

# KALMAN KNIZHNIK - DERIVATION OF THE STEFAN-BOLTZMANN LAW

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Here we derive the Stefan-Boltzmann law, which describes the energy density spectrum of blackbody radiation. We begin by modeling the energy spectrum of a photon gas as a simple harmonic oscillator, with energy levels given by

$$E_n = \hbar\omega(n + \frac{1}{2}) \quad (1)$$

The partition function for the simple harmonic oscillator is easily derived from this energy spectrum:

$$Z = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+\frac{1}{2})} = e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} \quad (2)$$

The average thermal energy of the photon gas is, then,

$$\langle U(\omega) \rangle = -\frac{d(\ln Z)}{d\beta} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \quad (3)$$

To obtain the total energy, we need to integrate equation 3 over all possible frequencies. Thus,

$$U = \int_0^{\infty} \langle U(\omega) \rangle g(\omega) d\omega \quad (4)$$

where  $g(\omega)$  is the multiplicity of the energy for a given frequency. In other words, there may be several frequencies that each have the same energy. To derive what  $g(\omega)$  is, we need to calculate the density of states of the wavenumber  $k$ :

$$g(k) = 2 \times \frac{V}{(2\pi)^3} 4\pi k^2 = \frac{V k^2}{\pi^2} \quad (5)$$

where the factor of 2 corresponds to the two possible polarizations of the photons, the factor of  $V$  is meant to restrict the number of wave numbers to a box of volume  $V$ , and the  $(2\pi)^3$  is generally conventional. Finally, the  $4\pi k^2$  corresponds to the area of a sphere of radius  $k$ . Thus,  $g(k)$  represents the number of wave vectors that correspond to the value  $k$ . To obtain  $g(\omega)$  from this, we note that  $g(\omega)d\omega = g(k)dk$ , and so

$$g(\omega) = g(k) \frac{dk}{d\omega} = \frac{V k^2}{\pi^2 c} = \frac{V \omega^2}{\pi^2 c^3} \quad (6)$$

where in the last two equalities I have used the fact that  $\omega = ck$ . We can now rewrite equation 4 as

$$U = \int_0^{\infty} \left[ \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \frac{V \omega^2}{\pi^2 c^3} \right] d\omega \quad (7)$$

We immediately see a problem, in that the first integral, which corresponds to the energy of the vacuum, obviously diverges. Oops. Fortunately, some smart people decided that we can redefine our energy scale such that the energy of the vacuum is zero, meaning that we are ignoring this infinite result. Odd, but it works. This integral now reduces to

$$U = \frac{\hbar V}{\pi^2 c^3} \int_0^{\infty} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} d\omega \quad (8)$$

If we rewrite this in terms of the energy *density*, this is:

$$u \equiv \frac{U}{V} = \int_0^{\infty} u_{\omega} d\omega \equiv \frac{\hbar}{\pi^2 c^3} \int_0^{\infty} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} d\omega \quad (9)$$

where

$$u_{\omega} = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \quad (10)$$

is known as a **blackbody distribution**. We can integrate equation 9 by changing variables, and letting  $x = \beta\hbar\omega$ . Then  $d\omega = dx/\beta\hbar$ . The integral becomes:

$$u = \frac{1}{\pi^2 c^3 \hbar^2} \frac{1}{\beta^4} \int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{k_B^4 T^4}{\pi^2 c^3 \hbar^2} \Gamma(4) \zeta(4) = \frac{\pi^2 k_B^3}{15 \hbar^2 c^3} T^4 \quad (11)$$

This equation tells us that the energy density of a photon gas depends only on the temperature, and specifically on the fourth power of the temperature. We are almost at the Stefan-Boltzmann law. To get there, we quote a result from kinetic theory, namely that the flux of particles striking a unit area of a container is

$$\Phi = \frac{1}{4} n \langle v \rangle = \frac{nc}{4} \quad (12)$$

where I have replaced  $v$  with  $c$ , and using the fact that power emitted is the energy  $\hbar\omega$  times the flux, we get

$$P = \hbar\omega\Phi = \frac{n\hbar\omega}{4} c \quad (13)$$

But  $n\hbar\omega = u$  is the energy density (remember that here  $n$  is not the quantum number but the number density). Thus, we arrive at the **Stefan-Boltzmann Law**

$$P = \frac{uc}{4} = \frac{\pi^2 k_B^3}{60 \hbar^2 c^2} T^4 \equiv \sigma T^4 \quad \square \quad (14)$$

where  $\sigma$  is the Stefan-Boltzmann constant.

Alternatively, we can skip the whole blackbody argument, and simply quote the result, also from kinetic theory, that the pressure of a gas of particles with number density  $n$  is

$$p = \frac{1}{3} nm \langle v^2 \rangle \Rightarrow p_\gamma = \frac{1}{3} nmc^2 \quad (15)$$

and we can interpret  $mc^2$  as the energy of a single photon. Then  $nmc^2$  is the energy density of a photon gas, so

$$p_\gamma = \frac{u}{3} \quad (16)$$

Now we use the first law of thermodynamics,  $dU = TdS - pdV$  to obtain

$$\left. \frac{\partial U}{\partial V} \right|_T = T \left. \frac{\partial S}{\partial V} \right|_T - p = T \left. \frac{\partial p}{\partial T} \right|_V - p \quad (17)$$

where the last equality follows from a Maxwell relation. Now, the left hand side is simply the energy density  $u$ , and plugging this in above, along with equation 16 gives

$$u = \frac{T}{3} \left. \frac{\partial u}{\partial T} \right|_V - \frac{u}{3} \Rightarrow \frac{du}{4u} = \frac{dT}{T} \quad (18)$$

leading immediately to  $u = AT^4$ , with  $A$  as some constant of integration. We then use equation 13, noting that  $n\hbar\omega \equiv u$ , to arrive, once again, at

$$P = \frac{1}{4} uc = \left(\frac{1}{4} Ac\right) T^4 \equiv \sigma T^4 \quad \square \quad (19)$$

which is again the Stefan-Boltzmann law. This derivation does not tell us what the constant is, but it is good enough to tell us the temperature dependence of the power emitted by a blackbody.