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Inventiones mathematicae

Pudet dicere, ad quot figurarum loca has computationes, otiosus eo tempore, produxi. Nam tunc sane nimis delectabar inventis hisce.

I. Newton, Letter to Oldenburg, October 24, 1676

Local Normal Forms of Functions

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Introduction

This article contains lists of normal forms of functions in the neighbourhoods of critical points (the classification of all singularities with the number of modules $m=0$, 1 and 2 or with multiplicity $\mu \leqq 16$ included) and the proofs of most of these classification theorems (the classification of unimodular singularities included).

For instance, the number ν of stable μ-equivalence classes (see the definitions below), is given for $\mu \leqq 16$ by the following table:

μ	**1**	**2**	3	**4**	5	6	7	**8**	9	10	11	12	13	14	15	**16**
ν	**1**	**1**	1	**2**	2	3	3	**4**	4	7	11	15	14	17	22	**32**

The motivations for and the applications of the classification of critical points of functions, and their relation to Coxeter groups, braids, automorphic functions, platonics, caustics, wavefronts and stationary phase method are discussed in [1].

The technical results are formulated in §§ 1 and 2. We begin with some general remarks, arising from these empirical facts.

1. *Normal Forms*

For many classes of singularities there exist simple normal forms.

The following definitions give an exact meaning to these words.

The group of germs (or jets) of diffeomorphisms of $\mathbb{C}^n$, leaving O invariant, acts on the space of germs (jets) of functions at O. A *singularity class* is a subset of the space of germs of functions, invariant under this action. Each orbit is such a class. Two germs (jets) belonging to the same orbit are *equivalent*.

Another example of a class is the *μ-equivalence class*. The multiplicity (or the Milnor number) μ of a critical point $O \in \mathbb{C}^n$ of a function f is the Poincaré index of the vector field grad f at the singular point O. Germs of functions f_0, f_1 at a critical point O are *μ-equivalent*, if there exist a holomorphical familie of functions f_t with critical point O of constant multiplicity μ (independent of t, $0 \leqq t \leqq 1$).

To define the *normal forms*, let us consider the space of polinomials, $M=\mathbb{C}[x_1,\dots,x_n]$, as a subset in the space of germs of functions $f(x_1,\dots,x_n)$ at O.

The *normal form for a given class K of functions* is a mapping $\Phi\colon B\to M$ from a finite dimensional linear "parameter space" B to the space of polynomials, satisfying three conditions:

1. $\Phi(B)$ intersects all the orbits, belonging to K;
2. the counterimage of every orbit in B is a finite set;
3. the counterimage of the whole complement to K lies in some hypersurface in B.

The normal form is *polynomial* (resp. *affine*) if Φ is a polynomial (resp. affine) mapping. An affine normal form is *simple*, if Φ is of the form:

$$\Phi(b_1,\dots,b_r)=\varphi+b_1\mathbf{x}^{\mathbf{m}_1}+\dots+b_r\mathbf{x}^{\mathbf{m}_r},$$

where φ is a fixed polynomial, b_i are complex numbers and $\mathbf{x}^{\mathbf{m}_i}$ are monomials. In most applications the polynomial φ itself is "simple", that is – a sum of *few* monomials.

The existence of a unique normal form (even a polynomial one) for the whole μ-equivalence class is by no mean evident a'priori. It is a rather surprising empirical fact, that such normal forms exist for all the singularities in the lists of § 1 (and hence for all the singularities with modules number $m=0,1$ and 2).

Most of our normal forms are simple; perhaps, all singularities of § 1 have simple normal forms. It is not known, how vast is the class of singularities, whose μ-equivalence classes admit simple (or polynomial) normal forms (here it is natural to consider stable equivalence classes, see 5).

2. *The Series of Singularities*

All singularities in our lists are divided into some *series*, such as A_k or $\mathbf{Z}_k$. We shall use light characters $A,\dots,Z$ to denote μ-equivalence classes and bold characters $\mathbf{A},\dots,\mathbf{Z}$ to denote singularity classes which contain more than one μ-equivalence classes.

