{B}$ with finite products and a functor $H\colon \mathcal{T} \to \mathcal{B}$ preserving finite products. H Induces a congruence $\underset{H}{\sim}$ on $\mathcal{T}$ by

$$u \underset{s,t}{\sim} v \ \text{ iff } \ H(u) = H(v)$$

for $u, v\colon s \rightrightarrows t$ in $\mathcal{T}$.

10.5 Remark.

1. Congruences on a given algebraic theory $\mathcal{T}$ are ordered in a natural way: we write $\sim \subseteq \sim'$ in case that for every pair s, t of objects of $\mathcal{T}$ the two subsets $\underset{s,t}{\sim}$ and $\underset{s,t}{\sim}'$ of $\mathcal{T}(s,t) \times \mathcal{T}(s,t)$ fulfil $\underset{s,t}{\sim} \subseteq \underset{s,t}{\sim}'$.

2. It is easy to see that a (set theoretical) intersection of congruences is a congruence. Consequently, for every set C of parallel pairs of morphisms of $\mathcal{T}$ there exists a smallest congruence $\sim$ on $\mathcal{T}$ containing C. We say that $\sim$ is *generated* by the set C.

10.6 Definition. A full subcategory $\mathcal{A}$ of $Alg\,\mathcal{T}$ is said to be *equationally specified* if there exists a set of parallel pairs (u_i, v_i) of morphisms of $\mathcal{T}$, $i \in I$, such that a $\mathcal{T}$-algebra A lies in $\mathcal{A}$ iff $A(u_i) = A(v_i)$ holds for all $i \in I$.

10.7 Remark. Let $\sim$ be the congruence generated by the set of all the pairs (u_i, v_i) above. It follows from 10.4 and 10.5 that a $\mathcal{T}$-algebra A lies in $\mathcal{A}$ iff it fulfils:

$$u \sim v \ \text{ implies } \ A(u) = A(v)\,.$$

Thus, every equationally specified subcategory is also specified by a congruence.

10.8 Definition. For every congruence $\sim$ on an algebraic theory $\mathcal{T}$ we denote by

$$\mathcal{T}/\sim$$

the algebraic theory on the same objects and with morphisms given by the congruence classes of morphisms of $\mathcal{T}$:

$$(\mathcal{T}/\sim)(s,t) = \mathcal{T}(s,t)/\underset{s,t}{\sim}\ .$$

Composition and identity morphisms are inherited from $\mathcal{T}$; more precisely, they are determined by the fact that we have a functor

$$Q\colon \mathcal{T} \to \mathcal{T}/\sim$$

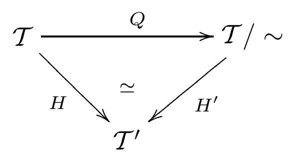
which is the identity map on objects and assigns to every morphism its congruence class.

10.9 Remark.

1. It is easy to verify that $\mathcal{T}/\sim$ has finite products (determined by those of $\mathcal{T}$) and, thus, $\mathcal{T}/\sim$ is an algebraic theory and Q is a theory morphism.

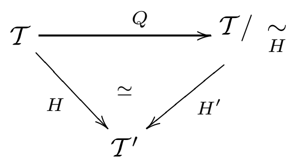
2. The morphism $Q\colon \mathcal{T} \to \mathcal{T}/\sim$ is full and essentially surjective. Following 10.1, $Alg\,Q\colon Alg\,(\mathcal{T}/\sim) \to Alg\,\mathcal{T}$ is full and faithful.

3. A morphism of theories $H\colon \mathcal{T} \to \mathcal{T}'$ factors through Q



   iff the congruence $\sim$ is contained in the congruence $\underset{H}{\sim}$ of 10.4. When this is the case, the factorization H' is essentially unique and is a theory morphism.

4. If in 3. we take $\sim$ equal to $\underset{H}{\sim}$ then the factorization H'

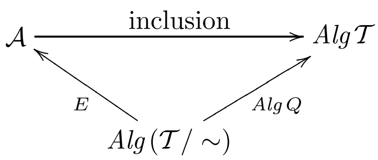


is faithful. Therefore, H is full and essentially surjective iff H' is an equivalence of categories.

10.10 Lemma. *Let $\sim$ be a congruence on an algebraic theory $\mathcal{T}$, and let $\mathcal{A}$ be the full subcategory of $Alg\,\mathcal{T}$ specified by $\sim$. There is an equivalence functor*

$$E\colon Alg\,(\mathcal{T}/\sim) \to \mathcal{A}$$

such that the triangle



commutes, up to natural isomorphism.

Proof. If B is in $Alg\,(\mathcal{T}/\sim)$ and $u \sim v$, then $B(Q(u)) = B[u] = B[v] = B(Q(v))$. This implies that $Alg\,Q$ factors through the full inclusion of $\mathcal{A}$ in $Alg\,\mathcal{T}$. Moreover, if A is in $\mathcal{A}$, then $A \simeq (Alg\,Q)(B)$ where B is the $(\mathcal{T}/\sim)$-algebra defined by $B[u] = A(u)$. This means that the factorization $E\colon Alg\,(\mathcal{T}/\sim) \to \mathcal{A}$ is essentially surjective. Since $Alg\,Q$ is full and faithful (10.9.2), E is an equivalence functor. ☐

10.11 Corollary. *Every equationally specified subcategory $\mathcal{A}$ of $Alg\,\mathcal{T}$*

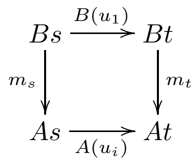
1. *is a* regular epireflective subcategory, *i.e., a reflective subcategory (with a reflector $R\colon Alg\,\mathcal{T} \to \mathcal{A}$ left adjoint to the inclusion $E\colon \mathcal{A} \to Alg\,\mathcal{T}$) with reflection maps $r(A)\colon A \to RA$ (i.e., components of the unit of the adjunction $R \dashv E$) regular epimorphisms,*

and

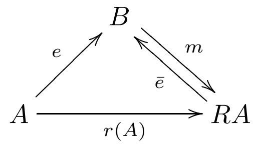
2. *is closed under sifted colimits (in particular, under directed unions), regular quotients, and subalgebras in $Alg\,\mathcal{T}$.*

Proof. (1) From 8.2 and 10.10 we see that $\mathcal{A}$ is a reflective subcategory closed under sifted colimits (and under directed unions, which are special filtered colimits).
(2) $\mathcal{A}$ is closed under subalgebras: given a monomorphism $m\colon B \to A$ with A in

$\mathcal{A}$, we prove that B is in $\mathcal{A}$ by verifying that every equation $u_1, u_2 \colon s \rightrightarrows t$ that A satisfies is also satisfied by B. We know from 1.8.2 that m_t is a monomorphism. From $A(u_1) = A(u_2)$ and the commutativity of



we conclude $B(u_1) = B(u_2)$.
(3) The reflection maps $r(A)$ are regular epimorphisms: let $r(A) = m \cdot e$



be a regular factorization of $r(A)$, see 3.4. By (2) $B \in \mathcal{A}$, thus there is a unique $\bar{e} \colon RA \to B$ such that $e = \bar{e} \cdot r(A)$. Since e is an epimorphism, we see that $\bar{e} \cdot m = \mathrm{id}_B$. Also $m \cdot \bar{e} = \mathrm{id}_{RA}$ due to the universal property of $r(A)$. Thus, m is an isomorphism and $r(A)$ a regular epimorphism.
(4) Let $e \colon A \to B$ be a regular epimorphism with A in $\mathcal{A}$. To prove that B is in $\mathcal{A}$, observe that a kernel pair $e_1, e_2 \colon N(e) \rightrightarrows A$ of e yields a subobject of $A \times A \in \mathcal{A}$, thus $N(e)$ is in $\mathcal{A}$. And since the pair e_1, e_2 is reflexive, B is a sifted colimit of the diagram $e_1, e_2 \colon N(e) \rightrightarrows A$ in $\mathcal{A}$, thus $B \in \mathcal{A}$. ☐

10.12 Theorem. *(Birkhoff's Variety Theorem) A full subcategory $\mathcal{A}$ of an algebraic category $Alg\,\mathcal{T}$ is equationally specified if and only if it is closed in $Alg\,\mathcal{T}$ under:*

(a) products: $\prod_{i \in I} A_i$ is in $\mathcal{A}$ if each A_i is in $\mathcal{A}$,

(b) subalgebras: given a monomorphism $m \colon B \to A$ with A in $\mathcal{A}$, then B also is in $\mathcal{A}$,

(c) regular quotients: given a regular epimorphism $e \colon A \to B$ with A in $\mathcal{A}$, then B also is in $\mathcal{A}$,

and

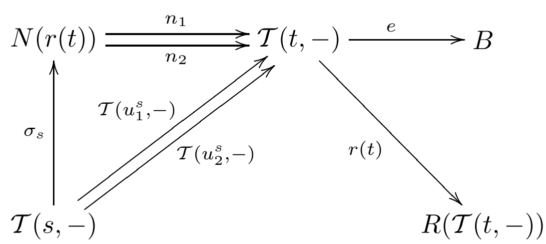
(d) directed unions: given a directed family of subalgebras $m_i \colon A_i \to A$ $(i \in I)$ with each A_i in $\mathcal{A}$, if A is the union, then A is in $\mathcal{A}$.

Proof. Every equationally specified subcategory is closed under (a)-(d): see 10.11. Conversely, let $\mathcal{A}$ be closed under (a)-(d).
(1) We observe that $\mathcal{A}$ is a regular epireflective subcategory. In fact, the embedding $E \colon \mathcal{A} \to Alg\,\mathcal{T}$ has a left adjoint because for every algebra B of $Alg\,\mathcal{T}$

we have a solution set consisting of all regular quotients $B \to A$ with $A \in \mathcal{A}$. By 3.3, this collection is small, and the regular factorization in $Alg\mathcal{T}$ together with $\mathcal{A}$ being closed under subalgebras proves that this is indeed a solution set for E. By the Adjoint Functor Theorem a reflection $r(A)\colon B \to RB$ of B in $\mathcal{A}$ exists. It is a regular epimorphism: see the proof of 10.11. (Explicitly, consider the product $\prod A$, indexed by all the regular epimorphisms $B \to A$ as above, and the induced morphism $b\colon B \to \prod A$. Then $r(B)\colon B \to RB$ is the regular epimorphic part part of the regular factorization of b.)
(2) We will prove that $\mathcal{A}$ is specified by the congruence $\sim$ defined, for every parallel pair $u_1, u_2\colon s \rightrightarrows t$ in $\mathcal{T}$, by

$$u_1 \underset{s,t}{\sim} u_2 \ \text{ iff } \ A(u_1) = A(u_2) \ \text{ for all } \ A \in \mathcal{A}\,.$$

This is indeed a congruence, see 10.4 and 10.5.2. It is our task to prove that every $\mathcal{T}$-algebra B such that $u_1 \underset{s,t}{\sim} u_2$ implies $B(u_1) = B(u_2)$ lies in $\mathcal{A}$.
(2a) Assume that B is a regular quotient of a representable algebra $\mathcal{T}(t,-)$. We thus have a regular epimorphism $e\colon \mathcal{T}(t,-) \to B$. Consider the diagram



where $N(r(t))$ is a kernel pair of the reflection morphism $r(t) = r(\mathcal{T}(t,-))$, and $(\sigma_s\colon \mathcal{T}(s,-) \to N(r(t)))$ exhibits $N(r(t))$ as a colimit of representable algebras (see 4.3). By Yoneda lemma, there exist $u_1^s, u_2^s\colon t \rightrightarrows s$ in $\mathcal{T}$ such that $n_i \cdot \sigma_s = \mathcal{T}(u_i^s,-)$ $(i = 1,2)$. Observe that $u_1^s \underset{t,s}{\sim} u_2^s$: every morphism $h\colon \mathcal{T}(t,-) \to A$ with $A \in \mathcal{A}$ factors through $r(t)$, so that $\mathcal{T}(u_1^s,-) \cdot h = \mathcal{T}(u_2^s,-) \cdot h$ and then $A(u_1^s) = A(u_2^s)$. By assumption on B, this implies $B(u_1^s) = B(u_2^s)$, and then $e \cdot n_1 \cdot \sigma_s = e \cdot n_2 \cdot \sigma_s$. Since the σ_s's are jointly epimorphic, we have $e \cdot n_1 = e \cdot n_2$. Since, by the first part of the proof, $r(t)$ is the coequalizer of its kernel pair, there is $f\colon R(\mathcal{T}(t,-)) \to B$ such that $f \cdot r(t) = e$. Finally, f is a regular epimorphism, because e is so, so that B, being a regular quotient of $R(\mathcal{T}(t,-))$, is in $\mathcal{A}$.
(2b) Let B be arbitrary. Express B as a colimit of representable algebras (4.3) and for each of the colimit morphism $\sigma_s\colon \mathcal{T}(s,-) \to B$ denote by B_s the image of σ_s which, by 3.4, is a subalgebra of B. Since $\mathcal{T}$ has finite products, the collection of these subalgebras of B is directed. By assumption (d) we only need to prove that B_s lies in $\mathcal{A}$. This follows from (2a): we know that B_s is a regular quotient of a representable algebra, and B_s has the desired property: given $u_1 \underset{s,t}{\sim} u_2$, we have $B(u_1) = B(u_2)$ and this implies $B_s(u_1) = B_s(u_2)$ since B_s is a subalgebra of B. □

We can prove the converse of 10.11.

10.13 Corollary. *Let $\mathcal{A}$ be a full subcategory of $Alg\,\mathcal{T}$. The following conditions are equivalent:*

1. *$\mathcal{A}$ is equationally specified;*
2. *$\mathcal{A}$ is a regular epireflective subcategory of $Alg\,\mathcal{T}$ closed under regular quotients and directed unions.*

Proof. The implication $1 \Rightarrow 2$ is 10.11. Conversely, following 10.12, we only need to prove that $\mathcal{A}$ is closed in $Alg\,\mathcal{T}$ under products (and this is obvious) and subalgebras: consider the diagram

$$\begin{array}{ccc} B & \xrightarrow{\ m\ } & A \\ {\scriptstyle r(B)}\downarrow & & \downarrow{\scriptstyle r(A)} \\ RB & \xrightarrow[R(m)]{} & RA \end{array}$$

where m is a monomorphism and A is in $\mathcal{A}$. Since A is in $\mathcal{A}$, $r(A)$ is an isomorphism and then $r(B)$ is a monomorphism. But $r(B)$ is also a regular epimorphism, so that it is an isomorphism and B is in $\mathcal{A}$. ☐

10.14 Example.

1. Let us consider the simplest two-sorted algebraic category $Set \times Set$ (no operations). This has an algebraic theory $\mathcal{T}_{\mathcal{C}}$ which is a free completion of the discrete category of objects s and t under finite products. Consider the subcategory $\mathcal{A}$ of $Set \times Set$ of all pairs $A = (A_s, A_t)$ with either $A_s = \emptyset$ or A_t has at most one element. This can be specified by the equations given by the parallel pair of the projections
$$s \times t \times t \rightrightarrows t\ .$$

2. Analogously, for $\mathbb{N}$-sorted sets we have the theory $\mathcal{T}_{\mathbb{N}}$, a free completion of $\mathbb{N}$ (considered as a discrete category) under finite products. Let $\mathcal{A}$ be the subcategory of $Set^{\mathbb{N}}$ of all $A = (A_n)_{n\in\mathbb{N}}$ such that either $A_n = \emptyset$ for all but finitely many indexes $n \in \mathbb{N}$, or A_n has at most one element for all $n \in \mathbb{N}$. This subcategory is closed under products, subalgebras and regular quotients – we omit the easy verification. However, it is not equationally specified, not being closed under directed unions. In fact, every $\mathbb{N}$-sorted set is a directed union of objects of $\mathcal{A}$.

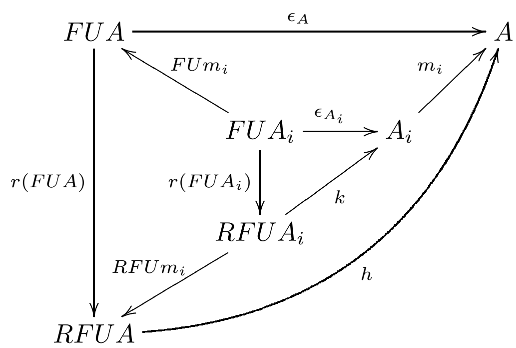
10.15 Remark.

1. For one-sorted algebraic categories the last condition, closure under directed unions, can be omitted in Birkhoff's Variety Theorem. In fact, let $\mathcal{T}$ be a one-sorted theory with the adjunction
$$Alg\,\mathcal{T} \underset{U}{\overset{F}{\rightleftarrows}} Set \qquad F \dashv U$$

as in 9.4.2. Let $\mathcal{A}$ be a regular epireflective subcategory of $Alg\,\mathcal{T}$ closed under subalgebras and regular quotients. Then for every directed union $A = \cup_{i\in I} A_i$ of subobjects $m_i\colon A_i \to A$ with $A_i \in \mathcal{A}$ for each i, we prove that A is in $\mathcal{A}$. We know that the counit $\epsilon_A\colon FUA \to A$ is a regular epimorphism (see the proof of 9.16), and we prove that it factorizes through $r(FUA)$ – the factorizing morphism $h\colon RFUA \to A$ is then a regular quotient, proving $A \in \mathcal{A}$. Since U preserves filtered colimits (9.6) and F, being a left adjoint, too, we see that FUA is a filtered colimit of FUA_i ($i \in I$) with the colimit cocone FUm_i ($i \in I$). We can assume $UA \neq \emptyset$ (if $UA = \emptyset$, then A is a subalgebra of the terminal algebra, and the $A \in \mathcal{A}$). For arbitrary elements $x, y \in FUA$ we prove

$$(*) \qquad \text{if } Ur(FUA)(x) = Ur(FUA)(y) \text{ then } U\epsilon_A(x) = U\epsilon_A(y)\,.$$

Since I is directed, there exists $i \in I$ such that x and y lie in the image of $UFUm_i$, say, $x = UFUm_i(x')$ and $y = UFUm_i(y')$. Since $A_i \in \mathcal{A}$ we have an homomorphism $k\colon RFUA_i \to A_i$ with $\epsilon_{A_i} = k \cdot r(FUA_i)$



Since $Ur(FUA)$ merges x and y, we see that $URFUm_i \cdot Ur(FUA_i)$ merges x' and y'. The crucial point is that Um_i can be considered to be a split monomorphism. In fact, this is true whenever $UA_i \neq \emptyset$, and we can assume that because I is directed and $UA \neq \emptyset$. Therefore, $URFUm_i$ is a monomorphism which implies that $Ur(UFA_i)$ merges x' and y'. Consequently, $U\epsilon_{A_i}$ merges them, too, and this implies that $U\epsilon_A$ merges x and y, as requested.

From $(*)$ it follows that $U\epsilon_A$ factorizes in *Set* through $Ur(FUA)$. It follows that ϵ_A factorizes in $Alg\,\mathcal{T}$ through $r(FUA)$ – the naturality conditions for the factorization map h follow from those of ϵ_A since $Ur(FUA)$ is an epimorphism.

2. A completely analogous argument to point 1. holds for S-sorted algebraic categories where S is a finite set. Here also Um_i can be considered as split monomorphisms since I is directed and S is finite.

3. For S infinite, Example 10.14.2 demonstrates that the condition of closedness under directed unions cannot be omitted. Surprisingly, in a number

of publications the simplified version, without that condition, is stated. Usually, "equations" means pairs of terms (u, v) in free many-sorted algebras on specified many-sorted sets X of generators. However: in case these sets X are understood to have finitely many elements, the subcategory $\mathcal{A}$ of 10.14.2 cannot be specified by such equations. If X is allowed to have infinitely many elements, then the formal equation "$u = v$" gets an infinite number of quantifiers and we thus left the realm of finitary logic.

Chapter 11

Many-sorted signatures

In the previous chapters, we have seen how equational specifications in algebraic categories are obtained from congruences on algebraic theories. In general algebra one usually starts from an "S-sorted signature" Σ and works with congruences on its algebraic theory $\mathcal{T}_\Sigma$ formed by terms (11.5). This chapter is devoted to a brief comparison of our approach with the usual one: we will find, for every S-sorted algebraic theory $\mathcal{T}$, an S-sorted signature Σ and a congruence $\sim$ on its algebraic theory $\mathcal{T}_\Sigma$ with $\mathcal{T}$ equivalent to $\mathcal{T}_\Sigma/\sim$.

