Problem 1

Compute

$$\int_{\Gamma} \frac{1}{z^2 + 1} dz$$

where $\Gamma = \{z \mid |z| = 2\}$.

Solution

Function $f(z) = \frac{1}{z^2+1}$ has two isolated singular points $z_1 = i; z_2 = -i$ that are simple poles. These points are inside the circle $|z| = 2$.

Let's find residues of the function f at points z_1, z_2.

$$res_{z=i} f(z) = \lim_{z\to i} (z - i) f(z) = \lim_{z\to i} \frac{1}{z + i} = \frac{1}{2i}$$

$$res_{z=-i} f(z) = \lim_{z\to -i} (z + i) f(z) = \lim_{z\to -i} \frac{1}{z - i} = -\frac{1}{2i}$$

Using the formula

$$\int_{\Gamma} f(z) dz = 2\pi i \big(res_{z=i} f(z) + res_{z=-i} f(z) \big) = 2\pi i \left(\frac{1}{2i} - \frac{1}{2i} \right) = 0$$

we get that integral equals to 0.

Problem 2

Compute

$$\int_\Gamma z e^{\frac{1}{z-1}} dz$$

where Γ is the boundary of the region $|z-1| < 2$.

Solution

Function $f(z) = ze^{\frac{1}{z-1}}$ has one isolated singular point $z = 1$ inside the circle $|z-1| = 2$. Laurent series for the function $f(z) = ze^{\frac{1}{z-1}}$ about a point $z = 1$ is given by

$$f(z) = z + 1 + \sum_{n=1}^{\infty} \frac{n+2}{(n+1)!\,(z-1)^n}, |z-1| > 0$$

Residual of function f at point $z = 1$ equals to the coefficient c_{-1} at $\frac{1}{z-1}$ of the Laurent series of the function f.

Thus $c_{-1} = res_{z=1} ze^{\frac{1}{z-1}} = \frac{3}{2}$. So

$$\int_\Gamma z e^{\frac{1}{z-1}} dz = 2\pi i \cdot res_{z=1} f(z) = 2\pi i \cdot \frac{3}{2} = 3\pi i$$