While the series undoubtely exist, they have yet no formal definition. Let us consider, for instance, series $\mathbf{A}$ and $\mathbf{D}$, formed by the orbits of germs

$$A_k=x^{k+1}+y^2\,(k\geqq 1),\qquad D_k=x^2y+y^{k-1}\,(k\geqq 4).$$

The A_k and D_k classes adjacences are

$$\begin{array}{ccccccccc} A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & A_4 & \leftarrow & \cdots \\ & & & & \uparrow & & \uparrow & & \\ & & & & D_4 & \leftarrow & D_5 & \leftarrow & \cdots. \end{array}$$

[A class L of singularities is *adjacent* to class K, $K\leftarrow L$, if every germ (sufficient jet) of $f\in L$ can be deformed into a germ (sufficient jet) in K by an arbitrary small deformation.]

It is clear, that A_k and D_k are two different series. What is, however, the exact meaning of this?

To define series **A** is to define such a rule of reversing adjacency arrows, that leads from A_k to A_{k+1} and not to D_{k+1}. In the case of **A** singularities the rule is easy to find (**A** singularities have corank 1). In more complicated cases (see §§ 1, 2) one can also find the rules of reversing arrows (different in different cases). So, series with one or more subscripts and parameters occur (*e.g.*, $T_{k,l,m}=axyz+x^k+y^l+z^m$).

In all these cases one can find the definition of the series *after* one have found the series, while no a'priori general definition is known. It is clear, however, that the series are connected with singularities of infinite multiplicity (*e.g.* $\mathbf{D}\sim x^2y$, $\mathbf{T}\sim xyz$). So the hierarchy of series reflects the hierarchy of nonisolated singularities.

3. *Periodicity*

The decomposition of many singularity classes into μ-equivalence classes is in some sence periodic. The whole decomposition (stratification) can be regarded as a chain of similar fragments ("animals"). Each animal is built of points (μ-equivalence classes), two of which are distinguished (the "head" and the "tail"). An animal can also contain some classes whose adjacency arrows join the tail to the head and some "legs" (infinite series of classes). The tail of each animal in the chain is adjacent to the head of the following one. For example, the stratification of the corank 2 singularities with the 3-jet equal to x^3 is a chain of animals each consisting of 5 points and 1 infinite leg:

$$\begin{tikzcd} \circ & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l] & \circ \arrow[l] \\ & \bullet \arrow[u] & \cdots \arrow[l] \end{tikzcd} \qquad (J_k \text{ in the list of } \S\,1).$$

The phenomenon of periodicity is only partialy explained and for quasihomogenous singularities only (the explanation is based upon some root technique for the quasihomodenous Lie algebra, related to that of Enriques and Demazure [2], see § 3). However this periodicity arises in all the calculations, connected with our classification theorems (so that it is sufficient to consider only first animal from every animal chain). This empirical periodicity have at present no satisfactory explanation. Like the existence of series, it suggests some algebraic structure of the set of all μ-equivalence classes.

4. *Small Modules Number Classes*

The main characteristic of a singularity class is from the point of view of applications its codimension c in the space of germs of functions with critical point O and critical value 0. Indeed, a generic function has only nondegenerate singularities (singularities with codimension $c=0$). Degenerate singularities appears inevitably only when *families* of functions depending upon parameters are considered. A class **K** of codimension c is unavoidable by small deformation if the number of parameters l is $\geqq c$.

So in all the applicatious to l-parametrical problems we need classification of all singularities *up to codimension* l (that is, we need classify such a set of classes, that the complement to their union be of codimension larger than l).

This problem is not to be confused with the classification of singularities with *orbit codimension* $\leqq l$ (that is, with $\mu \leqq l+1$). The only value of the last problem for applications is its relation with the former one.

From the topological point of view the main characteristic of a singularity is the *multiplicity* μ of the critical point (equal to the number of nondegenerate critical points, to which a given point bifurcate after a small variation; not to be confused with the algebraico-geometrical multiplicity of the hypersurface $f^{-1}(0)$ at O).