Recall that, for a given set S, we denote by S^* the set of all words on S, including the empty word ϵ.

11.1 Definition.

1. By an *S-sorted signature* Σ is meant a set Σ together with a function

   $$\mathrm{ar}\colon \Sigma \to S^* \times S$$

   called arity.

2. A *Σ-algebra* $(A, \sigma^A)_{\sigma\in\Sigma}$ is an S-sorted set $\langle A_s\rangle_{s\in S}$ together with operations

   $$\sigma^A\colon A_{s_1} \times \ldots \times A_{s_n} \to A_s$$

   for every $\sigma \in \Sigma$ of arity $\mathrm{ar}(\sigma) = (s_1 \ldots s_n, s)$.

3. Given Σ-algebras (A, σ^A) and (B, σ^B), an *homomorphism* $h\colon A \to B$ is an S-sorted function $h = \langle h_s\rangle_{s\in S}$ such that for each $\sigma \in \Sigma$ of arity $(s_1 \ldots s_n, s)$ the square

   $$\begin{array}{ccc} A_{s_1} \times \ldots \times A_{s_n} & \xrightarrow{\ \sigma^A\ } & A_s \\ {\scriptstyle h_{s_1}\times\ldots\times h_{s_n}}\big\downarrow & & \big\downarrow{\scriptstyle h_s} \\ B_{s_1} \times \ldots \times B_{s_n} & \xrightarrow[\ \sigma^B\ ]{} & B_s \end{array}$$

   commutes

11.2 Remark. If $S = 1$, a one-sorted signature is nothing but a set Σ together with an arity function

$$\mathrm{ar}\colon \Sigma \to \mathbb{N}\,.$$

A Σ-algebra $(A, \sigma^A)_{\sigma \in \Sigma}$ is a set A together with operations $\sigma^A\colon A^n \to A$ for every $\sigma \in \Sigma$ of arity $\mathrm{ar}(\sigma) = n$.

11.3 Notation. For every S-sorted signature Σ we denote by ΣAlg the category of Σ-algebras and homomorphisms. This is a concrete category over Set^S with the canonical forgetful functor

$$U_\Sigma\colon \Sigma Alg \to Set^S\,, \quad (A, \sigma^A) \mapsto A\,.$$

11.4 Proposition. *For every S-sorted signature Σ, the category ΣAlg is an S-sorted algebraic category.*

Proof. It is our task to prove that ΣAlg is cocomplete and that U_Σ is conservative, has a left adjoint and preserves sifted colimits.
(1) The functor U_Σ creates limits in the following sense: given a diagram $D\colon \mathcal{D} \to \Sigma Alg$ with a limit

$$h_d\colon A \to U_\Sigma Dd\,, \quad d \in obj\mathcal{D}$$

of $U_\Sigma \cdot D$ in Set^S, there is a unique structure of a Σ-algebra on A making each h_d an homomorphism: the following square

$$\begin{array}{ccc} A_{s_1} \times \ldots \times A_{s_n} & \xrightarrow{\ \sigma^A\ } & A_s \\ {\scriptstyle (h_d)_{s_1} \times \ldots \times (h_d)_{s_n}} \downarrow & & \downarrow {\scriptstyle (h_d)_s} \\ (Dd)_{s_1} \times \ldots \times (Dd)_{s_n} & \xrightarrow[\ \sigma^{Dd}\ ]{} & (Dd)_s \end{array}$$

determine (for $d \in obj\mathcal{D}$) the function σ^A uniquely due to the universal property of limits. It is easy to see that

$$h_d\colon (A, \sigma^A) \to Dd\,, \quad d \in obj\mathcal{D}$$

is a limit of D in ΣAlg.
(2) The functor U_Σ created sifted colimits in the following sense: given a sifted diagram $D\colon \mathcal{D} \to \Sigma Alg$ with a colimit

$$h_d\colon U_\Sigma Dd \to A\,, \quad d \in obj\mathcal{D}$$

of $U_\Sigma \cdot D$ in Set^S, there is a unique strucutre of a Σ-algebra on A making each h_d an homomorphism: since $A_{s_1} \times \ldots \times A_{s_n}$ is a colimit of $D_{s_1} \times \ldots \times D_{s_n}$ (where, given $s \in S$, we denote by $D_s\colon \mathcal{D} \to Set$ the composite of D with the s-projection of Set^S) the following squares

$$\begin{array}{ccc} (Dd)_{s_1} \times \ldots \times (Dd)_{s_n} & \xrightarrow{\ \sigma^{Dd}\ } & (Dd)_s \\ {\scriptstyle (h_d)_{s_1} \times \ldots \times (h_d)_{s_n}} \downarrow & & \downarrow {\scriptstyle (h_d)_s} \\ A_{s_1} \times \ldots \times A_{s_n} & \xrightarrow[\ \sigma^{A}\ ]{} & A_s \end{array}$$

determine (for $d \in obj\,\mathcal{D}$) the function σ^A uniquely. It is easy to see that

$$h_d\colon Dd \to (A, \sigma^A)\ , \quad d \in obj\,\mathcal{D}$$

is a colimit of D in ΣAlg.
(3) Due to (1) we see that ΣAlg is a complete category and U_Σ preserves limits. Moreover, U_Σ fulfils the solution-set condition: for every X in Set^S those arrows $f\colon X \to U_\Sigma(A, \sigma^A)$ such that no proper subalgebra of (A, σ^A) contains the image of f form a solution set (due to (1) intersections of subalgebras are formed on the level of sets). Consequently, U_Σ is a right adjoint. Also, due to (1), U_Σ is conservative.
(4) Due to (2), ΣAlg has sifted colimits and U_Σ preserves them. It remains to prove that ΣAlg has finite coproducts (as remarked at the beginning of Chapter 4, this implies cocompleteness). Thus, for every finite set I we are to show that the diagonal functor $\Delta\colon \Sigma Alg \to (\Sigma Alg)^I$ is a right adjoint. In fact, the verification of the solution-set condition for Δ is analogous to that for U_Σ above. □

11.5 Remark.

1. For a one-sorted signature Σ, the forgetful functor of ΣAlg has as left adjoint the functor

   $$F_\Sigma\colon Set \to \Sigma Alg$$

   assigning to every set X (of variables) the Σ-*term-algebra* $F_\Sigma X$. This is the smallest set

   - containing X, and
   - containing, for every n-ary symbol $\sigma \in \Sigma$, the term $\sigma(t_1, \ldots, t_n)$ for every n-tuple of terms $(t_1, \ldots, t_n) \in (F_\Sigma X)^n$.

   The operations are defined by the formal formation of terms $\sigma(t_1, \ldots, t_n)$, and the universal arrow $\eta_X\colon X \to F_\Sigma X$ is the embedding of X. In fact, F_Σ is easily seen to be a well-defined functor assigning to every function $f\colon X \to Z$ the homomorphism $F_\Sigma f\colon F_\Sigma X \to F_\Sigma Z$ which takes a term and replaces every occurence of a variable $x \in X$ by $f(x)$. The fact that F_Σ is left adjoint to U_Σ follows from the usual computation of terms in a Σ-algebra A: let $h\colon X \to U_\Sigma A$ be a function (interpreting all variables as elements of A). The unique homomorphism $h'\colon F_\Sigma X \to A$ with $h' \cdot \eta_X = h$ is given, necessarily, by $h'(\sigma(t_1, \ldots, t_n)) = \sigma^A(h(t_1), \ldots, h(t_n))$.

2. For every finite set X, the algebra $F_\Sigma X$ is projectively finitely presentable. This follows from 11.4 and the fact that U_Σ preserves sifted colimits. Moreover, these free algebras form a strong generator of ΣAlg: apply 6.12.1 to the adjunction $F_\Sigma \dashv U_\Sigma$.

3. It follows from 6.8 that, for a one-sorted signature Σ, the algebraic category ΣAlg has the following one-sorted algebraic theory $\mathcal{T}_\Sigma$: objects are the natural numbers, morphisms in $\mathcal{T}_\Sigma(n, 1)$ are the terms on $X = n$.

4. Analogously for S-sorted signatures: here

$$F_\Sigma \colon \mathit{Set}^S \to \Sigma \mathit{Alg}$$

assigns to every S-sorted set $X = \langle X_s \rangle_{s \in S}$ (of variables) the Σ-term-algebra of S-sorted terms. This is the smallest S-set such that

- every element $x \in X_s$ is a term of sort s, and
- for every symbol σ of arity $(s_1 \dots s_n, s)$ and for every n-tuple of terms $(t_1, \dots, t_n) \in (F_\Sigma X)^n$ with t_i of sort s_i, we have a term $\sigma(t_1, \dots, t_n)$ of sort s in $F_\Sigma X$.

This leads to an S-sorted algebraic theory $\mathcal{T}_\Sigma$ whose objects are all finite words in S and the morphisms in $\mathcal{T}_\Sigma(s_1, \dots s_n, s)$, for words $s_1 \dots s_n$ and s (a singleton), are the Σ-terms on $X = \{x_1, \dots, x_n\}$, where the x_i are pairwise distinct variables of sort s_i $(i = 1, \dots, n)$.

11.6 Remark.

1. Our definition of an S-sorted signature leads to the following concept of the *category of S-sorted signatures* as the slice category

$$\mathit{Sign}_S = \mathit{Set} \downarrow (S^* \times S) .$$

2. Consider the category Th_S of S-sorted theories (that is, forget the 2-cells) with morphisms

$$\begin{array}{ccc} & S & \\ {}^{\sigma_1}\swarrow & \simeq \varphi & \searrow^{\sigma_2} \\ \mathcal{T}_1 & \xrightarrow[H]{} & \mathcal{T}_2 \end{array}$$

given by pairs $(H \colon \mathcal{T}_1 \to \mathcal{T}_2, \varphi \colon H \cdot \sigma_1 \simeq \sigma_2)$ as in 9.23. There is a functor $\Lambda \colon \mathit{Th}_S \to \mathit{Sign}_S$ defined on objects by

$$\Lambda(\mathcal{T}, \sigma) = \coprod_{(s_1 \dots s_n, s) \in S^* \times S} \{\sigma(s_1) \times \dots \times \sigma(s_n) \xrightarrow{u} \sigma(s)\} \xrightarrow{\tau} S^* \times S$$

with $\tau(u) = (s_1 \dots s_n, s)$, and defined on morphisms in the obvious way.

3. The functor $\Lambda \colon \mathit{Th}_S \to \mathit{Sign}_S$ has a left adjoint $\mathcal{J} \colon \mathit{Sign}_S \to \mathit{Th}_S$ assigning to every S-sorted signature Σ the algebraic theory $\mathcal{T}_\Sigma$. Moreover, $\Sigma \mathit{Alg}$ and $\mathit{Alg}(\mathcal{J}(\Sigma))$ are equivalent categories, see 11.5.4.

11.7 Corollary. *Let $(\mathcal{T}, \sigma)$ be an S-sorted algebraic theory. Consider the counit $\varepsilon = \varepsilon_{(\mathcal{T}, \sigma)} \colon \mathcal{J}(\Lambda(\mathcal{T}, \sigma)) \to (\mathcal{T}, \sigma)$ of the adjunction $\mathcal{J} \dashv \Lambda$, and its factorization*

$$\begin{array}{ccc} \mathcal{J}(\Lambda(\mathcal{T}, \sigma)) & \xrightarrow{\qquad\qquad} & \mathcal{J}(\Lambda(\mathcal{T}, \sigma)) / \underset{\varepsilon}{\sim} \\ {}_{\varepsilon}\searrow & & \swarrow_{\varepsilon'} \\ & (\mathcal{T}, \sigma) & \end{array}$$

as in 10.9.4. The morphism ε' is an equivalence of S-sorted algebraic theories.

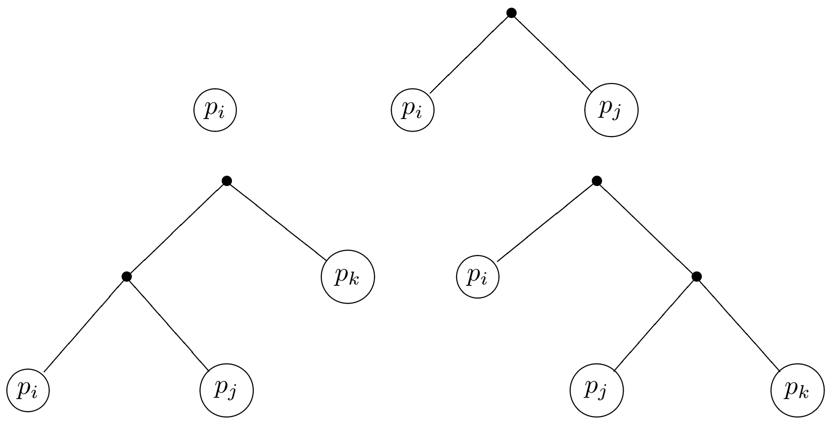
Proof. This follows from 10.9.4 and the previous discussion. □

11.8 Corollary. *Let $(\mathcal{T}, \sigma)$ be an S-sorted algebraic theory. The algebraic category $Alg\,\mathcal{T}$ is an equationally specified subcategory of $Alg\,\mathcal{J}(\Lambda(\mathcal{T}, \sigma))$.*

Proof. This follows from 10.10 and 11.7. □

11.9 Example.

1. Let $\mathcal{T}_2$ denote a one-sorted theory freely generated by a single binary operation $*$. That is, the morphisms of $\mathcal{T}_2$ are freely generated by $*\colon 2 \to 1$ and projections $p_1, \ldots, p_n \colon n \to 1$ making n a product of n copies of 1 (for every $n \in \mathbb{N}$). We have a direct description of $\mathcal{T}_2$ as follows: a morphism $u\colon n \to 1$ in $\mathcal{T}_2$ is a finite binary tree where leaves are labelled $p_1, \ldots, p_n$. Examples:

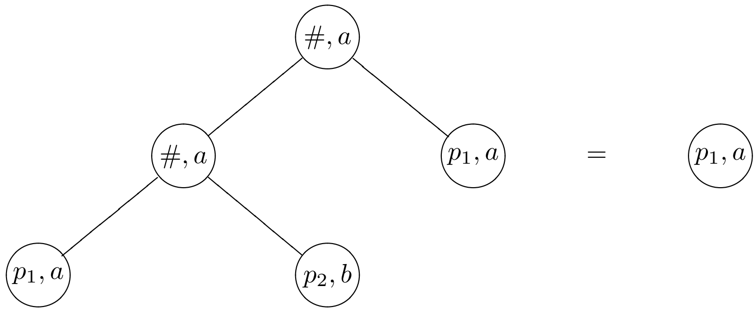
   A morphism $u\colon n \to k$ is a k-tuple of such binary trees. Composition is defined by tree-tupling: given $(u_1, \ldots, u_k)\colon n \to k$ and $(v_1, \ldots, v_r)\colon k \to r$ then their composition is the r-tuple $(w_1, \ldots, w_r)\colon n \to r$ of binary trees where w_i is the tree v_i in which every leaf of label p_1 is substituted by the subtree u_1, every leaf of label p_2 is substituted by u_2, etc. Consequently, the identity morphism of n is the n-tuple of trees

   $i = 1, \ldots, n.$

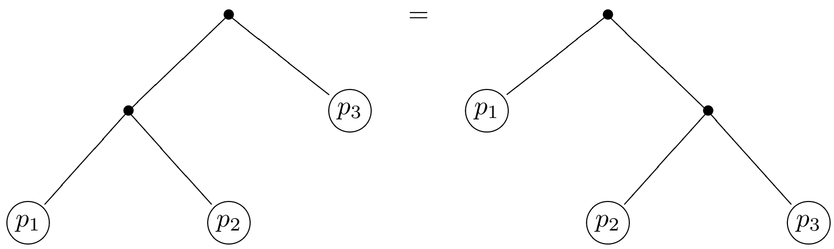
2. Analogously for many-sorted theories: here the labels of the nodes must also contains the sorts. For example, let $\mathcal{T}$ be the two-sorted theory freely generated by a single binary operation of arity (ab, a), that is, the first variable and the result have sort a and the second variable has sort b. The algebras are given by a pair of sets (X_a, X_b) and an operation

$\#\colon X_a \times X_b \to X_a$. The equation



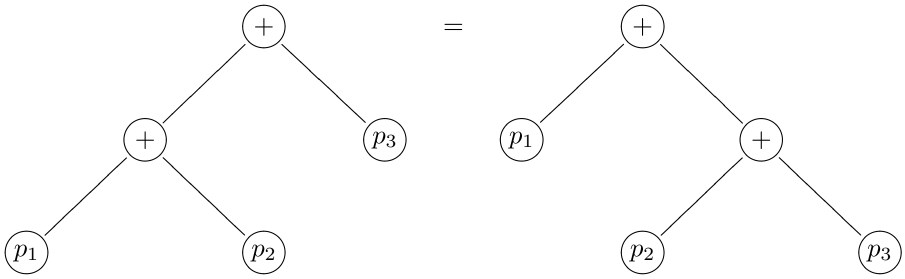
specifies those algebras in which $\#(\#(x, y), x) = x$ holds.

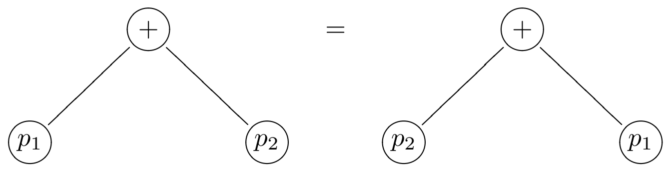
3. Recall that a *semigroup* is an algebra of one associative binary operation. This means that we consider the algebras of $\mathcal{T}_2$ which satisfy the equation

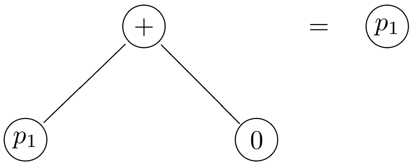


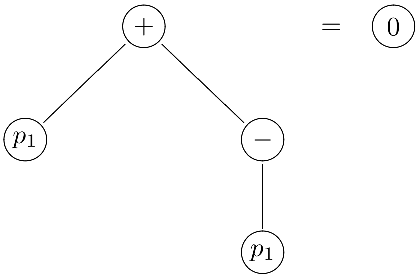
Thus, the theory of semigroups is the quotient theory $\mathcal{T}_2/\sim$ where $\sim$ is the congruence generated by the equation above.

4. Beside the algebraic theory $\mathcal{T}_{ab}$ of abelian groups of 1.3.2 we now have a different one, based on the usual equational presentation: let $\Sigma = \{+, -, 0\}$ with $+$ binary, $-$ unary and 0 nullary. Let $\sim$ be the congruence on $\mathcal{T}_\Sigma$ generated by the four equations



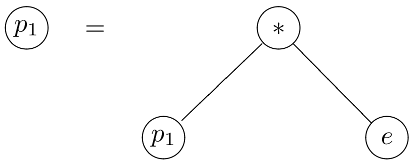


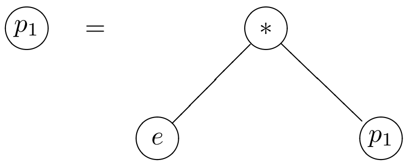




Then $\mathcal{T}_\Sigma/\sim$ is an algebraic theory of abelian groups.