An unexpected result of our calculation is the fact, that *algebraically the most natural classification theorems are not those of singularities of small codimension c or of small multiplicity* μ, *but the classification of singularities with small modules number, m.*

The *modules number m* of a germ of function $f\colon (\mathbb{C}^n, O) \to (\mathbb{C}, O)$ at critical point O is the minimal number M, such that some neighbourhood of a sufficient k-jet of the function f at O is covered by a finite number of no more than M-parametrical families of the orbits of the group of diffeomorphisms germs $(\mathbb{C}^n, O) \to (\mathbb{C}^n, O)$ acting on the space of functions with critical point O and critical value 0.

The modules number m is equal to the dimension of the $\mu=$const stratum in the base space of the versal deformation minus 1 (see Gabrielov [3]). So the codimension c of this stratum in the space of germs of functious with critical value 0 at the critical point O, the multiplicity μ and the modules number m are related by the formula

$$\mu = c + m + 1.$$

At present, the complete classifications are known for:

(i) all singularities with $c \leqq 10$;
(ii) all singularities with $\mu \leqq 16$;
(iii) all singularities with $m \leqq 2$.

The singularities with the modules number $m=0$, 1 and 2 are called *simple, unimodular* and *bimodular* respectively. Lists of them are given in § 1. A study of these lists leads to the following 4 conclusions:

1. The simple singularities are classified by the Coxeter groups A_k, D_k, E_6, E_7, E_8 (that is, by the platonic in the usual 3-space, see [4]).

2. The unimodular singularities form one infinite series $T_{k,l,m}$ and 14 "exceptional" families, generated by 14 quasihomogenous singularities ([5]).

All the quasihomogenous unimodular singularities can be constructed from 14 remarkable Lobatchevski triangles and 3 remarkable Euclidean triangles like simple singularities are constructed from platonics (Dolgatchev [6]).

3. The bimodular singularities form 8 infinite series and 14 "exceptional" families, generated by quasihomogenous singularities.

All the quasihomogenous bimodular singularities can be constructed from 6 Lobatchevski quadrangles and 14 Lobatchevski triangles. In the last case one has to consider 2-, 3- and 5-fold coverings of the surface, constructed in the canonical way (Dolgatchev).

4. All the singularities with the modules number $m=1$ and 2 are classified by the degenerations of elliptic curves, studied by Kodaira (Koulikov [7]). The Koulikov construction begins with a blowing up of 1, 2 or 3 points of the surface on the minimal resolution of the degenerate fiber; then one has to blow down this (resolved) fiber.

Unfortunately, all these coincidence of different classifications have only a' posteriori proofs and thus depend on our calculations.

5. *Stabilization*

Two germs $f: (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ and $g: (\mathbb{C}^m, O) \to (\mathbb{C}, 0)$ are *stably equivalent*, if they become equivalent after a direct addition to both of nondegenerate quadratic forms (*e.g.*, $f(x)=x^3$ is stably equivalent to $g(x, y)=x^3+y^2$, but not to $h(x, y)=x^3$). Two stably equivalent germs on equidimensional spaces are equivalent. In § 1 we shall not show explicitely the arguments of functions when all the arguments are presented in the formulae.

The *corank* of a germ of a function at a critical point is the nullity of its second differential. Every germ of corank r is stably equivalent to a germ of a function in r variables. Classifying functions up to the stable equivalence one normally chooses a normal form with a minimal number of variables (equal to the corank).

Our calculations show, however, that sometimes *one can obtain more natural normal forms with more variables.*

The simplest case is the inclusion of the singularities $x^p+ax^2y^2+y^q$ in the series T; series $\mathbf{W}^{\#} \sim (x^2+y^3)^2$ provides more interesting examples.

To formulate the general stabilisation problem we need some definitions. Let $f \in \mathbb{C}[[x_1, \ldots, x_n]]$. The *support* of f is the set $\operatorname{supp} f \subset \mathbb{Z}^n \subset \mathbb{R}^n$ of all indexes $\mathbf{m}$ of the monomials $\mathbf{x}^{\mathbf{m}}$ with nonzero coefficients in f. The *Newton polyhedron* $\Gamma=\Gamma(f)$ is the boundary of the convex hull of the union of all positive ortants $\mathbf{m}+\mathbb{R}^n_+$, $\mathbf{m} \in \operatorname{supp} f$.

A function f is *Γ-nondegenerate* if the multiplicity μ of its critical point O has the minimal possible value ν, for functions with a given Newton polyhedron Γ. An explicite formula for this *Newton number* $\nu(\Gamma)$ is given by A.G. Kouchnirenko (see [1]).

Most of our normal forms are Γ-nondegenerate functions. It is not known, how vast is the class of functions stably equivalent to Γ-nondegenerate ones. For Γ-nonedegenerate functions classification and reducing to normal forms can be done in a very explicit way: these problems are rather stereometrical (see [8]).

Our lists of normal forms are given in § 1. They contain all the singularities with the modules number $m=0$, 1 and 2, all the singularities with $\mu \leqq 16$, all the singularities of corank 2 with nonzero 4-jet, all the singularities of corank 3 with a 3-jet, which determine an irreducible cubic, and some other singularities.

§ 2 is an outline of the proofs of these results. It contains the statements of 105 theorems. These theorems together with the lists of § 1 form a kind of singularity determinator like plant determinators: an algorithm is described for finding every singularity's place in the lists of § 1.

Most of the theorems of § 2 are proved in § 3. All these proofs depend heavily on the geometrical technique, described in [8]. From this geometrical point of view, the formulae of §§ 1 and 2 are to be considered as descriptions of supports and Newton polyhedrons, and to prove the theorems one has rather to manipulate with these polyhedrons than to calculate.

§ 1. Lists of Normal Forms

Letters $A, \ldots, \mathbf{Z}$ stand here for stable equivalence classes of function germs (or families of function germs).

I. Singularities with Modules Number m=0, 1 and 2

I.0. Simple Singularities ($m=0$). There are 2 infinite series **A**, **D**, and 3 "exceptional" singularities E_6, E_7, E_8:

A_k	$x^{k+1}, k \geqq 1$
D_k	$x^2 y + y^{k-1}, k \geqq 4$
E_6	$x^3 + y^4$
E_7	$x^3 + x y^3$
E_8	$x^3 + y^5$

The adjacency diagram

$$\begin{tikzcd} \mathbf{A} & \mathbf{D} \arrow[l] & E_6 \arrow[l] & E_7 \arrow[l] & E_8 \arrow[l] \\ & & (\mathbf{P}) \arrow[u] & (\mathbf{X}) \arrow[u] & (\mathbf{J}) \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{A}=A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & \cdots \arrow[l] \\ & & (D) \arrow[u] & & \\ \mathbf{D}=D_4 & D_5 \arrow[l] & D_6 \arrow[l] & \cdots. \arrow[l] & \\ & (\mathbf{E}) \arrow[u] & & & \end{tikzcd}$$

The definitions of **P**, **X**, **J** are to be found below (see II).

I.1. Unimodular Singularities ($m=1$). There are 3 families of parabolical singularities, one series of hyperbolical singularities (with 3 subscripts), and 14 families of exceptional singularities.

The parabolical singularities

P_8	$x^3 + y^3 + z^3 + a x y z$	$a^3 + 27 \neq 0$
X_9	$x^4 + y^4 + a x^2 y^2$	$a^2 \neq 4$
J_{10}	$x^3 + y^6 + a x^2 y^2$	$4a^3 + 27 \neq 0$

The hyperbolical singularities

$$T_{p,q,r}\colon x^p + y^q + z^r + a x y z, \quad a \neq 0, \ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