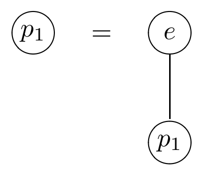
5. Recall that a *monoid* is a semigroup $(R, *)$ with a unit. We can consider the category of all monoids as the category $Alg\,(\mathcal{T}_\Sigma/\sim)$ where Σ has a binary symbol $*$ and a nullary symbol e, and $\sim$ is the congruence generated by associativity of $*$ and the equations

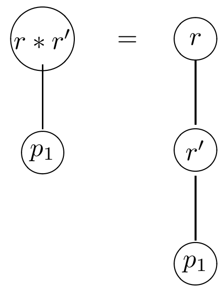




6. For every monoid R, an *R-set* is a pair (X, α) consisting of a set X and a monoid action $\alpha\colon R \times X \to X$ (the usual notation is rx in place of $\alpha(r, x)$) such that every element $x \in X$ satisfies $r(r'x) = (r * r')x$ for all $r, r' \in R$, and $ex = x$. The homomorphisms $f\colon (X, \alpha) \to (Z, \beta)$ of R-sets are the functions $f\colon X \to Z$ with $f(rx) = rf(x)$ for all $r \in R$ and $x \in X$. We can

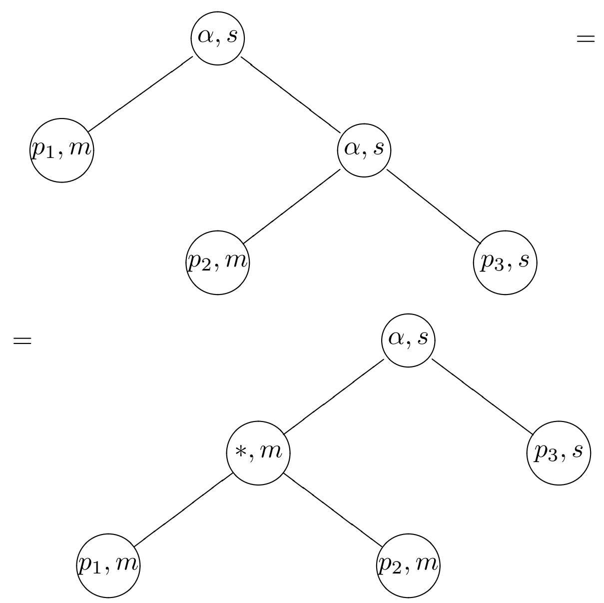
describe this category as $Alg(\mathcal{T}_\Sigma/\sim)$ where $\Sigma = R$ with all arities equal to 1, and $\sim$ is the congruence generated by the equations





for all $r, r' \in R$.

7. R-sets can also be constructed as two-sorted algebras of sorts m (monoid) and s (set). It is easy to see how all monoid equations and all equations for R-sets now become two-sorted equations. For example instead of $r(r'x) = (r * r')x$ we now have



Chapter 12

Morita equivalence

In this chapter we study the problem of different algebraic theories of a given algebraic category. This is inspired by the classical work of Kiiti Morita who in 1950's studied this problem for the categories *R-Mod* of left modules over a ring R. He completely characterized pairs of rings R and S such that *R-Mod* and *S-Mod* are equivalent categories; such rings are nowadays called *Morita equivalent.* We will recall the results of Morita below, and we will show in which way they generalize from *R-Mod* to $Alg\,\mathcal{T}$ where $\mathcal{T}$ is an algebraic theory.

We begin with a particularly simple example.

12.1 Example. In 1.3.1, we described a one-sorted algebraic theory $\mathcal{T}_1$ of Set : $\mathcal{T}_1$ is the full subcategory of Set^{op} whose objects are the natural numbers. Here is another one-sorted theory of Set : $\mathcal{T}_2$ is the full subcategory of Set^{op} whose objects are the even natural numbers $0, 2, 4, 6, \ldots$. $\mathcal{T}_2$ obviously has finite products. Observe that $\mathcal{T}_2$ is not idempotent-complete (consider the constant function $2 \to 2$) and that $\mathcal{T}_1$ is an idempotent completion of $\mathcal{T}_2$: for every natural number n we can find an idempotent function $f\colon 2n \to 2n$ with precisely n fixed points. Then n is obtained by splitting f. Following 7.10, $Alg\,\mathcal{T}_2 \simeq Alg\,\mathcal{T}_1 \simeq Set$. In fact, we can repeat the previous argument for every natural number $k > 0$. In this way we get a family $\mathcal{T}_k$, $k = 1, 2, \ldots$ of one-sorted algebraic theory of *Set.* We will prove later that, up to equivalence, there is no other one-sorted algebraic theory of *Set.*

Clearly, if $\mathcal{T}$ and $\mathcal{T}'$ are algebraic theories and if there is an equivalence $\mathcal{T} \simeq \mathcal{T}'$, then $Alg\,\mathcal{T}$ and $Alg\,\mathcal{T}'$ are equivalent categories. The previous example shows that the converse is not true. It is therefore sensible to give the following definition.

12.2 Definition. Two algebraic theories $\mathcal{T}$ and $\mathcal{T}'$ are called *Morita equivalent* if the corresponding categories $Alg\,\mathcal{T}$ and $Alg\,\mathcal{T}'$ are equivalent.

From 7.10 we already know a simple characterization of Morita equivalent algebraic theories: two theories are Morita equivalent iff they have equivalent idempotent completions. In case of S-sorted algebraic categories a much sharper result can be proved. Before doing so, let us recall the classical result of Morita.

12.3 Example. Let R be a unitary ring (not necessarily commutative) and write R-*Mod* for the category of left R-modules. There are two basic constructions:

1. Matrix ring $R^{[k]}$. This is the ring of all $k \times k$ matrices over R with the ususal addition, multiplication, and unit matrix. This ring $R^{[k]}$ is Morita equivalent to R for every $k > 0$, i.e., the category $R^{[k]}$-*Mod* is equivalent to R-*Mod*.

2. Idempotent modification uRu. Let u be an idempotent element of R, $uu = u$, and let uRu be the ring of all elements of the form uxu (i.e., all elements $x \in R$ with $x = uxu$) with the binary operation inherited from R and the neutral element u. This ring is Morita equivalent to R whenever u is pseudoinvertible, i.e., $eum = 1$ for some elements e and m of R.

Morita's original result is that the two operations above are sufficient: if a ring S is Morita equivalent to R, i.e., R-*Mod* and S-*Mod* are equivalent, then S is isomorphic to the ring $uR^{[k]}u$ for some pseudoinvertible idempotent $k \times k$ matrix u.

We generalize now Morita constructions to one-sorted algebraic theories. For one-sorted algebraic theories we take $\mathbb{N}$ as the set of objects, see 9.4.1.

12.4 Definition. Let $\mathcal{T}$ be a one-sorted algebraic theory.

1. The *matrix theory* $\mathcal{T}^{[k]}$, for $k = 1, 2, 3, \ldots$ is the one-sorted algebraic theory whose morphisms $f \colon p \to q$ are precisely the morphisms $f \colon kp \to kq$ of $\mathcal{T}$; composition and identity morphisms are defined as in $\mathcal{T}$.

2. Let $u \colon 1 \to 1$ be an idempotent morphism ($u \cdot u = u$). We call u *pseudoinvertible* provided that there exist morphisms $m \colon 1 \to n$ and $e \colon n \to 1$ such that $e \cdot u^n \cdot m = \mathrm{id}_1$. The idempotent modification of $\mathcal{T}$ is the theory $u\mathcal{T}u$ whose morphisms $f \colon p \to q$ are precisely the morphisms of $\mathcal{T}$ satisfying $f \cdot u^p = f = u^q \cdot f$. The composition is defined as in $\mathcal{T}$, the identity morphism on p is u^p.

12.5 Remark.

1. Both $\mathcal{T}^{[k]}$ and $u\mathcal{T}u$ are well defined. In fact, $\mathcal{T}^{[k]}$ has finite products with $p = 1 \times \ldots \times 1$: the i-th projection is obtained from the i-th projection in $\mathcal{T}$ of $kp = k \times \ldots \times k$. Also $u\mathcal{T}u$ has finite products with $p = 1 \times \ldots \times 1$: the i-projection $\pi_i \colon p \to 1$ of $\mathcal{T}$ yields a morphism $u \cdot \pi_i \colon p \to 1$ of $u\mathcal{T}u$ ($i = 1, \ldots, k$) and these morphisms form a product $p = 1 \times \ldots \times 1$ in $u\mathcal{T}u$.

2. Idempotent modifications are much more "concrete" than (the equality of) idempotent completions. This will be seen on examples illustrating Morita equivalence below.

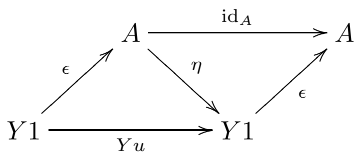
12.6 Theorem. *Let $\mathcal{T}$ be a one-sorted algebraic theory. Then:*

1. *the matrix theories $\mathcal{T}^{[k]}$ are Morita equivalent to $\mathcal{T}$ for all $k > 0$, and*
2. *the idempotent modifications $u\mathcal{T}u$ are Morita equivalent to $\mathcal{T}$ for all pseudoinvertible idempotent u.*

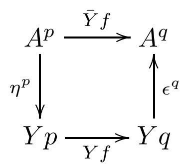
Proof. 1: Matrix theory $\mathcal{T}^{[k]}$. We have a full and faithful functor $\mathcal{T}^{[k]} \to \mathcal{T}$ defined on objects by $n \mapsto nk$ and on morphisms as the identity mapping. Every objects of $\mathcal{T}$ is a retract of an object coming from $\mathcal{T}^{[k]}$: in fact, for every n consider the diagonal morphism $\Delta\colon n \to nk = n \times \ldots \times n$. Consequently, $\mathcal{T}$ and $\mathcal{T}^{[k]}$ have the same idempotent completion. Thus, by 7.10, they are Morita equivalent.
2: Idempotent modification $u\mathcal{T}u$. Here we consider $\mathcal{T}$ as a full subcategory of $(Alg\,\mathcal{T})^{op}$ via the Yoneda embedding (1.4)

$$Y\colon \mathcal{T} \to (Alg\,\mathcal{T})^{op}\,, \quad t \mapsto \hom(t,-)\,.$$

Following 7.3.1, the idempotent $Yu\colon Y1 \to Y1$ has a splitting in $(Alg\,\mathcal{T})^{op}$, say



Consider also the subcategory $\mathcal{T}_A$ of $(Alg\,\mathcal{T})^{op}$ of all powers A^n, $n \in \mathbb{N}$. $\mathcal{T}_A$ is a one-sorted algebraic theory, and it is Morita equivalent to $\mathcal{T}$. In fact, every object of $\mathcal{T}$ is a retract of one in $\mathcal{T}_A$ and vice-versa – this clearly implies $\mathcal{T}$ and $\mathcal{T}_A$ have a joint idempotent completion (obtained by splitting their idempotents in $(Alg\,\mathcal{T})^{op}$). Indeed, since A is a retract of $Y1$, A^p is a retract of Yp. Conversely, consider $m\colon 1 \to n$ and $e\colon n \to 1$ in $\mathcal{T}$ such that $e \cdot u^n \cdot m = \mathrm{id}_1$ as in 12.4.2. Then $Y1$ is a retract of A^n via $\epsilon^n \cdot Y(m)\colon Y1 \to A^n$ and $Y(e) \cdot \eta^n\colon A^n \to Y1$, and then Yp is a retract of A^{np}.
To comlete the proof, we construct an equivalence functor $\bar{Y}\colon u\mathcal{T}u \to \mathcal{T}_1$. On objects, it is defined by $\bar{Y}(p) = A^p$, and on morphisms $f\colon p \to q$ by



in $(Alg\,\mathcal{T})^{op}$. $\bar{Y}(\mathrm{id}_p) = \mathrm{id}_{A^p}$ because $\epsilon \cdot \eta = \mathrm{id}_A$. Now, we check the equation

$$(*) \qquad\qquad Y(f) = \eta^q \cdot \bar{Y}(f) \cdot \epsilon^p\,.$$

Indeed:

$$Y(f) = Y(u)^q \cdot Y(f) \cdot Y(u)^p = \eta^q \cdot \epsilon^q \cdot Y(f) \cdot \eta^p \cdot \epsilon^p = \eta^q \cdot \bar{Y}(f) \cdot \epsilon^p\,.$$

From equation $(*)$, since ϵ^p is a (split) epimorphism and η^q is is a (split) monomorphism, we deduce that $\bar{Y}$ preserves composition (because Y preserves composition), and that $\bar{Y}$ is faithful (because Y is faithful). Since $\bar{Y}$ is surjective on objects, it remains to show that it is full: consider $h\colon A^p \to A^q$ in $(Alg\,\mathcal{T})^{op}$, we get $k = \eta^q \cdot h \cdot \epsilon^p \colon Y(p) \to Y(q)$. Since Y is full, there is a (unique) $f\colon p \to q$ in $\mathcal{T}$ such that $Y(f) = k$. Now:

$$\bar{Y}(f) = \epsilon^q \cdot Y(f) \cdot \eta^p = \epsilon^q \cdot \eta^q \cdot h \cdot \epsilon^p \cdot \eta^p = h\,.$$

It remains to check that f is in $u\mathcal{T}u$:

$$Y(f) \cdot Y(u^p) = \eta^q \cdot h \cdot \epsilon^p \cdot \eta^p \cdot \epsilon^p = \eta^q \cdot h \cdot \epsilon^p = k = Y(f)$$

and then $f \cdot u^p = f$ because Y is faithful; analogously, $u^q \cdot f = f$. □

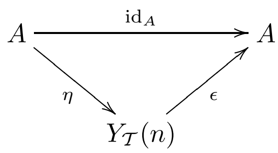
12.7 Theorem. *For two one-sorted algebraic theories $\mathcal{T}$ and $\mathcal{S}$ the following statements are equivalent:*

1. *$\mathcal{S}$ is Morita equivalent to $\mathcal{T}$;*
2. *$\mathcal{S}$ is equivalent, as a category, to an idempotent modification $u\mathcal{T}^{[k]}u$ of a matrix theory of $\mathcal{T}$, for some pseudoinvertible idempotent u of $\mathcal{T}^{[k]}$.*

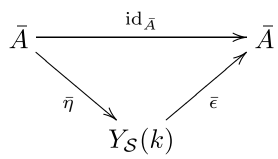
Proof. Consider an equivalence functor

$$E\colon Alg\,\mathcal{S} \to Alg\,\mathcal{T}$$

and the embeddings $Y_{\mathcal{S}}\colon \mathcal{S}^{op} \to Alg\,\mathcal{S}$, $Y_{\mathcal{T}}\colon \mathcal{T}^{op} \to Alg\,\mathcal{T}$ (recall, from 4.1, that $Y_{\mathcal{S}}$ and $Y_{\mathcal{T}}$ preserve finite coproducts). Since $Y_{\mathcal{S}}(1)$ is projectively finitely presentable in $Alg\,\mathcal{S}$ (5.5.4), then $A = E(Y_{\mathcal{S}}(1))$ is projectively finitely presentable in $Alg\,\mathcal{T}$, and then (5.11) it is a retract of $Y_{\mathcal{T}}(n)$ for some n in $\mathbb{N}$, say



There is a unique $u\colon n \to n$ in $\mathcal{T}$ such that $Y_{\mathcal{T}}(u) = \eta \cdot \epsilon$, and such a u is an idempotent. We consider u as an idempotent on 1 in $\mathcal{T}^{[n]}$ and prove that u is pseudoinvertible there. For this, chose an $\mathcal{S}$-algebra $\bar{A}$ and an isomorphism $i\colon Y_{\mathcal{T}}(n) \to E\bar{A}$. Since E is an equivalence, $\bar{A}$ is projectively finitely presentable, thus it is a retract of $Y_{\mathcal{S}}(k)$ for some $k \in \mathbb{N}$, say



Consider now the composites

$$Y_{\mathcal{T}}(n) \xrightarrow{i} E\bar{A} \xrightarrow{E\bar{\eta}} EY_{\mathcal{S}}(k) \simeq kA \xrightarrow{k\eta} kY_{\mathcal{T}}(n) \simeq Y_{\mathcal{T}}(nk)$$

$$Y_{\mathcal{T}}(nk) \simeq kY_{\mathcal{T}}(n) \xrightarrow{k\epsilon} kA \simeq EY_{\mathcal{S}}(k) \xrightarrow{E\bar{\epsilon}} E\bar{A} \xrightarrow{i^{-1}} Y_{\mathcal{T}}(n)$$

so that there exist unique morphisms $e\colon nk \to n$ and $m\colon n \to nk$ in $\mathcal{T}$ which $Y_{\mathcal{T}}$ maps on the above composites. One immediately checks that $Y_{\mathcal{T}}(e \cdot u^k \cdot m) = \mathrm{id}$, that is $e \cdot u^k \cdot m = \mathrm{id}$ and u is pseudoinvertible in $\mathcal{T}^{[n]}$.
To complete the proof, we construct an equivalence functor $\bar{E}\colon \mathcal{S} \to u\mathcal{T}^{[n]}u$. It is the identity on objects. If $f\colon p \to q$ is a morphism in $\mathcal{S}$, $\bar{E}(f)$ is the unique morphism $np \to nq$ in $\mathcal{T}$ such that

$$\begin{array}{ccc} qY_{\mathcal{T}}(n) \simeq Y_{\mathcal{T}}(nq) & \xrightarrow{Y_{\mathcal{T}}(\bar{E}(f))} & Y_{\mathcal{T}}(np) \simeq pY_{\mathcal{T}}(n) \\ {\scriptstyle q\epsilon}\downarrow & & \uparrow{\scriptstyle p\eta} \\ qA \simeq E(Y_{\mathcal{S}}(q)) & \xrightarrow[E(Y_{\mathcal{S}}(f))]{} & E(Y_{\mathcal{S}}(p)) \simeq pA \end{array}$$

commutes. Using once again $Y_{\mathcal{T}}(u) = \eta \cdot \epsilon$ and the faithfulness of $Y_{\mathcal{T}}$, one easily checks that $u^p \cdot \bar{E}(f) \cdot u^q = \bar{E}(f)$, so that $\bar{E}(f)$ is a morphism $p \to q$ in $u\mathcal{T}^{[n]}u$. The proof that $\bar{E}$ is a well defined, full and faithful functor is analogous to that in Theorem 12.6 and is left to the reader. ☐

12.8 Example. All one-sorted theories of *Set*. There are, up to equivalence of categories, precisely the theories $\mathcal{T}_k$ of 12.1. In fact, it ie easy to see that $\mathcal{T}_k \simeq \mathcal{T}_1^{[k]}$ is the matrix theory for every $k \geq 1$. Moreover, given an idempotent $u\colon 1 \to 1$ of $\mathcal{T}_k$, then the function $u\colon k \to k$ in *Set* is pseudoinvertible iff it is invertible, thus $u = \mathrm{id}$. Consequently, there are no other one-sorted theories of *Set*.

12.9 Example. Let R be a ring with unit. Generalizing 1.3.2, we can describe a one-sorted theory $\mathcal{T}_R$ of *R-Mod* : $\mathcal{T}_R$ is the full subcategory of *R-Mod*^{op} of the finitely generated free R-modules R^n ($n \in \mathbb{N}$). Every one-sorted algebraic theory of *R-Mod* is equivalent to $\mathcal{T}_S$ for some ring S which is Morita equivalent to R. Indeed, the two constructions of 12.3 fully correspond to the two constructions of 12.4:

1. $\mathcal{T}_{(R^{[k]})}$ is equivalent to $(\mathcal{T}_R)^{[k]}$;
2. given an idempotent element $u \in R$, the corresponding module homomorphism $\bar{u}\colon R \to R$ with $\bar{u}(x) = ux$ fulfils: uRu is equivalent to $\bar{u}(\mathcal{T}_R)\bar{u}$.

12.10 Example. For every monoid M, we denote by *M-Set* the corresponding category Set^{M} (where M is viewed as a one-object category). Two monoids M and N are called *Morita equivalent* if *M-Set* and *N-Set* are equivalent categories. Here we need just one operation on monoids: if N is Morita equivalent

to M, then N is isomorphic to an idempotent modification uMu for some pseudoinvertible idempotent u of M.
In contrast with the situation of 12.9, M-*Set* has, in general, many one-sorted theories not connected to any Morita equivalent monoid. (This is true even for $M = \{*\}$, since M-*Set* = *Set* has infinitely many categorically non-equivalent theories, see 12.1.) However, all *unary theories* of M-*Set* have the form which correspond to Morita equivalent monoids. By a unary theory we mean a one-sorted algebraic theory $\mathcal{T}$ on objects T^n $(n \in \mathbb{N})$ which is a free completion of its full subcategory on $\{T\}$ (i.e., of the endomorphism monoid of T) under finite products. The category M-*Set* has an obvious one-sorted theory $\mathcal{T}_{[M]}$: the theory of free M-sets $M + \ldots + M$ on n generators $(n \in \mathbb{N})$ as a full subcategory of $(M\text{-}Set)^{op}$. Consequently, for every Morita equivalent monoid N we have a unary theory $\mathcal{T}_{[N]}$ for the category M-*Set*. And these are, up to categorical equivalence, all unary theories. In fact, let $\mathcal{T}$ be a unary theory with $Alg\,\mathcal{T}$ categorically equivalent to M-*Set*. For the monoid $N = \mathcal{T}(T, T)$, there is an obvious categorical equivalence between $Alg\,\mathcal{T}$ and N-*Set* : every N-set $A \colon \mathcal{T}(T, T) \to Set$ has an essentially unique extension to a $\mathcal{T}$-algebra $A' \colon \mathcal{T} \to Set$, and $(-)'$ is the desired equivalence functor. Therefore, N is Morita equivalent to M, and $\mathcal{T}$ is categorically equivalent to $\mathcal{T}_{[N]}$.