The 14 exceptional families

E_{12}	$x^3+y^7+axy^5$	E_{13}	$x^3+xy^5+ay^8$
E_{14}	$x^3+y^8+axy^6$	Z_{11}	$x^3y+y^5+axy^4$
Z_{12}	$x^3y+xy^4+ax^2y^3$	Z_{13}	$x^3y+y^6+axy^5$
W_{12}	$x^4+y^5+ax^2y^3$	W_{13}	$x^4+xy^4+ay^6$
Q_{10}	$x^3+y^4+yz^2+axy^3$	Q_{11}	$x^3+y^2z+xz^3+az^5$
Q_{12}	$x^3+y^5+yz^2+axy^4$	S_{11}	$x^4+y^2z+xz^2+ax^3z$
S_{12}	$x^2y+y^2z+xz^3+az^5$	U_{12}	$x^3+y^3+z^4+axyz^2$

Some of the adjacencies between the unimodular singularities

$$\begin{tikzcd}
 & & J_{10}=T_{2,3,6} \arrow[r] & (E_8) & & \\
\cdots \arrow[r] & T_{2,3,8} \arrow[r] & T_{2,3,7} \arrow[u] & E_{12} \arrow[l] & E_{13} \arrow[l] & E_{14} \arrow[l] & (\mathbf{J}_3) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_9=T_{2,4,4} \arrow[r] & (E_7) & & \\
\cdots \arrow[r] & T_{2,4,6} \arrow[r] & T_{2,4,5} \arrow[u] & Z_{11} \arrow[l] & Z_{12} \arrow[l] & Z_{13} \arrow[l] & (\mathbf{Z}_1^1) \arrow[l] \\
\cdots \arrow[r] & T_{2,5,6} \arrow[r] \arrow[u] & T_{2,5,5} \arrow[u] & W_{12} \arrow[l] \arrow[u] & W_{13} \arrow[l] \arrow[u] & (\mathbf{W}_{1,0}, \mathbf{N}) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & P_8=T_{3,3,3} \arrow[r] & (E_6) & & \\
\cdots \arrow[r] & T_{3,3,5} \arrow[r] & T_{3,3,4} \arrow[u] & Q_{10} \arrow[l] & Q_{11} \arrow[l] & Q_{12} \arrow[l] & (\mathbf{Q}_2) \arrow[l] \\
\cdots \arrow[r] & T_{3,4,5} \arrow[r] \arrow[u] & T_{3,4,4} \arrow[u] & S_{11} \arrow[l] \arrow[u] & S_{12} \arrow[l] \arrow[u] & (\mathbf{S}_{1,0}) \arrow[l] \\
\cdots \arrow[r] & T_{4,4,5} \arrow[r] \arrow[u] & T_{4,4,4} \arrow[u] & U_{12} \arrow[l] \arrow[u] & (\mathbf{U}_{1,0}, \mathbf{V}) \arrow[l] & \\
 & & (\mathbf{O}) \arrow[u] & & &
\end{tikzcd}$$

The singularity classes enclosed in brackets are not unimodular.

I.2. Bimodular Singularities ($m=2$). There are 8 infinite series and 14 exceptional families. In all the formulae of this section $\mathbf{a}=a_0+a_1y$.

The 4 infinite series of bimodular singularities of corank 2

Notation	Normal form	Restrictions	Multiplicity μ
$J_{3,0}$	$x^3+bx^2y^3+y^9+cxy^7$	$4b^3+27\neq 0$	16
$J_{3,p}$	$x^3+x^2y^3+\mathbf{a}y^{9+p}$	$p>0,\ a_0\neq 0$	$16+p$
$Z_{1,0}$	$y(x^3+dx^2y^2+cxy^5+y^6)$	$4d^3+27\neq 0$	15
$Z_{1,p}$	$y(x^3+x^2y^2+\mathbf{a}y^{6+p})$	$p>0,\ a_0\neq 0$	$15+p$
$W_{1,0}$	$x^4+\mathbf{a}x^2y^3+y^6$	$a_0^2\neq 4$	15
$W_{1,p}$	$x^4+x^2y^3+\mathbf{a}y^{6+p}$	$p>0,\ a_0\neq 0$	$15+p$
$W^{\#}_{1,2q-1}$	$(x^2+y^3)^2+\mathbf{a}xy^{4+q}$	$q>0,\ a_0\neq 0$	$15+2q-1$
$W^{\#}_{1,2q}$	$(x^2+y^3)^2+\mathbf{a}x^2y^{3+q}$	$q>0,\ a_0\neq 0$	$15+2q$