Chapter 13

Free cocompletion under reflexive coequalizers

This chapter is devoted to study the free FC-conservative cocompletion under reflexive coequalizers of a small category with finite coproducts. (Recall that "FC-conservative" refers to the preservation of finite coproducts.) This completion, together with the exact completion of Chapter 14, will be used to characterize algebraic categories and their finitary localizations among exact categories in Chapter 15.

The following definition is based on the fact that a category has finite coproducts and reflexive coequalizers iff it is cocomplete (see Chapter 4).

13.1 Definition. Let $\mathcal{C}$ be a category with finite coproducts. A functor $P\colon \mathcal{C} \to Rec\,\mathcal{C}$ is a *free FC-conservative cocompletion of* $\mathcal{C}$ *under reflexive coequalizers* if

1. $Rec\,\mathcal{C}$ has finite colimits and P preserves finite coproducts

and

2. for every functor $F\colon \mathcal{C} \to \mathcal{B}$ preserving finite coproducts, where $\mathcal{B}$ is a finitely cocomplete category, there exists an assentially unique functor $F^*\colon Rec\,\mathcal{C} \to \mathcal{B}$ preserving finite colimits with F naturally isomorphic to $F^* \cdot P$.

We give now a first description of $P\colon \mathcal{C} \to Rec\,\mathcal{C}$.

13.2 Definition. Given a category $\mathcal{C}$ with finite coproducts, we define the category $Rec\,\mathcal{C}$ as follows:

1. Objects of $Rec\,\mathcal{C}$ are reflexive pairs $x_1, x_2\colon X_1 \rightrightarrows X_0$ in $\mathcal{C}$ (that is, there exists $d\colon X_0 \to X_1$ such that $x_1 \cdot d = \mathrm{id}_{X_0} = x_2 \cdot d$, see 3.9).

2. Consider the following diagram in $\mathcal{C}$

$$\begin{array}{ccc} & & Z_1 \\ & & {\scriptstyle z_1}\downarrow\downarrow{\scriptstyle z_2} \\ V & \overset{f}{\underset{g}{\rightrightarrows}} & Z_0 \end{array}$$

with z_1, z_2 a reflexive pair. We write

$$h\colon f \mapsto g$$

if there exists a morphism $h\colon V \to Z_1$ such that $z_1 \cdot h = f$ and $z_2 \cdot h = g$. This is a reflexive relation in the hom-set $\mathcal{C}(V, Z_0)$. We write $f \sim g$ if f and g are in the equivalence relation generated by the above reflexive relation.

3. A premorphism in $Rec\mathcal{C}$ is a morphism f in $\mathcal{C}$ as in the diagram

$$\begin{array}{ccc} X_1 & & Z_1 \\ {\scriptstyle x_1}\downarrow\downarrow{\scriptstyle x_2} & & {\scriptstyle z_1}\downarrow\downarrow{\scriptstyle z_2} \\ X_0 & \underset{f}{\longrightarrow} & Z_0 \end{array}$$

such that $f \cdot x_1 \sim f \cdot x_2$.

4. A morphism in $Rec\mathcal{C}$ is an equivalence class $[f]$ of premorphisms with respect to the equivalence $\sim$ of 2.

5. Composition and identities in $Rec\mathcal{C}$ are the obvious ones.

6. The functor $P\colon \mathcal{C} \to Rec\mathcal{C}$ is defined by

$$P(X \xrightarrow{f} Z) \quad = \quad \begin{array}{ccc} X & & Z \\ {\scriptstyle \mathrm{id}}\downarrow\downarrow{\scriptstyle \mathrm{id}} & & {\scriptstyle \mathrm{id}}\downarrow\downarrow{\scriptstyle \mathrm{id}} \\ X & \underset{[f]}{\longrightarrow} & Z \end{array}$$

13.3 Remark.

1. Consider

$$\begin{array}{ccc} & & Z_1 \\ & & {\scriptstyle z_1}\downarrow\downarrow{\scriptstyle z_2} \\ V & \overset{f}{\underset{g}{\rightrightarrows}} & Z_0 \end{array}$$

as in 12.2. Explicitly, $f \sim g$ means that there exists a zig-zag

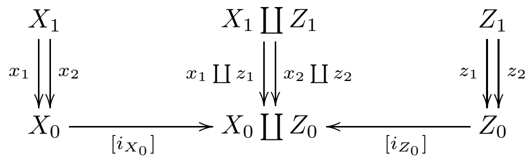
$$\begin{array}{ccccccccc} & & f_1 & & & & & f_n & \\ & {\scriptstyle h_1}\swarrow & & \searrow{\scriptstyle h_2} & & & {\scriptstyle h_n}\swarrow & & \searrow{\scriptstyle h_{n+1}} \\ f & & & & f_2 \quad \cdots\cdots & f_{n-1} & & & g \end{array}$$

2. Using the explicit description of $f \sim g$, it is straightforward to prove that $Rec\mathcal{C}$ is a category and $P\colon \mathcal{C} \to Rec\mathcal{C}$ is a full and faitful functor.

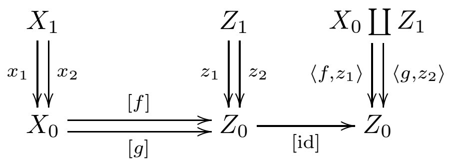
3. The above description of $P\colon \mathcal{C} \to Rec\mathcal{C}$ does not depend on the existence of finite coproducts in $\mathcal{C}$

13.4 Lemma. *Let $\mathcal{C}$ be a category with finite coproducts. The category $Rec\mathcal{C}$ of 13.2 has finite colimits and $P\colon \mathcal{C} \to Rec\mathcal{C}$ preserves finite coproducts.*

Proof. (1) Finite coproducts in $Rec\mathcal{C}$ are computed componentwise, i.e., if $x_1, x_2\colon X_1 \rightrightarrows X_0$ and $z_1, z_2\colon Z_1 \rightrightarrows Z_0$ are objects of $Rec\mathcal{C}$, their coproduct is

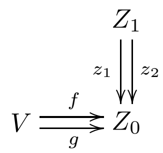


(2) Reflexive coequalizers in $Rec\mathcal{C}$ are depicted in the following diagram



∎

13.5 Lemma. *Consider the diagram*



as in 13.2. If a morphism $w\colon Z_0 \to W$ is such that $w \cdot z_1 = w \cdot z_2$ and if $f \sim g$, then $w \cdot f = w \cdot g$.

Proof. Clearly if $h\colon f \mapsto g$, then $w \cdot f = w \cdot g$. The claim follows now from the fact that to be coequalized by w is an equivalence relation in $\mathcal{C}(V, Z_0)$. ∎

13.6 Proposition. *Let $\mathcal{C}$ be a category with finite coproducts. The functor*

$$P\colon \mathcal{C} \to Rec\mathcal{C}$$

of 13.2 is a free FC-conservative cocompletion of $\mathcal{C}$ under reflexive coequalizers.

Proof. Let $F\colon \mathcal{C} \to \mathcal{B}$ as in 13.1. We define $F^*\colon Rec\mathcal{C} \to \mathcal{B}$ on objects by the following coequalizer in $\mathcal{B}$:

$$FX_1 \underset{Fx_2}{\overset{Fx_1}{\rightrightarrows}} FX_0 \longrightarrow F^*(x_1, x_2)\ .$$

Lemma 13.5 makes it clear how to define F^* on morphisms. The argument for the essential uniqueness of F^* is stated in 13.7 for future references. The rest of the proof is straightforward. ☐

13.7 Remark. For every reflexive pair $x_1, x_2\colon X_1 \rightrightarrows X_0$ in $\mathcal{C}$, the diagram

$$PX_1 \underset{Px_2}{\overset{Px_1}{\rightrightarrows}} PX_0 \xrightarrow{[\mathrm{id}_{X_0}]} (X_1 \underset{x_2}{\overset{x_1}{\rightrightarrows}} X_0)$$

is a reflexive coequalizer in $Rec\mathcal{C}$. Therefore, if $F, G\colon Rec\mathcal{C} \to \mathcal{B}$ preserve reflexive coequalizers and $F \cdot P \simeq G \cdot P$, then $F \simeq G$.

13.8 Proposition. *Let $\mathcal{C}$ be a small category with finite coproducts.*

1. *There is an equivalence of categories*

$$Ind(Rec\mathcal{C}) \simeq Sind\mathcal{C}\,.$$

2. *$Rec\mathcal{C}$ is equivalent to the full subcategory of all finitely presentable objects in $Alg(\mathcal{C}^{op})$.*

Proof. 1: Let $\mathcal{B}$ be a cocomplete category. By 4.17, functors $Ind(Rec\mathcal{C}) \to \mathcal{B}$ preserving colimits correspond to functors $Rec\mathcal{C} \to \mathcal{B}$ preserving finite colimits and then, by 13.6, to functors $\mathcal{C} \to \mathcal{B}$ preserving finite coproducts. On the other hand, functors $\mathcal{C} \to \mathcal{B}$ preserving finite coproducts correspond, by 4.13, to functors $Sind\mathcal{C} \to \mathcal{B}$ preserving colimits. Since both $Ind(Rec\mathcal{C})$ and $Sind\mathcal{C}$ are cocomplete (4.17.1 and 4.5), we can conclude that $Ind(Rec\mathcal{C})$ and $Sind\mathcal{C}$ are equivalent categories.
2: Let $\mathcal{A} = Alg(\mathcal{C}^{op}) = Sind\mathcal{C}$ (4.3). Since $\mathcal{A}$ is locally finitely presentable (6.16), then $\mathcal{A} \simeq Ind(\mathcal{A}_{fp})$ (6.15). Therefore, $Ind(Rec\mathcal{C}) \simeq Ind(\mathcal{A}_{fp})$. By Gabriel-Ulmer duality (8.20), we deduce that $Rec\mathcal{C}$ and $\mathcal{A}_{fp}$ are equivalent categories. ☐

13.9 Remark. Let $\mathcal{C}$ be a small category with finite coproducts. Following 5.15 and 13.8.2, we have that $Rec\mathcal{C}$ is equivalent to the full subcategory of $Alg(\mathcal{C}^{op})$ (or of $Set^{\mathcal{C}^{op}}$, see 3.3) given by reflexive coequalizers of representable functors. Moreover, 13.8.2 and 5.9.2 imply that $Rec\mathcal{C}$ is closed in $Alg(\mathcal{C}^{op})$ under finite colimits.

Chapter 14

Free exact categories

We know that every algebraic category is an exact category having enough regular projective objects (see 3.12 and 5.12). In the present chapter, we study *free exact completions* and prove that every algebraic category is a free exact completion of its full subcategory of all regular projectives (and even of its full subcategory of free algebras). This will be used in the next chapter to characterize algebraic categories among exact categories, and to describe all finitary localizations of algebraic categories. The aim of the present chapter is to introduce the concept of a free exact completion, and describe a construction of it. The main point here is that the universal property of a free exact completion is based on *left covering functors.* These are functors which play, for categories with weak finite limits, the role that finitely continuous functors play for finitely complete categories. The trouble with regular projective objects in an algebraic category is namely that they are not closed under finite limits. Luckily they have weak finite limits (defined as finite limits except that the uniqueness of the factorization is not requested).

We will be concerned with regular epimorphisms (3.2) in an exact category (3.11). For the comfort of the reader, we start by listing some of their (easy but) important properties. In diagrams, regular epimorphisms are denoted by $\twoheadrightarrow$.

14.1 Lemma. *Let $\mathcal{A}$ be an exact category.*

1. *Any morphism factorizes as a regular epimorphism followed by a monomorphism.*

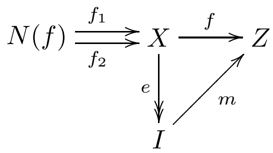
2. *Consider a morphism $f\colon X \to Z$. The following conditions are equivalent:*

    (a) *f is a regular epimorphism;*

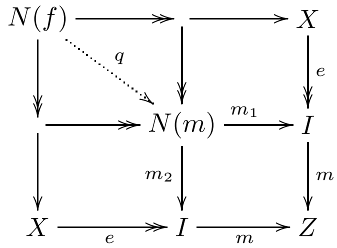
    (b) *f is a strong epimorphism (6.1);*

    (c) *f is an extremal epimorphism (6.1).*

Proof. 1: Consider a morphism $f\colon X \to Z$ and its factorization through the coequalizer of its kernel pair



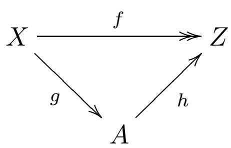
We have to prove that m is a monomorphism. For this, consider the following diagram, where each square is a pullback



Since in $\mathcal{A}$ regular epimorphisms are pullback stable, the diagonal q is an epimorphism. Now, $m_1 \cdot q = e \cdot f_1 = e \cdot f_2 = m_2 \cdot q$, so that $m_1 = m_2$. This means that m is a monomorphism.
2: a $\Rightarrow$ b: If f is the coequalizer of a pair (x, y), then u also coequalizes x and y (because $v \cdot f = m \cdot u$ and m is a monomorphism).
b $\Rightarrow$ c: Just take $v = 1$ in condition (b) (recall that a monomorphism which is also a split epimorphism is an isomorphism).
c $\Rightarrow$ a: Just take a regular epi-mono factorization $f = m \cdot e$ (which exists by part 1); if condition (c) holds, then m is an isomorphism and therefore f is a regular epimorphism. ☐

14.2 Corollary. *Let $\mathcal{A}$ be an exact category.*

1. *The factorization stated in Lemma 14.1 is essentially unique;*
2. *The composite of two regular epimorphisms is a regular epimorphism;*
3. *If the triangle*



   commutes and f is a regular epimorphism, then h is a regular epimorphism;

4. *If a morphism is a regular epimorphism and a monomorphism, then it is an isomorphism.*

Proof. Everything follows easily from condition 2.b of Lemma 14.1. □

14.3 Lemma. *Every exact category has the following properties:*

1. *The product of two regular epimorphisms is a regular epimorphism;*

2. *Consider the following diagram*

$$\begin{array}{ccc} A_0 & \overset{a_1}{\underset{a_2}{\rightrightarrows}} & A_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ B_0 & \overset{b_1}{\underset{b_2}{\rightrightarrows}} & B_1 \end{array}$$

   with $f_1 \cdot a_i = b_i \cdot f_0$ for $i = 1, 2$. If f_0 is a regular epimorphism and f_1 is a monomorphism, then the unique extension to the equalizers is a regular epimorphism;

3. *Consider the following commutative diagram*

$$\begin{array}{ccccc} A_0 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_1 \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ B_0 & \xrightarrow[b_1]{} & B & \xleftarrow[b_2]{} & B_1 \end{array}$$

   If f_0 and f_1 are regular epimorphisms and f is a monomorphism, then the unique extension to the pullbacks is a regular epimorphism.

Proof. 1: Observe that $f \times \mathrm{id}$ is the pullback of f along the suitable projection, and the same holds for $\mathrm{id} \times g$. Now $f \times g = (f \times \mathrm{id}) \cdot (\mathrm{id} \times g)$.
2: Since f_1 is a monomorphism, the pullback along f_0 of the equalizer of (b_1, b_2) is the equalizer of (a_1, a_2).
3: This follows from 1. and 2., using the usual construction of pullbacks via products and equalizers. □

14.4 Exercise. State and prove the n-ary generalization of 14.3.

For the sake of generality, let us point out that in 14.1, 14.2 and 14.3 we do not need that in $\mathcal{A}$ equivalence relations are effective.

From Propositions 3.12 and 5.12 we know that an algebraic category is an exact category having enough regular projective objects. In fact, each algebra is a regular quotient of a free (and thus regular projective) algebra. In the following we study categories having enough regular projectives; we introduce the concept of a regular projective cover for a subcategory of regular projectives in case there is "enough of them".

14.5 Definition. Let $\mathcal{A}$ be a category. A *regular projective cover* of $\mathcal{A}$ is a full and faithful functor $I \colon \mathcal{P} \to \mathcal{A}$ such that:

1. every object of $\mathcal{P}$ is regular projective in $\mathcal{A}$;
2. for every object A of $\mathcal{A}$, there is an object P in $\mathcal{P}$ and a regular epimorphism $P \to A$ (we write P instead of IP and we call $P \to A$ a $\mathcal{P}$-*cover* of A).

Recall that a functor is *exact* if it preserves finite limits and regular epimorphisms. The present chapter is devoted to the study of exact functors defined on an exact category $\mathcal{A}$ having a regular projective cover $\mathcal{P} \to \mathcal{A}$. First of all, observe that regular projective objects are not closed under finite limits, so that we cannot hope that $\mathcal{P}$ inherits finite limits from $\mathcal{A}$. Nevertheless, a "trace" of finite limits remains in $\mathcal{P}$. In fact, $\mathcal{P}$ has *weak finite limits.* (Weak limits are defined as limits, but without the uniqueness of the factorization. Observe that, unlike limits, weak limits are very much "non-unique". For example, any non-empty set is a weak terminal object in the category *Set*.)

14.6 Lemma. *If $\mathcal{P} \to \mathcal{A}$ is a regular projective cover of a finitely complete category $\mathcal{A}$, then $\mathcal{P}$ has weak finite limits.*

Proof. Consider a finite diagram $D\colon \mathcal{D} \to \mathcal{P}$. If

$$\langle\, \pi_X : L \to DX \,\rangle_{X \in \mathcal{D}}$$

is a limit of D in $\mathcal{A}$, then it is possible to find a $\mathcal{P}$-cover $l\colon P \to L$. The resulting cone

$$\langle\, \pi_X \cdot l \colon P \to DX \,\rangle_{X \in \mathcal{D}}$$

is a weak limit of D in $\mathcal{P}$. □

In the situation of the previous lemma, apply an exact functor $G\colon \mathcal{A} \to \mathcal{B}$. Since G preserves finite limits, the factorization of the cone

$$\langle\, G(\pi_X \cdot l) \colon G(P) \to G(DX) \,\rangle_{X \in \mathcal{D}}$$

through the limit in $\mathcal{B}$ is $G(l)\colon G(P) \to G(L)$, which is a regular epimorphism because G is exact.

We can formalize this property in the following definition.

14.7 Definition. Let $\mathcal{B}$ be an exact category and let $\mathcal{P}$ be a category with weak finite limits. A functor $F\colon \mathcal{P} \to \mathcal{B}$ is *left covering* if, for any finite diagram $D\colon \mathcal{D} \to \mathcal{P}$ with weak limit W, the canonical comparison morphism $F(W) \to \mathit{lim}\, F \cdot D$ is a regular epimorphism.

14.8 Remark. To avoid any ambiguity in the previous definition, let us point out that if the comparison $w\colon F(W) \to \mathit{lim}\, F \cdot D$ is a regular epimorphism for a certain weak limit W of D, then the comparison $w'\colon F(W') \to \mathit{lim}\, F \cdot D$ is a regular epimorphism for any other weak limit W' of D. This follows from Corollary 14.2 because w factorizes through w'.

14.9 Example.

1. Any functor preserving weak finite limits is left covering.
2. If $\mathcal{P} \to \mathcal{A}$ is a regular projective cover of an exact category $\mathcal{A}$, then it is a left covering functor.
3. The composition of a left covering functor with an exact functor is a left covering functor.

14.10 Example. Let $\mathcal{P}$ be a category with weak finite limits, and consider the (possibly illegitimate) functor category $[\mathcal{P}^{op}, Set]$. The canonical Yoneda embedding $Y\colon \mathcal{P} \to [\mathcal{P}^{op}, Set]$ is a left covering functor.

Proof. Consider a finite diagram $D\colon \mathcal{D} \to \mathcal{P}$ in $\mathcal{P}$, a weak limit W of D and a limit L of $Y \cdot D$. The canonical comparison $\tau\colon Y(W) \to L$ is a regular epimorphism whenever, for all $Z \in \mathcal{P}$, $\tau(Z)\colon Y(W)(Z) \to L(Z)$ is surjective. Since the limit L is computed point-wise in *Set*, an element of $L(Z)$ is a cone from Z to $\mathcal{L}$, so that the surjectivity of $\tau(Z)$ is just the weak universal property of W. □

14.11 Remark. In one of our main results we characterize categories which are free exact completions of other categories, see Proposition 14.32: they are precisely the exact categories which have a regular projective cover. This is one of the results that request working with (instead of the seemingly more natural condition of preservation of weak finite limits) the left covering property. In fact, the basic example 14.9.2 would not be true otherwise. This is illustrated by the category of rings: the inclusion of the full subcategory $\mathcal{P}$ of all regular projective rings does not preserve weak finite limits. For example, the ring $\mathbb{Z}$ of integers is a weak terminal object in $\mathcal{P}$, but it is not a weak terminal object in $\mathcal{A}$, because the unique morphism from $\mathbb{Z}$ to the one-element ring does not have a section.

A remarkable fact about left covering functors is that they classify exact functors. Before stating this fact in a precise way, see 14.29, we need some facts about left covering functors and pseudoequivalences. A pseudoequivalence is defined "almost" as an equivalence relation, but (a) using a weak pullback instead of a pullback to express the transitivity, and (b) without the assumption that the graph is jointly monic.

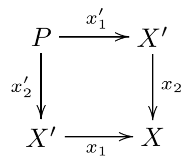
14.12 Definition. Let $\mathcal{P}$ be a category with weak pullbacks. A *pseudoequivalence* is a parallel pair

$$X' \underset{x_2}{\overset{x_1}{\rightrightarrows}} X$$

which is

1. reflexive, i.e., there is $r\colon X \to X'$ such that $x_1 \cdot r = \mathrm{id}_X = x_2 \cdot r$,
2. symmetric, i.e., there is $s\colon X' \to X'$ such that $x_1 \cdot s = x_2$ and $x_2 \cdot s = x_1$, and

3. transitive, i.e., in an arbitrary weak pullback



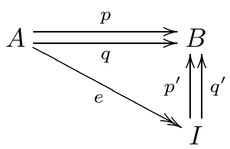
there exists $t\colon P \to X'$ such that $x_1 \cdot t = x_1 \cdot x_1'$ and $x_2 \cdot t = x_2 \cdot x_2'$. The morphism t is called a *transitivity morphism* of x_0 and x_1.

14.13 Example. Any internal groupoid in a category with pullbacks is a pseudoequivalence.

14.14 Remark.

1. Observe that the existence of a transitivity morphism of x_0 and x_1 does not depend on the choice of a weak pullback of x_0 and x_1.

2. Recall that a regular factorization of a morphism is a factorization as a regular epimorphism followed by a monomorphism. In a category with binary products, we speak about *regular factorization of a parallel pair* $p, q\colon A \rightrightarrows B$. What we mean is a factorization of (p, q) as in the following diagram, where e is a regular epimorphism and (p', q') is a jointly monomorphic parallel pair,

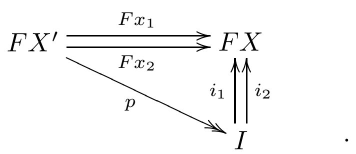


obtained by a regular factorization of $\langle p, q\rangle\colon A \to B \times B$. Since jointly monomorphic parallel pairs are also called relations, we call (p', q') the *relation induced* by (p, q).

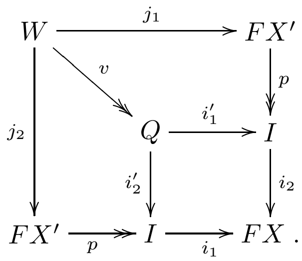
3. If $\mathcal{P}$ has finite limits, then equivalence relations precisely are those parallel pairs which are, at the same time, relations and pseudoequivalences. The next result, which is the main link between pseudoequivalences and left covering functors, shows that any pseudoequivalence in an exact category is a composition of an equivalence relation with a regular epimorphism. (The converse is not true. Consider the category of rings, the unique equivalence relation on the one-element ring 0, and the unique morphism $\mathbb{Z} \to 0$. The parallel pair $\mathbb{Z} \rightrightarrows 0$ cannot be reflexive, because there are no morphisms from 0 to $\mathbb{Z}$.)

14.15 Lemma. *Let $F\colon \mathcal{P} \to \mathcal{B}$ be a left covering functor. For every pseudoequivalence $x_1, x_2\colon X' \rightrightarrows X$ in $\mathcal{P}$, the relation in $\mathcal{B}$ induced by (Fx_1, Fx_2) is an equivalence relation.*

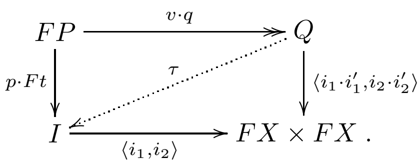
Proof. Consider a regular factorization in $\mathcal{B}$



Since the reflexivity and transitivity are obvious, we only check the transitivity of (i_1, i_2). The pullback of $i_1 \cdot p$ and $i_2 \cdot p$ factorizes through the pullback of i_1 and i_2, and the factorization, v, is a regular epimorphism (because p is a regular epimorphism and $\mathcal{B}$ is an exact category):



Consider also a transitivity morphism $t\colon P \to X'$ of (x_1, x_2) as in Definition 14.12. Since $F\colon \mathcal{P} \to \mathcal{B}$ is left covering, the factorization $q\colon FP \to W$ such that $j_1 \cdot q = Fx_1'$ and $j_2 \cdot q = Fx_2'$ is a regular epimorphism. Finally, we have the following commutative diagram



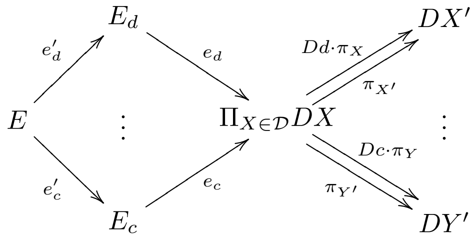
Since $v \cdot q$ is a regular epimorphism and $\langle i_1, i_2 \rangle$ is a monomorphism, there exists $\tau\colon Q \to I$ such that $\langle i_1, i_2 \rangle \cdot \tau = \langle i_1 \cdot i_1', i_2 \cdot i_2' \rangle$. This implies that (i_1, i_2) is transitive. ☐

14.16 Lemma. *A functor $F\colon \mathcal{P} \to \mathcal{B}$ (where $\mathcal{P}$ has weak finite limits and $\mathcal{B}$ is exact) is left covering if and only if it is left covering with respect to weak finite products and weak equalizers.*

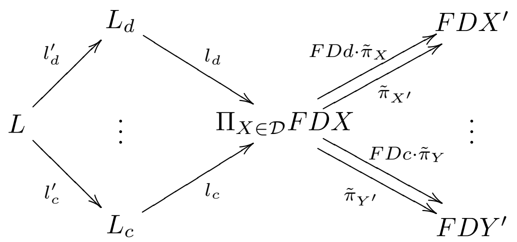
14.17 Remark. The meaning of "left covering with respect to weak finite products" is obvious, we do not need to spell this out. Observe that this is equivalent to being left covering with respect to weak binary products and weak terminal objects.

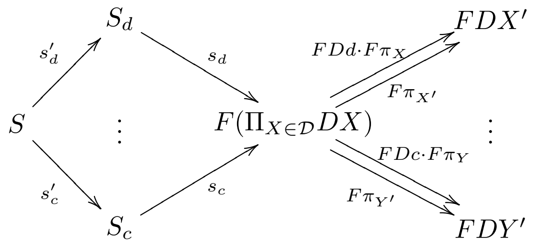
Proof. 1. Using Lemma 14.3 and working by induction, one extends the left covering character of F to joint equalizers of parallel n-tuples, and then to

multiple pullbacks.
2. Consider a finite diagram $D\colon \mathcal{D} \to \mathcal{P}$. We can construct a weak limit of D using a weak product $\Pi_{X\in\mathcal{D}}DX$, weak equalizers E_d, one for each morphism $d\colon X \to X'$ in $\mathcal{D}$, and a weak multiple pullback E as in the following diagram



Perform now the same constructions in $\mathcal{B}$ to get limits as in the following diagrams

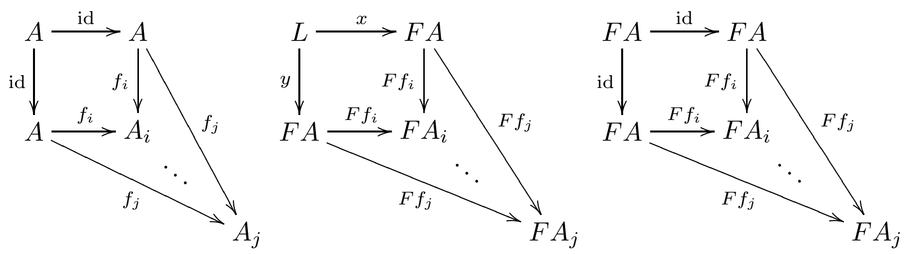




By assumption, the canonical factorization $q_d\colon FE_d \to S_d$ is a regular epimorphism. By Lemma 14.3, this gives rise to a regular epimorphism $q\colon Q \to S$, where Q is the multiple pullback of the Fe_d. By part 1., the canonical factorization $t\colon FE \to Q$ is a regular epimorphism. Finally, a diagram chase shows that the pullback of $l_d \cdot l'_d$ along the canonical factorization $p\colon F(\Pi_{X\in\mathcal{D}}DX) \to \Pi_{X\in\mathcal{D}}FDX$ is $s_d \cdot s'_d$. By part 1., p is a regular epimorphism, so that we get a regular epimorphism $p'\colon S \to L$. The regular epimorphism $p' \cdot q \cdot t\colon FE \to Q \to S \to L$ shows that F is left covering. □

14.18 Lemma. *A left covering functor $F\colon \mathcal{P} \to \mathcal{B}$ preserves finite jointly monomorphic sources.*

Proof. A family of morphisms $(f_i\colon A \to A_i)_{i\in I}$ is jointly monomorphic if and only if the span formed by $\mathrm{id}_A, \mathrm{id}_A$ is a limit of the corresponding diagram:



Now apply F and consider the canonical factorization $q\colon FA \to L$, where L is a limit in $\mathcal{B}$ of the corresponding diagram. By assumption, q is a regular epimorphism. It is also a monomorphism, because $x \cdot q = \mathrm{id}$, and so it is an isomorphism. This implies that $\mathrm{id}_{FA}, \mathrm{id}_{FA}$ is a limit, thus the family $(Ff_i\colon FA \to FA_i)_{i\in I}$ is jointly monomorphic. ☐

14.19 Lemma. *Consider a functor $F\colon \mathcal{P} \to \mathcal{B}$. Assume that $\mathcal{P}$ has finite limits and $\mathcal{B}$ is exact. Then F is left covering if and only if it preserves finite limits.*

Proof. One implication is clear, see 14.9.1. Thus, let us assume that F is left covering and consider a finite non-empty diagram $D\colon \mathcal{D} \to \mathcal{P}$. Let $(\pi_X\colon L \to DX)_{X\in\mathcal{D}}$ be a limit of D and $(\tilde{\pi}_X\colon \tilde{L} \to FDX)_{X\in\mathcal{D}}$ a limit of $F \cdot D$. Since the family $(\pi_X)_{X\in\mathcal{D}}$ is jointly monomorphic, by Lemma 14.18 also the family $(F\pi_X)_{X\in\mathcal{D}}$ is monomorphic. This implies that the canonical factorization $p\colon FL \to \tilde{L}$ is a monomorphism. But it is a regular epimorphism by assumption, so that it is an isomorphism.
The argument for the terminal object T is different. In $\mathcal{P}$, the product of T with itself is T with the identity morphisms as projections. Then the canonical factorization $FT \to FT \times FT$ is a (regular) epimorphism. This implies that the two projections $\pi_1, \pi_2\colon FT \times FT \rightrightarrows FT$ are equal. But the pair (π_1, π_2) is the kernel pair of the unique morphism q to the terminal object of $\mathcal{B}$, so that q is a monomorphism. Since F is left covering, q is a regular epimorphism, thus an isomorphism. ☐

Let us point out that in 14.15 and 14.16, we do not need to assume that in $\mathcal{B}$ equivalence relations are effective. Moreover, if in Definition 14.7 we replace *regular epimorphism* by *strong epimorphism*, then 14.18 and 14.19 hold for all categories $\mathcal{B}$ with finite limits.

14.20 Definition. Let $\mathcal{P}$ be a category with weak finite limits. A *free exact completion* of $\mathcal{P}$ is an exact category $\mathcal{P}_{ex}$ with a left covering functor

$$\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$$

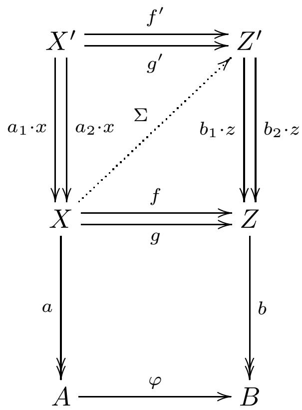
such that, for any exact category $\mathcal{B}$ and for any left covering functor $F\colon \mathcal{P} \to \mathcal{B}$, there is an essentially unique exact functor $\hat{F}\colon \mathcal{P}_{ex} \to \mathcal{B}$ such that $\hat{F} \cdot \Gamma$ is naturally isomorphic to F.

Note that, since a free exact completion is defined via a universal property, it is determined uniquely up to equivalence.

Let us explain now, in an informal way, why pseudoequivalences enter in the description of the free exact completion (see Proposition 14.32 for the precise statement). Let $\mathcal{P} \to \mathcal{A}$ be a regular projective cover of an exact category $\mathcal{A}$. Fix an object A in $\mathcal{A}$ and consider a $\mathcal{P}$-cover $a\colon X \to A$, its kernel pair $a_1, a_2\colon N(a) \rightrightarrows X$, and again a $\mathcal{P}$-cover $x\colon X' \to N(a)$. In the resulting diagram

$$X' \underset{a_2\cdot x}{\overset{a_1\cdot x}{\rightrightarrows}} X \xrightarrow{\;a\;} A$$

the left-hand part is a pseudoequivalence in $\mathcal{P}$ (not in $\mathcal{A}$!) and A is its coequalizer. Consider now the following diagram



Using the regular projectivity of X and X' and the universal property of the kernel pair of b, we get a pair (f', f) such that $\varphi \cdot a = b \cdot f$ and $f \cdot a_i \cdot x = b_i \cdot z \cdot f'$ for $i = 1, 2$. Conversely, a pair (f', f) such that $f \cdot a_i \cdot x = b_i \cdot z \cdot f'$ for $i = 1, 2$ induces a unique extension to the quotient. Moreover, two such pairs (f', f) and (g', g) have the same extension if and only if there is a morphism $\Sigma\colon X \to Z'$ such that $b_2 \cdot z \cdot \Sigma = f$ and $b_2 \cdot z \cdot \Sigma = g$.

Keeping the previous situation in mind, we give a first description of the free exact completion $\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$.

14.21 Definition. Given a category $\mathcal{P}$ with weak finite limits, we define the category $\mathcal{P}_{ex}$ as follows:

1. Objects of $\mathcal{P}_{ex}$ are pseudoequivalences $x_1, x_2\colon X' \rightrightarrows X$ in $\mathcal{P}$ (we sometimes denote such an object by X/X').

2. A premorphism in $\mathcal{P}_{ex}$ is a pair of morphisms (f', f) as in the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\;f'\;} & Z' \\ {\scriptstyle x_1}\downarrow\downarrow{\scriptstyle x_2} & & {\scriptstyle z_1}\downarrow\downarrow{\scriptstyle z_2} \\ X & \xrightarrow[\;f\;]{} & Z \end{array}$$

such that $f \cdot x_1 = z_1 \cdot f'$ and $f \cdot x_2 = z_2 \cdot f'$.

3. A morphism in $\mathcal{P}_{ex}$ is an equivalence class $[f', f]\colon X/X' \to Z/Z'$ of premorphisms. Two parallel premorphisms (f', f) and (g', g) are equivalent if there exists a morphism $\Sigma\colon X \to Z'$ such that $z_1 \cdot \Sigma = f$ and $z_2 \cdot \Sigma = g$.

4. Composition and identities are obvious.

14.22 Notation. We denote by $\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$ the embedding of $\mathcal{P}$ into $\mathcal{P}_{ex}$ assigning to a morphism $f\colon X \to Z$ the following morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \mathrm{id}\downdownarrows\mathrm{id} & & \mathrm{id}\downdownarrows\mathrm{id} \\ X & \xrightarrow[f]{} & Z \end{array}$$

14.23 Remark.

1. The fact that the above relation among premorphisms is an equivalence relation can be proved (step by step) using the assumption that the codomain $z_1, z_2\colon Z' \rightrightarrows Z$ is a pseudoequivalence. Observe also that the class of (f', f) depends on f only (compose f with a reflexivity of (z_1, z_2) to show that (f', f) and (f'', f) are equivalent); for this reason, we often write $[f]$ instead of $[f', f]$.

2. The fact that composition is well-defined is obvious.

3. Γ is a full and faithful functor. This is easy to verify.

4. Observe that if $\mathcal{P}$ is small (locally small), then $\mathcal{P}_{ex}$ also is small (locally small, respectively).

14.24 Remark. The equivalence relation among premorphisms in $\mathcal{P}_{ex}$ can be thought of as a kind of "homotopy" relation. And in fact, this is the case in a particular example: if X is a topological space, the evaluation maps $ev_0, ev_1\colon X^{[0,1]} \rightrightarrows X$ constitute a pseudoequivalence. This gives rise to a functor $\mathcal{E}\colon \mathbf{Top} \to \mathbf{Top}_{ex}$. Now two continuous maps $f, g\colon X \to Z$ are homotopic in the usual sense precisely when $\mathcal{E}(f)$ and $\mathcal{E}(g)$ are equivalent in the sense of Definition 14.21. More precisely, $\mathcal{E}$ factorizes through the homotopy category, and the factorization $\mathcal{E}'\colon \mathbf{HTop} \to \mathbf{Top}_{ex}$ is full and faithful (and left covering).

We are going to prove that the functor $\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$ just described is the free exact completion of $\mathcal{P}$, in the sense of Definition 14.20. For this, it is useful to have an equivalent description of $\mathcal{P}_{ex}$ as a full subcategory of the functor category $[\mathcal{P}^{op}, Set]$.

14.25 Lemma. *Let $\mathcal{P}$ be a category with weak finite limits, and let $Y\colon \mathcal{P} \to [\mathcal{P}^{op}, Set]$ be the Yoneda embedding. The following properties of a functor $A\colon \mathcal{P}^{op} \to Set$ are equivalent:*

1. *A is a regular quotient of a representable object modulo a preequivalence in $\mathcal{P}$, i.-e., there exists a preequivalence $x_1, x_2 \colon X' \rightrightarrows X$ in $\mathcal{P}$ and a coequalizer*

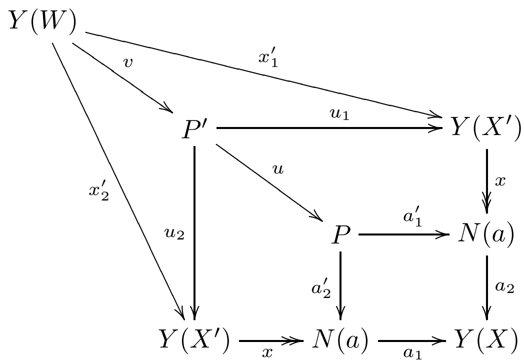
$$Y(X') \underset{x_2}{\overset{x_1}{\rightrightarrows}} Y(X) \twoheadrightarrow A$$

in $[\mathcal{P}^{op}, Set]$;

2. *A is a regular quotient of a representable object modulo a regular epimorphism $a \colon Y(X) \to A$ such that $N(a)$, the domain of a kernel pair of a, is also a regular quotient of a representable object:*

$$Y(X') \overset{x}{\twoheadrightarrow} N(a) \underset{a_2}{\overset{a_1}{\rightrightarrows}} Y(X) \overset{a}{\twoheadrightarrow} A \qquad \textit{(for some } X' \textit{ in } \mathcal{P}\textit{).}$$

Proof. Consider the previous diagram $[\mathcal{P}^{op}, Set]$. Since a is the coequalizer of $(a_1 \cdot x, a_2 \cdot x)$, we have to prove that $(a_1 \cdot x, a_2 \cdot x)$ is a pseudoequivalence in $\mathcal{P}$. Let us check the transitivity: consider the following diagram



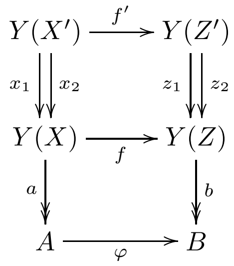
where P and P' are pullbacks, and W is a weak pullback. Since $Y(W)$ is regular projective and x is a regular epimorphism, the transitivity morphism $t \colon P \to N(a)$ of (a_1, a_2) extends to a morphism $t' \colon Y(W) \to Y(X')$ such that $t \cdot u \cdot v = x \cdot t'$. This morphism t' is a transitivity for $(a_1 \cdot x, a_2 \cdot x)$. The converse implication follows from Lemma 14.15, since $Y \colon \mathcal{P} \to [\mathcal{P}^{op}, Set]$ is left covering. □

Note that the fact that (x_1, x_2) is a pseudoequivalence in $\mathcal{P}$ does not mean that $(Y(x_1), Y(x_2))$ is a pseudoequivalence in $[\mathcal{P}^{op}, Set]$, because Y does not preserve weak pullbacks.