The 4 infinite series of bimodular singularities of corank 3

Notation	Normal form	Restrictions	Multiplicity μ
$Q_{2,0}$	$x^3+yz^2+\mathbf{a}x^2y^2+xy^4$	$a_0^2\neq 4$	14
$Q_{2,p}$	$x^3+yz^2+x^2y^2+\mathbf{a}z^{6+p}$	$p>0, a_0\neq 0$	$14+p$
$S_{1,0}$	$x^2z+yz^2+y^5+\mathbf{a}zy^3$	$a_0^2\neq 4$	14
$S_{1,p}$	$x^2z+yz^2+x^2y^2+\mathbf{a}y^{5+p}$	$p>0, a_0\neq 0$	$14+p$
$S^{\#}_{1,2q-1}$	$x^2z+yz^2+zy^3+\mathbf{a}xy^{2+q}$	$q>0, a_0\neq 0$	$14+2q-1$
$S^{\#}_{1,2q}$	$x^2z+yz^2+zy^3+\mathbf{a}x^2y^{2+q}$	$q>0, a_0\neq 0$	$14+2q$
$U_{1,0}$	$x^3+xz^2+xy^3+\mathbf{a}y^3z$	$a_0(a_0^2+1)\neq 0$	14
$U_{1,2q-1}$	$x^3+xz^2+xy^3+\mathbf{a}y^{1+q}z^2$	$q>0, a_0\neq 0$	$14+2q-1$
$U_{1,2q}$	$x^3+xz^2+xy^3+\mathbf{a}y^{3+q}z$	$q>0, a_0\neq 0$	$14+2q$

The 14 exceptional families

E_{18}	$x^3+y^{10}+\mathbf{a}xy^7$	E_{19}	$x^3+xy^7+\mathbf{a}y^8$
E_{20}	$x^3+y^8+\mathbf{a}xy^8$	Z_{17}	$x^3y+y^8+\mathbf{a}xy^6$
Z_{18}	$x^3y+xy^6+\mathbf{a}y^9$	Z_{19}	$x^3y+y^9+\mathbf{a}xy^7$
W_{17}	$x^4+xy^5+\mathbf{a}y^7$	W_{18}	$x^4+y^7+\mathbf{a}x^2y^4$
Q_{16}	$x^3+yz^2+y^7+\mathbf{a}xy^5$	Q_{17}	$x^3+yz^2+xy^5+\mathbf{a}y^8$
Q_{18}	$x^3+yz^2+y^8+\mathbf{a}xy^6$	S_{16}	$x^2z+yz^2+xy^4+\mathbf{a}y^6$
S_{17}	$x^2z+yz^2+y^6+\mathbf{a}zy^4$	U_{16}	$x^3+xz^2+y^5+\mathbf{a}x^2y^2$

All the functions of all these families are bimodular (when the restrictions hold).

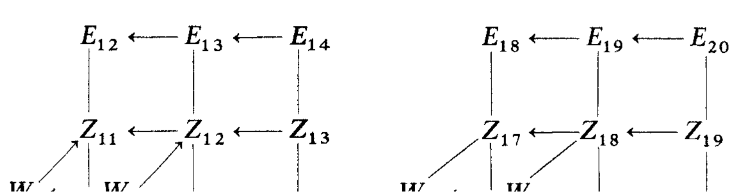
Some of the adjacencies between the bimodular singularities

$$\begin{array}{ccccccccc} J_{3,0} & \leftarrow & J_{3,1} & \leftarrow & E_{18} & \leftarrow & E_{19} & \leftarrow & E_{20}, \\ & & \uparrow & & & & & & \\ & & \cdots & & & & & & \end{array} \qquad \begin{array}{ccccccccc} Z_{1,0} & \leftarrow & Z_{1,1} & \leftarrow & Z_{17} & \leftarrow & Z_{18} & \leftarrow & Z_{19}, \\ & & \uparrow & & & & & & \\ & & \cdots & & & & & & \end{array}$$

$$\begin{tikzcd} & W_{1,1} \arrow[dl] & \cdots \arrow[l] & \\ W_{1,0} & & W_{17} \arrow[ul] \arrow[dl] & W_{18} \arrow[l] \\ & W^{\#}_{1,1} \arrow[ul] & \cdots \arrow[l] & \end{tikzcd} \qquad \begin{tikzcd} & S_{1,1} \arrow[dl] & \cdots \arrow[l] & \\ S_{1,0} & & S_{16} \arrow[ul] \arrow[dl] & S_{17} \arrow[l] \\ & S^{\#}_{1,1} \arrow[ul] & \cdots \arrow[l] & \end{tikzcd}$$

$$\begin{array}{ccccccccc} Q_{2,0} & \leftarrow & Q_{2,1} & \leftarrow & Q_{16} & \leftarrow & Q_{17} & \leftarrow & Q_{18}, \\ & & \uparrow & & & & & & \\ & & \cdots & & & & & & \end{array} \qquad \begin{array}{ccccc} U_{1,0} & \leftarrow & U_{1,1} & \leftarrow & U_{16}. \\ & & \uparrow & & \\ & & \cdots & & \end{array}$$

The pyramids of exceptional singularities with modules numbers $m=1$ and 2



Each vertical joins the singularities of the same Kodaira class in the Koulikov's construction [7].

II. The Corank 2 Singularities with Nonzero 4-Jets

Through this section $\mathbf{a}=a_0+\dots+a_{k-2}y^{k-2}$ for $k>1$ and $\mathbf{a}=0$ for $k=1$.

II$_1$. The Corank 2 Singularities with Nonzero 3-Jets. These are the simple singularities **A**, **D**, E_6, E_7, E_8 and the singularities of the following infinite series of classes.

$$\begin{tikzcd}[column sep=small, row sep=small]
\mathbf{J}=\mathbf{J}_2 \leftarrow \mathbf{J}_3 \leftarrow \cdots, \text{ where } \mathbf{J}_k= \leftarrow J_{k,0} & J_{k,1} \arrow[l] & E_{6k} \arrow[l] & E_{6k+1} \arrow[l] & E_{6k+2} \arrow[l] & (\mathbf{J}_{k+1}) \arrow[l] \\
 & J_{k,2} \arrow[u] & J_{k,3} \arrow[l] & \cdots . \arrow[l]
\end{tikzcd}$$

Notation	Normal form	Restrictions	Multiplicity μ	Modules number m
$J_{k,0}$	$x^3+bx^2y^k+y^{3k}+\mathbf{c}xy^{2k+1}$	$k>1, 4b^3+27\neq 0$	$6k-2$	$k-1$
$J_{k,i}$	$x^3+x^2y^k+\mathbf{a}y^{3k+i}$	$k>1, i>0, a_0\neq 0$	$6k+i-2$	$k-1$
E_{6k}	$x^3+y^{3k+1}+\mathbf{a}xy^{2k+1}$	$k\geqq 1$	$6k$	$k-1$
E_{6k+1}	$x^3+xy^{2k+1}+\mathbf{a}y^{3k+2}$	$k\geqq 1$	$6k+1$	$k-1$
E_{6k+2}	$x^3+y^{3k+2}+\mathbf{a}xy^{2k+2}$	$k\geqq 1$	$6k+2$	$k-1$

Here $\mathbf{c}=c_0+\dots+c_{k-3}y^{k-3}$ for $k>2$; $\mathbf{c}=0$ for $k=2$.