14.26 Notation. The full subcategory of $[\mathcal{P}^{op}, Set]$ of all objects satisfying (i) or (ii) of the above lemma is denoted by $\mathcal{P}'_{ex}$. In the next lemma, the codomain restriction to $\mathcal{P}'_{ex}$ of the Yoneda embedding $Y \colon \mathcal{P} \to [\mathcal{P}^{op}, Set]$ is again denoted by Y, and Γ is the functor from 14.22.

14.27 Lemma. *There is an equivalence of categories $\mathcal{E} \colon \mathcal{P}_{ex} \to \mathcal{P}'_{ex}$ such that $\mathcal{E} \cdot \Gamma = Y$.*

Proof. Consider the functor $\mathcal{E}\colon \mathcal{P}_{ex} \to \mathcal{P}'_{ex}$ sending a morphism $[f]\colon X/X' \to Z/Z'$ to the extension φ to the coequalizers as in the following diagram



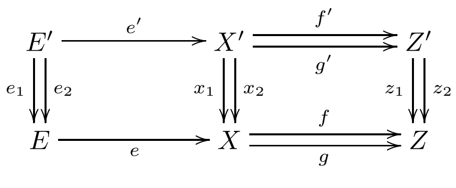
The functor $\mathcal{E}$ is well-defined because a is an epimorphism and b coequalizes y_0 and y_1. Moreover, $\mathcal{E}$ is essentially surjective by definition of $\mathcal{P}'_{ex}$. Let us prove that $\mathcal{E}$ is faithful: if $\mathcal{E}[f] = \mathcal{E}[g]$, then the pair (f, g) factorizes through the kernel pair $N(b)$ of b, which is a regular factorization of (y_0, y_1). Since $Y(X)$ is regular projective, this factorization extends to a morphism $Y(X) \to Y(Z')$, which shows that $[f] = [g]$.
$\mathcal{E}$ is full: given $\varphi\colon A \to B$, we get $f\colon Y(X) \to Y(Z)$ by regular projectivity of $Y(X)$. Since $b \cdot f \cdot x_1 = b \cdot f \cdot x_2$, we get $\overline{f}\colon Y(X') \to N(b)$. Since $N(b)$ is the regular factorization of (z_1, z_2) and $Y(X')$ is regular projective, $\overline{f}$ extends to $f'\colon Y(X') \to Y(Z')$. Clearly, $\mathcal{E}[f', f] = \varphi$. □

14.28 Proposition. *For every category $\mathcal{P}$ with weak finite limits, the functor*

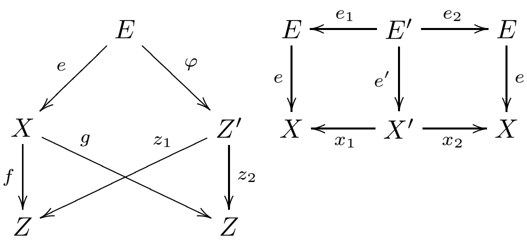
$$\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$$

of 14.22 is a left covering functor into an exact category. Moreover, it is a regular projective cover of $\mathcal{P}_{ex}$.

Proof. 1: $\mathcal{P}_{ex}$ has finite limits. Since the construction of the other basic types of finite limits is completely analogous, we explain in details the case of equalizers, and we just mention the construction for binary products and terminal object.
1a: Equalizers: Consider a parallel pair in $\mathcal{P}_{ex}$ together with what we want to be their equalizer



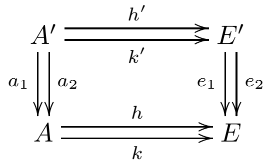
This means that we need the following equations: $x_1 \cdot e' = e \cdot e_1$ and $x_2 \cdot e' = e \cdot e_2$. Moreover, we ask for $f \cdot e$ and $g \cdot e$ being equivalent, that is, we need a morphism $\varphi\colon E \to Z'$ such that $z_1 \cdot \varphi = f \cdot e$ and $z_2 \cdot \varphi = g \cdot e$. So, we take E and E' to be the following weak limits



(i) It is straightforward to check that (e_1, e_2) is a pseudoequivalence in $\mathcal{P}$ (just use the fact that (x_1, x_2) is a pseudoequivalence).

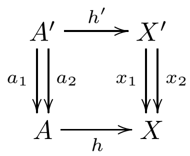
(ii) To show that $[e]$ equalizes $[f]$ and $[g]$, use the morphism $\varphi\colon E \to Z'$.

(iii) The morphism $[e]$ is a monomorphism: in fact, consider two morphisms in $\mathcal{P}_{ex}$



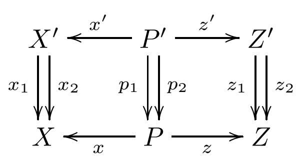
such that $[e] \cdot [h] = [e] \cdot [k]$. This means that there is a morphism $\Sigma\colon A \to X'$ such that $x_1 \cdot \Sigma = e \cdot h$ and $x_2 \cdot \Sigma = e \cdot k$. By the weak universal property of E', we have a morphism $\Sigma'\colon A \to E'$ such that $e_1 \cdot \Sigma' = h$ and $e_2 \cdot \Sigma' = k$. This means that $[h] = [k]$.

(iv) We prove that every morphism



in $\mathcal{P}_{ex}$ such that $[f] \cdot [h] = [g] \cdot [h]$ factors through $[e]$. We know that there is $\Sigma\colon A \to Z'$ such that $z_1 \cdot \Sigma = f \cdot h$ and $z_2 \cdot \Sigma = g \cdot h$. The weak universal property of E yields then a morphism $k\colon A \to E$ such that $e \cdot k = h$ and $\Sigma \cdot k = \varphi$. Now, $x_1 \cdot h' = e \cdot k \cdot a_1$ and $x_2 \cdot h' = e \cdot k \cdot a_2$. The weak universal property of E' yields a morphism $k'\colon A' \to E'$ such that $e_1 \cdot k' = k \cdot a_1$ and $e_2 \cdot k' = k \cdot a_2$. Finally, the needed factorization is $[k', k]\colon A/A' \to E/E'$.

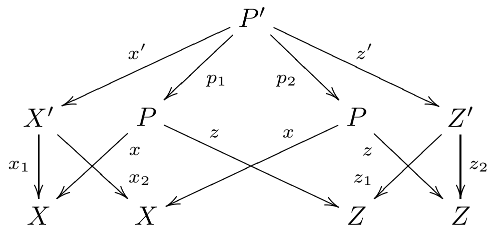
1b: Products: Consider two objects $x_1, x_2\colon X' \rightrightarrows X$ and $z_1, z_2\colon Z' \rightrightarrows Z$ in $\mathcal{P}_{ex}$. Their product is given by



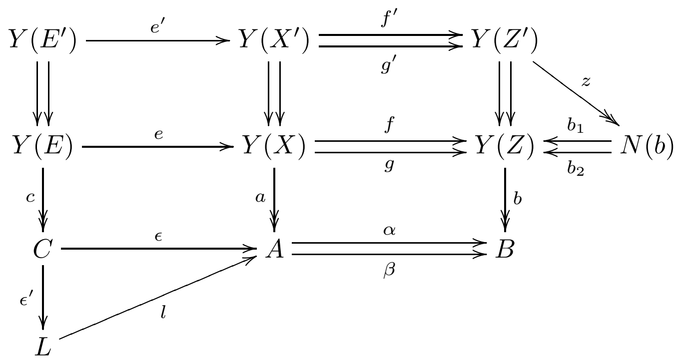
where

$$X \xleftarrow{x} P \xrightarrow{z} Z$$

is a weak product of X and Z in $\mathcal{P}$, and P' is the following weak limit



1c: Terminal object: For any object T of $\mathcal{P}$, the projections from a weak product $\pi_1, \pi_2 \colon T \times T \rightrightarrows T$ form a pseudoequivalence. If T is a weak terminal object in $\mathcal{P}$, then (π_0, π_1) is a terminal object in $\mathcal{P}_{ex}$.
2: $\mathcal{P}_{ex}$ is closed under finite limits in $[\mathcal{P}^{op}, Set]$. In fact, by Lemma 14.27, we can identify $\mathcal{P}_{ex}$ with $\mathcal{P}'_{ex}$. We prove that the full inclusion of $\mathcal{P}_{ex}$ into $[\mathcal{P}^{op}, Set]$ preserves finite limits. Because of Lemma 14.19, it is enough to prove that the inclusion is left covering. We give the argument for equalizers, since that for products and terminal object is similar (and easier). With the notations of part 1., consider the following diagram, where ϵ, α and β are extensions to the coequalizers, the triangle on the right is a regular factorization, and the triangle at the bottom is the factorization through the equalizer



We have to prove that ϵ' is a regular epimorphism. Using $\varphi \colon E \to Z'$, we check that $\alpha \cdot a \cdot e = \beta \cdot a \cdot e$, so that there is $p \colon Y(E) \to L$ such that $l \cdot p = a \cdot e$, and then $p = \epsilon' \cdot c$. So, it is enough to prove that p is a regular epimorphism, that is, the components $p(P) \colon Y(E)(P) \to L(P)$ are surjective. This means that, given a morphism $u \colon Y(P) \to A$ such that $\alpha \cdot u = \beta \cdot u$, we need a morphism $\hat{u} \colon P \to E$ such that $l \cdot p \cdot \hat{u} = u$. First of all, observe that, since a is a regular epimorphism and $Y(P)$ is regular projective, there is $u' \colon P \to X$ such that $a \cdot u' = u$. Now, $b \cdot f \cdot u' = b \cdot g \cdot u'$, so that there is $u'' \colon Y(P) \to N(b)$ such that $b_1 \cdot u'' = f \cdot u'$ and $b_2 \cdot u'' = g \cdot u'$. Moreover, since z is a regular epimorphism and $Y(P)$ is regular

projective, there is $\tilde{u}\colon P \to Z'$ such that $z \cdot \tilde{u} = u''$. Finally, $z_1 \cdot \tilde{u} = f \cdot u'$ and $z_2 \cdot \tilde{u} = g \cdot u'$, so that there is $\hat{u}\colon P \to E$ such that $\varphi \cdot \hat{u} = \tilde{u}$ and $e \cdot \hat{u} = u'$. This last equation implies that $l \cdot p \cdot \hat{u} = u$.
3: $\mathcal{P}_{ex}$ is closed in $[\mathcal{P}^{op}, Set]$ under coequalizers of equivalence relations. In fact, consider an equivalence relation in $\mathcal{P}_{ex}$, with its coequalizer in $[\mathcal{P}^{op}, Set]$

$$B \underset{\beta}{\overset{\alpha}{\rightrightarrows}} A \xrightarrow{c} C$$

We have to prove that C is in $\mathcal{P}_{ex}$. For this, consider the following diagram:

$$\begin{array}{ccccc}
K & \longrightarrow & B'' & \longrightarrow & Y(X) \\
\downarrow & & \downarrow & & \downarrow a \\
B' & \longrightarrow & B & \xrightarrow{\alpha} & A \\
\downarrow & & \beta \downarrow & & \downarrow c \\
Y(X) & \xrightarrow[a]{} & A & \xrightarrow[c]{} & C
\end{array}$$

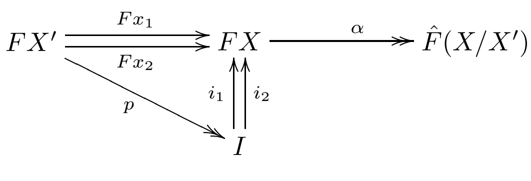
with each square except, possibly, the right-hand bottom one, is a pullback. The remaining square is, then, also a pullback because $[\mathcal{P}^{op}, Set]$ is exact, $X \in \mathcal{P}$ and a is a regular epimorphism. Since A, B and $Y(X)$ are in $\mathcal{P}_{ex}$, which is closed in $[\mathcal{P}^{op}, Set]$ under finite limits (see part 2.), also K is in $\mathcal{P}_{ex}$. So, K is a regular quotient of a representable object. But K is also the kernel pair of the regular epimorphism $c \cdot a\colon Y(X) \to C$. By Lemma 14.25, this means that C is in $\mathcal{P}_{ex}$. □

14.29 Theorem. *For every category $\mathcal{P}$ with weak finite limits the functor*

$$\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$$

of 14.22 is a free exact completion of $\mathcal{P}$.

Proof. 1: Extension of a left covering functor $F\colon \mathcal{P} \to \mathcal{B}$ to a functor $\hat{F}\colon \mathcal{P}_{ex} \to \mathcal{B}$. To define $\hat{F}$ on objects, consider an object $X/X' = (x_1, x_2\colon X' \rightrightarrows X)$ in $\mathcal{P}_{ex}$ and the relation (i_1, i_2) induced by $Fx_1, Fx_2\colon FX' \rightrightarrows X$ in $\mathcal{B}$



We define $\hat{F}(X/X')$ to be a coequalizer of (i_1, i_2) (or of (Fx_1, Fx_2)), which exists because, by Lemma 14.15, (i_1, i_2) is an equivalence relation in the exact category $\mathcal{B}$. To define $\hat{F}$ on morphisms in $\mathcal{P}_{ex}$

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Z' \\ {\scriptstyle x_1}\downdownarrows{\scriptstyle x_2} & & {\scriptstyle z_1}\downdownarrows{\scriptstyle z_2} \\ X & \xrightarrow[f]{} & Z \end{array}$$

let $\hat{F}[f]$ be the unique extension to the quotients as in the following diagram

$$\begin{array}{ccc} FX' & \xrightarrow{Ff'} & FZ' \\ {\scriptstyle Fx_1}\downdownarrows{\scriptstyle Fx_2} & & {\scriptstyle Fz_1}\downdownarrows{\scriptstyle Fz_2} \\ FX & \xrightarrow[Ff]{} & FZ \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ \hat{F}(X/X') & \xrightarrow[\hat{F}[f]]{} & \hat{F}(Z/Z') \end{array}$$

This definition does not depend on the choice of the premorphism (f', f). Indeed, if $\Sigma\colon X \to Z'$ gives an equivalence between premorphisms (f', f) and (g', g), then a diagram chase shows that $\hat{F}[f] \cdot \alpha = \hat{F}[g] \cdot \alpha$, so that $\hat{F}[f] = \hat{F}[g]$ because α is an epimorphism. Finally, the preservation of composition and identity morphisms by $\hat{F}$ comes from the uniqueness of the extension to the quotients. It is clear that $\hat{F} \cdot \Gamma$ is naturally isomorphic to F.
2: The extension $\hat{F}\colon \mathcal{P}_{ex} \to \mathcal{B}$ is essentially unique. More precisely, $\hat{F}$ is the essentially unique exact functor such that $\hat{F} \cdot \Gamma$ is naturally isomorphic to F. For this, consider an object $X/X' = (x_1, x_2\colon X' \rightrightarrows X)$ in $\mathcal{P}_{ex}$. From Lemma 14.25, we know that X/X' is a coequalizer as in the following diagram, where (i_1, i_2) is the relation induced by $(\Gamma x_1, \Gamma x_2)$

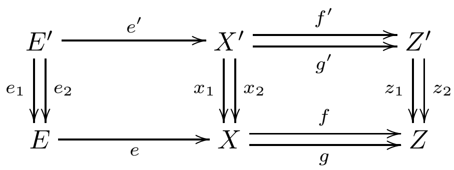
$$\begin{array}{ccccc} \Gamma X' & \overset{\Gamma x_1}{\underset{\Gamma x_2}{\rightrightarrows}} & \Gamma X & \overset{a}{\twoheadrightarrow} & X/X' \\ & {\scriptstyle e}\searrow & {\scriptstyle i_1}\upuparrows{\scriptstyle i_2} & & \\ & & I & & \end{array}$$

Moreover, since Γ is left covering, we know, due to Lemma 14.15, that (i_1, i_2) is an equivalence relation in the exact category $\mathcal{P}_{ex}$, thus it is a kernel pair of its coequalizer. Now, if $\hat{F}$ is exact and $\hat{F} \cdot \Gamma \simeq F$, then in the following diagram $(\hat{F}i_1, \hat{F}i_2)$ is the kernel pair of $\hat{F}a$, and $\hat{F}e$ and $\hat{F}a$ are regular epimorphisms:

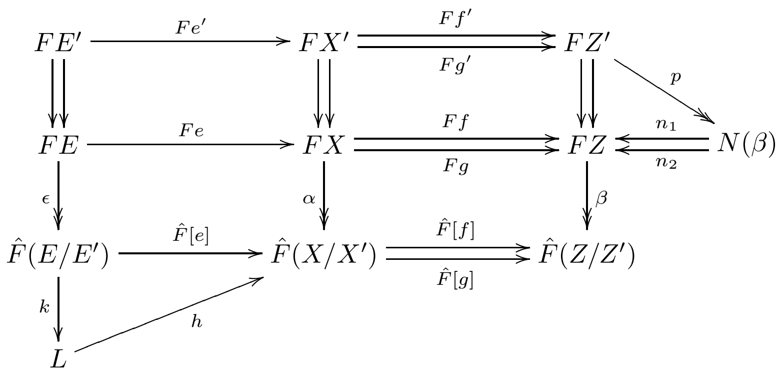
$$FX' \simeq \hat{F}(\Gamma X') \xrightarrow{\hat{F}e} \hat{F}I \overset{\hat{F}i_1}{\underset{\hat{F}i_2}{\rightrightarrows}} \hat{F}(\Gamma X) \simeq FX \xrightarrow{\hat{F}a} \hat{F}(X/X')$$

This implies that $\hat{F}(X/X')$ is necessarily a coequalizer of (Fx_1, Fx_2). In a similar way one shows that $\hat{F}$ is uniquely determined on morphisms.

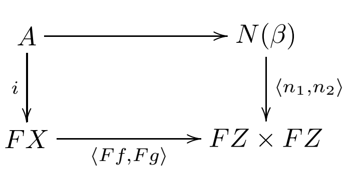
3: The extension $\hat{F}\colon \mathcal{P}_{ex} \to \mathcal{B}$ preserves finite limits. In fact, it is sufficient to show that $\hat{F}$ is left covering with respect to the terminal object, binary products and equalizers of pairs. By Lemma 14.16 and Lemma 14.19, this implies that $\hat{F}$ preserves finite limits. For each case, we use the description of the corresponding limit in $\mathcal{P}_{ex}$ given in the proof of Proposition 14.28, part 1.
3a: Equalizers: consider a parallel pair in $\mathcal{P}_{ex}$ together with its equalizer

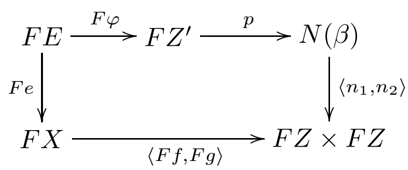


Consider also the following diagram in $\mathcal{B}$, where the triangle on the right is a regular factorization, and the triangle at the bottom is the factorization of $\hat{F}[e]$ through the equalizer L of $\hat{F}[f]$ and $\hat{F}[g]$:

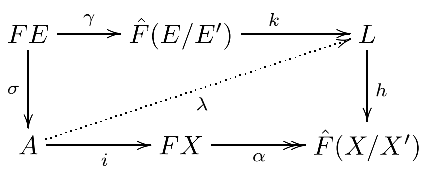


We have to prove that k is a regular epimorphism. For this, consider the following diagrams: the first one is a pullback, and the second one commutes.



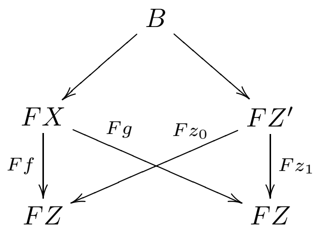


The unique factorization $\sigma\colon FE \to A$ makes the following diagram

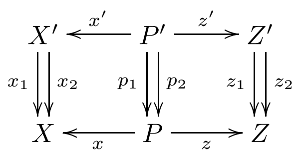


commutative. Below we will prove that σ is a regular epimorphism. Since h is a monomorphism, there is a morphism λ making the above diagram commutative.

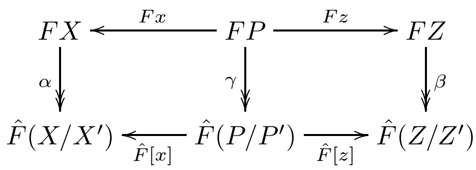
To prove that k is a regular epimorphism, it is enough to prove that λ is a regular epimorphism, and for this it suffices to show that λ is the pullback of α along h. If two morphisms x and y are such that $h \cdot x = \alpha \cdot y$, then $\beta \cdot Ff \cdot y = \beta \cdot Fg \cdot y$. But (n_1, n_2) is the kernel pair of β (by Lemma 14.15 and because $\mathcal{B}$ is exact), so that there is a morphism θ such that $n_1 \cdot \theta = Ff \cdot y$ and $n_2 \cdot \theta = Fg \cdot y$. This implies that the pair θ, y factorizes through A, and then also the pair x, y factorizes through A because h is a monomorphism. This factorization is certainly unique because i is a monomorphism.
It remains to prove that $\sigma \colon FE \to A$ is a regular epimorphism. Consider the following limit in $\mathcal{B}$



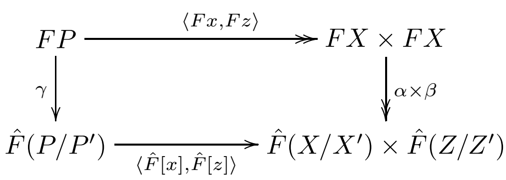
Since F is left covering, the canonical factorization $b \colon FE \to B$ is a regular epimorphism. But B is also a pullback of $\langle Ff, Fg \rangle \colon FX \to FZ \times FZ$ along $\langle Fz_1, Fz_2 \rangle \colon FZ' \to FZ \times FZ$, so that the canonical factorization $a \colon B \to A$ is a regular epimorphism. Finally, composing with the monomorphism i, one checks that $\sigma = a \cdot b$.
3b: Products: consider two objects $x_1, x_2 \colon X' \rightrightarrows X$ and $z_1, z_2 \colon Z' \rightrightarrows Z$ in $\mathcal{P}_{ex}$ and their product



Applying $\hat{F}$, we have the following diagram in $\mathcal{B}$



from which we get the following commutative diagram



The top morphism is a regular epimorphism because F is left covering, and the right-hand morphism is a regular epimorphism by Lemma 14.3, so that the bottom morphism also is a regular epimorphism, as requested.
3c: Terminal object: consider a weak terminal object T of $\mathcal{P}$, the terminal object $T/T \times T$ of $\mathcal{P}_{ex}$, and a terminal object T' of $\mathcal{B}$. The unique morphism $FT \to T'$ is a regular epimorphism (because F is left covering) and factorizes through the unique morphism $\hat{F}(T/T \times T) \to T'$ (by definition of $\hat{F}$). This implies that $\hat{F}(T/T \times T) \to T'$ is a regular epimorphism.
4: The extension $\hat{F} \colon \mathcal{P}_{ex} \to \mathcal{B}$ preserves regular epimorphisms. Consider an object A in $\mathcal{P}_{ex}$, presented as a coequalizer of a pseudoequivalence in $\mathcal{P}$:

$$\Gamma X' \underset{x_2}{\overset{x_1}{\rightrightarrows}} \Gamma X \overset{a}{\twoheadrightarrow} A$$

and a regular epimorphism $e \colon A \to B$. Cover the kernel pair $z_1, z_2 \colon N(e \cdot a) \rightrightarrows \Gamma X$ by a regular epimorphism $n \colon \Gamma Z' \to N(e \cdot a)$ (which exists because $\Gamma(\mathcal{P})$ is a regular projective cover of $\mathcal{P}_{ex}$). The unique factorization of (x_1, x_2) through (z_1, z_2) extends to a morphism $n' \colon \Gamma X' \to \Gamma Z'$ making the following diagram

$$\begin{array}{ccc}
\Gamma X' & \xrightarrow{n'} & \Gamma Z' \\
{\scriptstyle x_1}\downdownarrows{\scriptstyle x_2} & & {\scriptstyle z_1 \cdot n}\downdownarrows{\scriptstyle z_2 \cdot n} \\
\Gamma X & \xrightarrow[\mathrm{id}]{} & \Gamma X \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e \cdot a} \\
A & \underset{e}{\twoheadrightarrow} & B
\end{array}$$

commutative, where e is the extension to the coequalizers, and $(z_1 \cdot n, z_2 \cdot n)$ is a pseudoequivalence in $\mathcal{P}$ by Lemma 14.25. Applying $\hat{F}$, we get the following commutative diagram, which shows that $\hat{F}e$ is a regular epimorphism

$$\begin{array}{ccc}
FX & \xrightarrow{\mathrm{id}} & FX \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\
\hat{F}A & \xrightarrow[\hat{F}e]{} & \hat{F}B
\end{array}$$

□

14.30 Remark. Since the composition of the left covering functor $\Gamma \colon \mathcal{P} \to \mathcal{P}_{ex}$ with an exact functor $\mathcal{P}_{ex} \to \mathcal{B}$ clearly gives a left covering functor $\mathcal{P} \to \mathcal{B}$, the previous theorem can be restated in the following way: Composition with Γ induces an equivalence

$$- \cdot \Gamma \colon \mathit{Ex}[\mathcal{P}_{ex}, \mathcal{B}] \to \mathit{Lco}[\mathcal{P}, \mathcal{B}]\,,$$

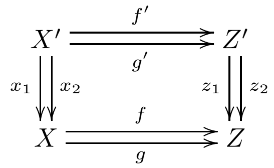
where $\mathit{Ex}[\mathcal{P}_{ex}, \mathcal{B}]$ is the category of exact functors from $\mathcal{P}_{ex}$ to $\mathcal{B}$ and natural transformations, and $\mathit{Lco}[\mathcal{P}, \mathcal{B}]$ is the category of left covering functors from $\mathcal{P}$ to $\mathcal{B}$ and natural transformations.

14.31 Remark. For later use, let us point out a simple consequence of the previous theorem. Consider the free exact completion $\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$ and a functor $K\colon \mathcal{P}_{ex} \to \mathcal{B}$, with $\mathcal{B}$ exact. If K preserves coequalizers of equivalence relations and $K \cdot \Gamma$ is left covering, then K is exact.

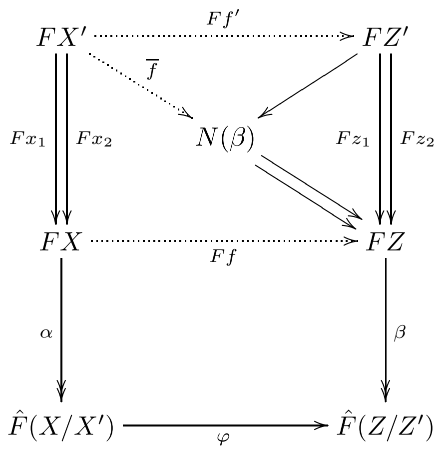
Our last result in this chapter is a characterization of exact categories which occur as free exact completions of categories with weak finite limits. From Proposition 14.28, we already know that $\mathcal{P}_{ex}$ has a regular projective cover (given by $\Gamma\colon \mathcal{P} \to \mathcal{P}_{ex}$). The converse also is true:

14.32 Proposition. *An exact category $\mathcal{A}$ is a free exact completion of a category with weak finite limits if and only if $\mathcal{A}$ has a regular projective cover.*

Proof. Let $\mathcal{A}$ be an exact category with a regular projective cover $F\colon \mathcal{P} \to \mathcal{A}$. The exact extension $\hat{F}\colon \mathcal{P}_{ex} \to \mathcal{A}$ of the full inclusion F is an equivalence of categories.
1: $\hat{F}$ is faithful: consider a parallel pair in $\mathcal{P}_{ex}$



If $\hat{F}[f] = \hat{F}[g]$, then the pair (Ff, Fg) factorizes through the kernel pair of the coequalizer $\beta\colon FZ \to \hat{F}(Z/Z')$ of Fz_1 and Fz_2. Since FX is regular projective, we get a morphism $\sigma\colon FX \to FZ'$ such that $Fz_1 \cdot \sigma = Ff$ and $Fz_2 \cdot \sigma = Fg$. Since F is full, there is $\Sigma\colon X \to Z'$ such that $F(\Sigma) = \sigma$. Since F is faithful, this shows that $[f] = [g]$.
2: $\hat{F}$ is full: consider two objects $x_1, x_2\colon X' \rightrightarrows X$ and $z_1, z_2\colon Z' \rightrightarrows Z$ in $\mathcal{P}_{ex}$, and a morphism $\varphi\colon \hat{F}(X/X') \to \hat{F}(Z/Z')$ as in the following diagram



Since FX is regular projective, we get $f: X \to Z$. Using that the equivalence relation $N(\beta) \rightrightarrows FZ$ is a kernel pair of its coequalizer β (because $\mathcal{A}$ is exact), we get $\overline{f}: FX' \to N(\beta)$. Using that FX' is regular projective, we get $f': X' \to Z'$. Clearly, $\hat{F}[f', f] = \varphi$.
3: $\hat{F}$ is essentially surjective: let A be an object in $\mathcal{B}$ and consider the following diagram

$$FX' \xrightarrow{x} N(a) \underset{a_2}{\overset{a_1}{\rightrightarrows}} FX \xrightarrow{a} A$$

where a and x are regular epimorphisms and $N(a)$ is a kernel pair of a. We have unique morphisms $x_i: X' \to X$ such that $Fx_i = a_i \cdot x$, for $i = 1, 2$. The pair $x_1, x_2: X' \rightrightarrows X$ is a pseudoequivalence in $\mathcal{P}$. In fact, we only need to check the transitivity. Consider the following diagram

$$\begin{array}{ccccc} FW & \xrightarrow{m} & P' & \xrightarrow{n} & P \\ {\scriptstyle F\tau}\downarrow & & & & \downarrow{\scriptstyle t} \\ FX' & & \xrightarrow{\quad x \quad} & & N(a) \end{array}$$

where P is a pullback of a_1 and a_2, t is the transitivity of (a_1, a_2), P' is a pullback of Fx_1 and Fx_2, W is a weak pullback of x_1 and x_2, and m and n are the canonical factorizations. Since FW is regular projective, x is a regular epimorphism and F is full, there is $\tau: W \to X'$ making the diagram commutative. A diagram chase using that F is faithful shows that τ is a transitivity for (x_1, x_2). □

14.33 Corollary.

1. *Let $\mathcal{A}$ and $\mathcal{B}$ be exact categories and $I: \mathcal{P} \to \mathcal{A}$ a regular projective cover. Consider two exact functors $G, G': \mathcal{A} \rightrightarrows \mathcal{B}$. If $G \cdot I \simeq G' \cdot I$, then $G \simeq G'$.*

2. *Let $\mathcal{A}$ and $\mathcal{A}'$ be exact categories, $\mathcal{P} \to \mathcal{A}$ a regular projective cover of $\mathcal{A}$ and $\mathcal{P}' \to \mathcal{A}'$ a regular projective cover of $\mathcal{A}'$. Any equivalence $\mathcal{P} \simeq \mathcal{P}'$ extends to an equivalence $\mathcal{A} \simeq \mathcal{A}'$.*

Chapter 15

Finitary localizations of algebraic categories

In 6.8, we characterized algebraic categories among cocomplete categories by the existence of a suitable generator. In this chapter, we will analogously characterize algebraic categories among exact categories.

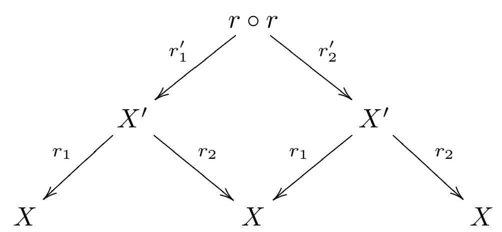
Let us recall from 8.19 that a limit preserving functor between algebraic categories preserves sifted colimits if and only if it preserves filtered colimits and regular epimorphisms (i.e., it is finitary and regular). This holds for more general categories:

15.1 Theorem. *Let $\mathcal{E}$ be a cocomplete exact category, $\mathcal{A}$ a category with colimits and finite limits, and $F\colon \mathcal{E} \to \mathcal{A}$ a finite limit preserving functor. Then F preserves sifted colimits if and only if it is finitary and regular.*

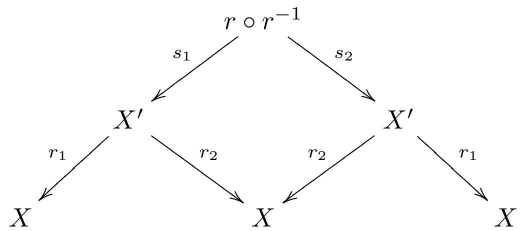
Proof. Necessity is evident, because filtered colimits and reflexive coequalizers are sifted colimits. Let F be finitary and regular. Then F preserves coequalizers of equivalence relations. Since every pseudoequivalence in $\mathcal{E}$ can be decomposed as a regular epimorphism followed by an equivalence relation (cf. 14.15), F preserves coequalizers of pseudoequivalences. Consider a reflexive and symmetric pair $r = (r_1, r_2\colon X' \rightrightarrows X)$ of morphisms in $\mathcal{E}$. We construct a pseudoequivalence $\overline{r}$ containing r (the *transitive hull* of r) as a (filtered) colimit of the chain of compositions

$$r \circ r \circ \ldots \circ r \qquad n\text{-times}$$

(the composition $r \circ r$ is depicted in the following diagram, where the square is a pullback)

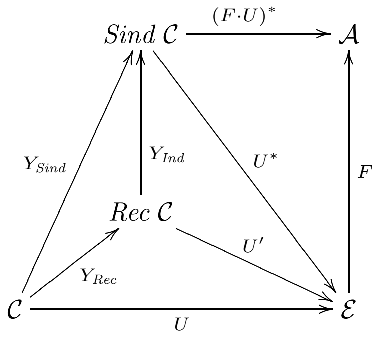


Since F preserves filtered colimits and finite limits, we have $\overline{F(r)} = F(\overline{r})$. The pseudoequivalence $\overline{r}$ has a coequalizer, which is preserved by F. But a coequalizer of $\overline{r}$ is also a coequalizer of r, and so F preserves coequalizers of reflexive and symmetric pairs of morphisms. If $r = (r_1, r_2)$ is just a reflexive pair, then a reflexive and symmetric pair containing r is given by $r \circ r^{-1}$, that is



Once again, a coequalizer of $r \circ r^{-1}$ is also a coequalizer of r, so that F preserves reflexive coequalizers.

Let $D \colon \mathcal{D} \to \mathcal{E}$ be a sifted diagram. Let $\mathcal{C}$ be the closure of $D(\mathcal{D})$ in $\mathcal{E}$ under finite coproducts and $U \colon \mathcal{C} \to \mathcal{E}$ the inclusion. Consider the following diagram of categories and functors



where U^* and $(F \cdot U)^*$ are the extensions of U and $F \cdot U$, respectively, preserving sifted colimits (4.16), and U' is the extension of U preserving finite colimits (13.6) - here we use the formula $\mathit{Ind}\ \mathit{Rec}\ \mathcal{C} = \mathit{Sind}\ \mathcal{C}$ of 13.8. Since

$$(F \cdot U)^* \cdot Y_{Ind} \cdot Y_{Rec} = (F \cdot U)^* \cdot Y_{Sind} = F \cdot U = F \cdot U' \cdot Y_{Rec}$$

and the functors $(F \cdot U)^*, U', Y_{Ind}$ and F preserve reflexive coequalizers (see 13.6, 13.9 and the first part of the proof), we have $(F \cdot U)^* \cdot Y_{Ind} \simeq F \cdot U'$ (13.7). Since

$$U^* \cdot Y_{Ind} \cdot Y_{Rec} = U^* \cdot Y_{Sind} = U = U' \cdot Y_{Rec}$$

and, once again, the functors U^*, Y_{Ind} and U' preserve reflexive coequalizers, we have $U^* \cdot Y_{Ind} \simeq U'$. Finally, since

$$F \cdot U^* \cdot Y_{Ind} = F \cdot U' = (F \cdot U)^* \cdot Y_{Ind}$$

and the functors F, U^* and $(F \cdot U)^*$ preserve filtered colimits, we have $F \cdot U^* \simeq (F \cdot U)^*$ (4.9). Hence $F(colim\ D) = colim\ (F \cdot D)$ and we have proved that F preserves sifted colimits. □

We can now improve 4.9.

15.2 Corollary. *In a cocomplete exact category, projectively finitely presentable objects are precisely finitely presentable regular projectives.*

Proof. One implication is established in 5.4. For the converse implication, apply 15.1 to the hom-functor $\hom(G, -)$ of a finitely presentable regular projective object G. □

15.3 Corollary. *A category is algebraic if and only if it is cocomplete, exact and has a strong generator consisting of finitely presentable regular projectives.*

Proof. Necessity follows from 3.12 and 6.8. Sufficiency follows from 15.2 and 6.8. □

In the previous corollary, the assumption of cocompleteness can be reduced to asking the existence of coequalizers of kernel pairs, which is part of the exactness of the category, and the existence of coproducts of objects from the generator. In fact, we have the following general lemma.

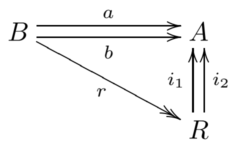
15.4 Lemma. *Let $\mathcal{A}$ be a well-powered exact category with a regular projective cover $\mathcal{P} \to \mathcal{A}$. If $\mathcal{P}$ has coproducts, then $\mathcal{A}$ is cocomplete.*

Proof. 1: The functor $\mathcal{P} \to \mathcal{A}$ preserves coproducts. Indeed, consider a coproduct

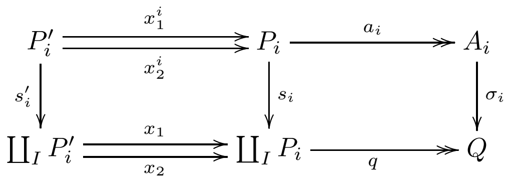
$$s_i \colon P_i \to \coprod_I P_i$$

in $\mathcal{P}$, and a family of morphisms $\langle x_i \colon P_i \to X \rangle_I$ in $\mathcal{A}$. Let $q \colon Q \to X$ be a regular epimorphism, with $Q \in \mathcal{P}$. For each $i \in I$, consider a morphism $y_i \colon P_i \to Q$ such that $q \cdot y_i = x_i$. Since Q is in $\mathcal{P}$, there is $y \colon \coprod_I P_i \to Q$ such that $y \cdot s_i = y_i$, and then $q \cdot y \cdot s_i = x_i$, for all $i \in I$.
As far as the uniqueness of the factorization is concerned, consider a pair of morphisms $f, g \colon \coprod_I P_i \rightrightarrows X$ such that $s_i \cdot f = s_i \cdot g$ for all i. Consider also $f', g' \colon \coprod_I P_i \rightrightarrows Q$ such that $q \cdot f' = f$ and $q \cdot g' = g$. Since $q \cdot f' \cdot s_i = q \cdot g' \cdot s_i$, there is $t_i \colon P_i \to N(q)$ such that $q_1 \cdot t_i = f' \cdot s_i$ and $q_2 \cdot t_i = g' \cdot s_i$, where $q_1, q_2 \cdot N(q) \rightrightarrows Q$ is a kernel pair of q. From the first part of the proof, we

obtain a morphism $t\colon \coprod_I P_i \to N(q)$ such that $t \cdot s_i = t_i$ for all i. Moreover, $q_1 \cdot t \cdot s_i = f' \cdot s_i$ for all i, so that $q_1 \cdot t = f'$ because Q is in $\mathcal{P}$. Analogously, $q_2 \cdot t = g'$. Finally, $f = q \cdot f' = q \cdot q_1 \cdot t = q \cdot q_2 \cdot t = q \cdot g' = g$.
2: Recall that $Sub_{\mathcal{A}}(A)$ is the ordered class of subobjects of A. For every category $\mathcal{A}$, we denote by $\theta(\mathcal{A})$ its “ordered reflection”, i.e., the ordered class obtained from the preorder on the objects of $\mathcal{A}$ given by $A \leq B$ iff $\mathcal{A}(A,B)$ is nonempty. We are going to prove that for any object A of $\mathcal{A}$, $Sub_{\mathcal{A}}(A)$ and $\theta(\mathcal{P}/A)$ are isomorphic ordered classes. In fact, given a monomorphism $m\colon X \to A$, we consider a $\mathcal{P}$-cover $q\colon Q \to X$ and we get an element in $\theta(\mathcal{P}/A)$ from the composition $m \cdot q$. Conversely, given an object $f\colon Q \to A$ in $\mathcal{P}/A$, the monomorphic part of its regular factorisation gives an element in $Sub_{\mathcal{A}}(A)$.
3: $\mathcal{A}$ has coequalizers. Consider a parallel pair (a, b) in $\mathcal{A}$ and its regular factorization



Consider now the equivalence relation $a_1, a_2\colon A' \rightrightarrows A$ generated by (i_1, i_2), that is the intersection of all the equivalence relations on A containing (i_1, i_2). Such an intersection exists: by part 2., $Sub_{\mathcal{A}}(A)$ is isomorphic to $\theta(\mathcal{P}/A)$, which is cocomplete because $\mathcal{P}$ has coproducts. Since, by assumption, $\mathcal{A}$ is well-powered, $Sub_{\mathcal{A}}(A)$ is a small set, and a cocomplete ordered set is also complete. Since $\mathcal{A}$ is exact, (a_1, a_2) has a coequalizer, which is also a coequalizer of (i_1, i_2) and then of (a, b).
4: $\mathcal{A}$ has coproducts. Consider a family of objects $(A_i)_I$ in $\mathcal{A}$. Each of them can be seen as a coequalizer of a pseudoequivalence in $\mathcal{P}$ as in the following diagram, where the first and the second columns are coproducts in $\mathcal{P}$ (and then in $\mathcal{A}$, because of part 1.), x_0 and x_1 are the extensions to the coproducts, the bottom row is a coequalizer (which exists by part 3.), and σ_i is the extension to the coequalizers.



Since coproducts commute with coequalizers, the third column is a coproduct of the family $(A_i)_I$. ☐

15.5 Corollary. *A category is algebraic if and only if it is exact and has a strong generator $\mathcal{G}$ consisting of finitely presentable regular projectives such that coproducts of objects of $\mathcal{G}$ exist.*

Proof. Let $\mathcal{A}$ be an exact category and $\mathcal{G}$ a strong generator consisting of regular projectives. Since a coproduct of regular projectives is regular projective, the full subcategory $\mathcal{P}$ consisting of coproducts of objects from $\mathcal{G}$ is a regular projective cover of $\mathcal{A}$. Following I.4.5.15 in [BOR], $\mathcal{A}$ is well-powered because it has a strong generator. By 15.4, $\mathcal{A}$ is cocomplete. □

From Propositions 3.12 and 6.16, we know that an algebraic category is exact and locally finitely presentable. The converse is not true because of the lack of projectivity of the generator. In the remaining part of this chapter we want to state in a precise way the relationship between algebraic categories and exact, locally finitely presentable categories.

15.6 Definition. Given a category $\mathcal{A}$, by a *localization* of $\mathcal{A}$ is meant a full, reflective subcategory whose reflector preserves finite limits. It is called a *finitary localization* if, moreover, it is closed in $\mathcal{A}$ under filtered colimits.

15.7 Remark. More loosely, we speak about localizations of $\mathcal{A}$ as categories equivalent to full subcategories having the above property. We use the notation

$$\mathcal{A} \underset{I}{\overset{R}{\rightleftarrows}} \mathcal{A}$$

that is, R is left adjoint to I and I is full and faithful.

Let us start with a general lemma.

15.8 Lemma. *Consider a reflection*

$$\mathcal{A} \underset{I}{\overset{R}{\rightleftarrows}} \mathcal{A}$$

1. *If I preserves filtered colimits and an object $P \in \mathcal{A}$ is finitely presentable, then $R(P)$ is finitely presentable;*

2. *If the reflection is a localization and $\mathcal{A}$ is exact, then $\mathcal{A}$ is exact.*

Proof. 1: Same argument as in the proof of 6.12.1.
2: Let $r_1, r_2 \colon A' \rightrightarrows A$ be an equivalence relation in $\mathcal{A}$. Its image in $\mathcal{A}$ is an equivalence relation, so that it has a coequalizer Q and it is the kernel pair of its coequalizer (because $\mathcal{A}$ is exact)

$$IA' \underset{I(r_2)}{\overset{I(r_1)}{\rightrightarrows}} IA \overset{q}{\twoheadrightarrow} Q$$

If we apply the functor R to this diagram, we obtain a coequalizer (because R is a left adjoint) and a kernel pair (because R preserves finite limits)

$$R(IA') \simeq A' \underset{r_2}{\overset{r_1}{\rightrightarrows}} A \simeq R(IA) \overset{Rq}{\twoheadrightarrow} RQ$$

and this means that (r_1, r_2) is effective. It remains to prove that regular epimorphisms are stable under pullbacks. For this, consider the following diagrams:

$$\begin{array}{ccc} P & \xrightarrow{f'} & C \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccccc} IP & \xrightarrow{e'} & Q & \xrightarrow{m'} & IC \\ {\scriptstyle Ig'}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Ig} \\ IA & \xrightarrow[e]{} & E & \xrightarrow[m]{} & IB \end{array}$$

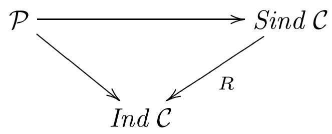
The first one is a pullback in $\mathcal{A}$, with f a regular epimorphism. The second one is the image of the first one in $\mathcal{A}$, computed as a two-step pullback of Ig along the regular factorization $m \cdot e$ of If, so that e' is a regular epimorphism. If we apply the functor R to the second one, we come back to the original pullback, computed now as a two-step pullback (because R preserves finite limits)

$$\begin{array}{ccccc} P \simeq R(IP) & \xrightarrow{Re'} & RQ & \xrightarrow{Rm'} & R(IC) \simeq C \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle Rh} & & \downarrow{\scriptstyle g} \\ A \simeq R(IA) & \xrightarrow[Re]{} & RE & \xrightarrow[Rm]{} & R(IB) \simeq B \end{array}$$

Now observe that Rm is a monomorphism (because R preserves finite limits) and also a regular epimorphism (because f is a regular epimorphism, and $Rm \cdot Re = f$), so that it is an isomorphism. It follows that Rm' is an isomorphism. Moreover, Re' is a regular epimorphism (because R, being a left adjoint, preserves regular epimorphisms). Finally, f' is a regular epimorphism because $f' = Rm' \cdot Re'$. □

15.9 Theorem. *Finitary localizations of algebraic categories are precisely the exact, locally finitely presentable categories.*

Proof. Since an algebraic category is exact and locally finitely presentable, necessity follows from 6.12.1 and 8.9. For the sufficiency, let $\mathcal{A}$ be an exact and locally finitely presentable category. Following 6.13, $\mathcal{A}$ is equivalent to $\mathit{Ind}\,\mathcal{C}$, where $\mathcal{C}$ consists of finitely presentable objects in $\mathcal{A}$. Since $\mathcal{C}$ is finitely cocomplete, $\mathcal{C}^{op}$ is an algebraic theory, and, following 4.3, $\mathit{Sind}\,\mathcal{C} = \mathit{Alg}(\mathcal{C}^{op})$. Following 6.18, $\mathit{Ind}\,\mathcal{C}$ is a reflective subcategory of $\mathit{Sind}\,\mathcal{C}$. Consider the full subcategory $\mathcal{P}$ of $\mathit{Sind}\,\mathcal{C}$ consisting of regular projective objects. Such an object P is a retract of a coproduct of representable algebras (5.11.2). Since every coproduct is a filtered colimit of its finite subcoproducts, and a finite coproduct of representable algebras is representable (4.1), P is a retract of a filtered colimit of representable algebras. Following 4.17, we have that $\mathcal{P}$ is contained in $\mathit{Ind}\,\mathcal{C}$. Moreover, following 5.12, $\mathcal{P}$ is a regular projective cover of $\mathit{Sind}\,\mathcal{C}$. Since, by 14.32, the free exact completion $\Gamma \colon \mathcal{P} \to \mathcal{P}_{ex}$ of $\mathcal{P}$ is equivalent to the full inclusion of $\mathcal{P}$ into $\mathit{Sind}\,\mathcal{C}$ and, by assumption, $\mathit{Ind}\,\mathcal{C}$ is exact, it remains just to prove that the inclusion $\mathcal{P} \to \mathit{Ind}\,\mathcal{C}$ is left covering. Once this done, we can apply 14.31 to the following situation



where R is the reflector, and we conclude that R is an exact functor. But the inclusion $\mathcal{P} \to Sind\,\mathcal{C} \simeq \mathcal{P}_{ex}$ is left covering, and $Ind\,\mathcal{C}$ is closed in $Sind\,\mathcal{C}$ under limits, so that also the inclusion $\mathcal{P} \to Ind\,\mathcal{C}$ is left covering. □

Chapter 16

Abelian categories

The categories *R-Mod* of left modules over a unitary ring R are algebraic and abelian. The aim of the present chapter is to prove that these are the only one-sorted abelian algebraic categories. We also prove the many-sorted generalization of this result.

16.1 Remark. In the following we use the standard terminology of the theory of abelian categories:

1. A *zero object* is an object 0 which is initial as well terminal. For two objects A, B the composite $A \to 0 \to B$ is denoted by $0\colon A \to B$.

2. A *biproduct* of objects A and B is a product $A \times B$ with the property that the morphisms

   $$\langle \mathrm{id}_A, 0\rangle\colon A \to A \times B \quad \text{and} \quad \langle 0, \mathrm{id}_B\rangle\colon B \to A \times B$$

   form a coproduct of A and B.

3. A category is called *preadditive* if it is enriched over the category Ab of abelian groups, i.e., if every hom-set carries the structure of an abelian group such that composition is a group homomorphism.

4. In a preadditive category, an object is a zero object iff it is terminal, and a product of two objects is a biproduct. A preadditive category with finite products is called *additive*.

5. A functor $F\colon \mathcal{A} \to \mathcal{A}'$ between preadditive categories is called *additive* if it is enriched over Ab, i.e., the derived functions $\mathcal{A}(A, B) \to \mathcal{A}'(FA, FB)$ are group homomorphisms. In case of additive categories this is equivalent to the preservation of finite products.

6. Finally, a category is called *abelian* if it is exact and additive.

16.2 Example. Just as one-object categories are precisely the monoids, one-object preadditive categories are precisely the unitary rings. Every left R-module M defines an additive functor $\overline{M}\colon R \to Ab$ with $\overline{M}(*) = M$ and $\overline{M}(r) = r\cdot - \colon M \to M$, for $r \in R$. Conversely, every additive functor $F\colon R \to Ab$ is naturally isomorphic to $\overline{M}$, for $M = F(*)$.

For a small, preadditive category $\mathcal{C}$, we denote by $Add\,[\mathcal{C}, Ab]$ the category of all additive functors into Ab (and all natural transformations). The previous example implies that R-Mod is equivalent to $Add\,[R, Ab]$.

16.3 Theorem. *The following conditions on a category $\mathcal{A}$ are equivalent:*

1. *$\mathcal{A}$ is an abelian algebraic category;*
2. *$\mathcal{A}$ is equivalent to $Add\,[\mathcal{C}, Ab]$ for a small additive category $\mathcal{C}$;*
3. *$\mathcal{A}$ is equivalent to $Add\,[\mathcal{C}, Ab]$ for a small preadditive category $\mathcal{C}$.*

Proof. $2 \Rightarrow 1$: For every small additive category $\mathcal{C}$, we prove that $Add\,[\mathcal{C}, Ab]$ is equivalent to $Alg\,\mathcal{C}$. For this, consider the forgetful functor $U\colon Ab \to Set$. Since U preserves finite products, it induces a functor

$$\widehat{U} = U \cdot - : Add\,[\mathcal{C}, Ab] \to Alg\,\mathcal{C}$$

Let us prove that $\widehat{U}$ is an equivalence functor.
(a) $\widehat{U}$ is faithful: obvious, because U is faithful.
(b) $\widehat{U}$ is full: in fact, we first observe that $\widehat{U}$ preserves sifted colimits. This follows from the fact that sifted colimits commute in Ab (as in any algebraic category, see 2.4.1) with finite products, and the functor $U = \hom(\mathbb{Z}, -)$ preserves sifted colimits. Let objects $C \in \mathcal{C}, G \in Add\,[\mathcal{C}, Ab]$ and a natural transformation $\alpha\colon \hom(C, -) \to \widehat{U}(G)$ be given. By the Yoneda lemma, for all $X \in \mathcal{C}$ and for all $x\colon C \to X$, we have $\alpha_X(x) = G(x)(a)$, where $a = \alpha_C(\mathrm{id}_C)$, so that α_X is a group homomorphism. The general case of a morphism $\beta\colon \widehat{U}(F) \to \widehat{U}(G)$ reduces to the previous one using the fact that F is a sifted colimit of representables and that $\widehat{U}$ preserves sifted colimits. To see that F is a filtered colimit of representables, observe that, following 4.3, $\widehat{U}(F)$ is a sifted colimit of representables. Now

$$\widehat{U}(F) = colim\, \hom(C_i, -) = colim\,(U \cdot \hom(C_i, -)) =$$

$$= colim\, \widehat{U}(\mathrm{Hom}(C_i, -)) = U(colim\, \mathrm{Hom}(C_i, -))\,.$$

This implies $F = colim\, \mathrm{Hom}(C_i, -)$ because $\widehat{U}$ reflects sifted colimits (since it preserves sifted colimits and reflects isomorphisms).
(c) $\widehat{U}$ is essentially surjective on objects: first, consider objects $C, C' \in \mathcal{C}$ and the representable functor $\hom(C, -)\colon \mathcal{C} \to Set$, which is an object of $Alg\,\mathcal{C}$. Since $\mathcal{C}$ is preadditive, $\hom(C, C')$ is an abelian group, and $\hom(C, -)$ factorizes as

$$\mathcal{C} \xrightarrow{\mathrm{Hom}(C,-)} Ab \xrightarrow{U} Set$$

with $\mathrm{Hom}(C, -)\colon \mathcal{C} \to Ab$ additive. Once again, the general case follows from the previous one using the fact that any $\mathcal{C}$-algebra is a sifted colimit of representable $\mathcal{C}$-algebras and that $\widehat{U}$ preserves sifted colimits.
$1 \Rightarrow 2$: Let $\mathcal{T}$ be an algebraic theory, and assume that $Alg\,\mathcal{T}$ is abelian. Since $\mathcal{T}^{op}$ embeds into $Alg\,\mathcal{T}$, $\mathcal{T}$ is preadditive (with finite products), and then it is a small additive category. Following the first part of the proof, $Alg\,\mathcal{T}$ is equivalent to $Add\,[\mathcal{T}, Ab]$.
$3 \Rightarrow 2$: Let $\mathcal{C}$ ba a small preadditive category. We can construct the small and preadditive category $Mat\,(\mathcal{C})$ of matrices over $\mathcal{C}$ as follows:

- Objects are finite (possibly empty) families $(X_i)_{i\in I}$ of objects of $\mathcal{C}$;
- Morphisms from $(X_i)_{i\in I}$ to $(Z_j)_{j\in J}$ are matrices $M = (m_{i,j})_{(i,j)\in I\times J}$ of morphisms $m_{i,j}\colon X_i \to Z_j$ in $\mathcal{C}$;
- The matrix multiplication, the identity matrices, and matrix addition, as well known from Linear Algebra, define the composition, the identity morphisms and the preadditive structure, respectively.

This new category $Mat\,(\mathcal{C})$ is additive. Indeed, it has a zero object given by the empty family, and biproducts $\oplus$ given by disjoint unions. Let us check that the obvious embedding $\mathcal{C} \to Mat\,(\mathcal{C})$ induces an equivalence between $Add\,[Mat\,(\mathcal{C}), Ab]$ and $Add\,[\mathcal{C}, Ab]$. Indeed, given $F \in Add\,[\mathcal{C}, Ab]$, we get an extension $F' \in Add\,[Mat\,(\mathcal{C}), Ab]$ in the following way: $F'(M)$ is the unique morphism such that the following square

$$\begin{array}{ccc} \bigoplus_I F(X_i) & \xrightarrow{F'(M)} & \bigoplus_J F(Z_j) \\ \uparrow & & \downarrow \\ F(X_i) & \xrightarrow[F(m_{i,j})]{} & F(Z_j) \end{array}$$

commutes for all $(i, j) \in I \times J$, where the vertical morphisms are injections in the coproduct and projections from the product, respectively. It is easy to verify that the functor $F \mapsto F'$ is an equivalence from $Add\,[\mathcal{C}, Ab]$ to $Add\,[Mat\,(\mathcal{C}), Ab]$.
$2 \Rightarrow 3$: Obvious. □

16.4 Corollary. *The following conditions on a category $\mathcal{A}$ are equivalent:*

1. *$\mathcal{A}$ is equivalent to $Add\,[\mathcal{C}, Ab]$ for a small additive category $\mathcal{C}$;*
2. *$\mathcal{A}$ is additive, cocomplete, and has a strong generator consisting of projectively finitely presentable objects.*

Proof. It follows from 6.8 and 16.3. □

16.5 Remark. Observe that an object G of an additive, cocomplete category $\mathcal{A}$ is projectively finitely presentable iff its enriched hom-functor $\mathrm{Hom}(G, -)\colon \mathcal{A} \to Ab$ preserves colimits. (Compare with the absolutely presentable objects of

5.6.) In fact, if G is projectively finitely presentable, then $\mathrm{Hom}(G, -)$ preserves finite coproducts (because they are finite products) and reflexive coequalizers (because $U: Ab \to Set$ reflects them). This implies that $\mathrm{Hom}(G, -)$ preserves finite colimits. Indeed, given a parallel pair $a, b: X \rightrightarrows Z$ in $\mathcal{A}$, its coequalizer is precisely the coequalizer of the reflexive pair $(a, \mathrm{id}_Z), (b, \mathrm{id}_Z): X + Z \rightrightarrows Z$. Finally, $\mathrm{Hom}(G, -)$ preserves arbitrary colimits because they are filtered colimits of finite colimits.

16.6 Example. The group $\mathbb{Z}$ is projectively finitely presentable in *Ab*; indeed, $\mathrm{Hom}(\mathbb{Z}, -): Ab \to Ab$ is naturally isomorphic to the identity functor. Observe that $\mathbb{Z}$ is of course not absolutely presentable.

16.7 Corollary. *One-sorted abelian algebraic categories are precisely the categories equivalent to R-Mod for a unitary ring R.*

Proof. Following 16.3, a one-sorted abelian algebraic category $\mathcal{A}$ is of the form $Add\,[\mathcal{T}, Ab]$ for $\mathcal{T}$ a one-sorted additive algebraic theory with objects T^n ($n \in \mathbb{N}$). Any $F \in Add\,[\mathcal{T}, Ab]$ restricts to an additive functor $\mathcal{T}(T, T) \to Ab$, where the ring $\mathcal{T}(T, T)$ is seen as a preadditive category with a single object. Moreover, F is uniquely determined by such a restriction, because each object of $\mathcal{T}$ is a finite product of T. Finally, $Add\,[\mathcal{T}(T, T), Ab]$ is equivalent to $\mathcal{T}(T, T)$-*Mod*. ☐

16.8 Corollary. *Abelian, locally finitely presentable categories are precisely the finitary localizations of the categories* $Add\,[\mathcal{C}, Ab]$.

Proof. Let $\mathcal{A}$ be an abelian, locally finitely presentable category. Following the proof of 15.9, we have that $\mathcal{A} = Ind\,\mathcal{C}$ is a finitary localization of $Sind\,\mathcal{C}$, with $\mathcal{C}$ an additive algebraic theory. Following the proof of 16.3, $Sind\,\mathcal{C}$ is equivalent to $Add\,[\mathcal{C}, Ab]$. ☐