II$_2$. The Corank 2 Singularities with Zero 3-Jets and Nonzero 4-Jets. These singularities form one infinite series of classes

$$\mathbf{X}=\mathbf{X}_1 \leftarrow \mathbf{X}_2 \leftarrow \cdots$$

where

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \mathbf{Y_k} \arrow[dl] & & \\
\mathbf{X_k}=X_{k,0} & \mathbf{X_k^*} \arrow[l] & & \mathbf{W_k} \arrow[ul] \arrow[dl] & \leftarrow ; \\
 & & \mathbf{Z_k} \arrow[ul] & &
\end{tikzcd}$$

here

$$\begin{array}{ll}
\quad\ \ \downdownarrows & \qquad\quad\ \downarrow \\
\mathbf{X_k^*}=\leftarrow X_{k,1}\leftarrow X_{k,2}\leftarrow\cdots, & \mathbf{Y_k}=\leftarrow Y^k_{1,1}\longleftarrow Y^k_{2,1}\leftleftarrows\cdots, \\
\mathbf{Z_k}=\leftarrow \mathbf{Z^k_0}\longleftarrow \mathbf{Z^k_1}\longleftarrow\cdots, & \mathbf{Z^k_0}=\leftarrow Z^k_{12k-1}\leftarrow Z^k_{12k}\leftarrow Z^k_{12k+1}, \\
\qquad\ \ \uparrow &
\end{array}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
\mathbf{Z^k_i}=\leftarrow Z^k_{i,0} & Z^k_{i,1} \arrow[l] & Z^k_{12k+6i-1} \arrow[l] & Z^k_{12k+6i} \arrow[l] & Z^k_{12k+6i+1} \arrow[l] & (\mathbf{Z^k_{i+1}}), \quad i>0 \arrow[l] \\
 & Z^k_{i,2} \arrow[u] & Z^k_{i,3} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & W_{k,1} \arrow[dl] & W_{k,2} \arrow[l] & \cdots \arrow[l] & \\
\mathbf{W_k}=\leftarrow W_{12k} & W_{12k+1} \arrow[l] & W_{k,0} \arrow[l] & & W_{12k+5} \arrow[ul] \arrow[dl] & W_{12k+6} \arrow[l] & (\mathbf{X_{k+1}}) \arrow[l] \\
 & & & W^{\#}_{k,1} \arrow[ul] & W^{\#}_{k,2} \arrow[l] & \cdots . \arrow[l] &
\end{tikzcd}$$

Classes **X** *and* **Y**

Notation	Normal Form	Restrictions	Multiplicity μ	Modules number m
$X_{k,0}$	$x^4+\mathbf{b}x^3y^k+\mathbf{a}x^2y^{2k}+xy^{3k}$	$\Delta\neq 0, a_0b_0\neq 9$ $k>1$	$12k-3$	$3k-2$
$X_{k,p}$	$x^4+\mathbf{a}x^3y^k+x^2y^{2k}+\mathbf{b}y^{4k+p}$	$a_0^2\neq 4, b_0\neq 0$ $p>0, k>1$	$12k-3+p$	$3k-2$
$Y^k_{r,s}$	$[(x+\mathbf{a}y^k)^2+\mathbf{b}y^{2k+s}](x^2+y^{2k+r})$	$1\leqq s\leqq 7, k>1$ $a_0\neq 0\neq b_0$	$12k-3+r+s$	$3k-2$

The case $k=1$:

$X_{1,0}$	$x^4+a_0x^2y^2+y^4$	$a_0^2\neq 4$	9	1
$X_{1,p}$	$x^4+x^2y^2+a_0y^{4+p}$	$a_0\neq 0$	$9+p$	1
$Y^1_{r,s}$	$x^{4+r}+a_0x^2y^2+y^{4+s}$	$a_0\neq 0$	$9+r+s$	1

Of course, $X_{1,0}=X_9$, $X_{1,p}=T_{2,4,4+p}$, $Y^1_{r,s}=T_{2,4+r,4+s}$. Here

$$\Delta=4(a_0^3+b_0^3)-a_0^2b_0^2-18a_0b_0+27,$$

$$\mathbf{b}=b_0+\cdots+b_{2k-2}y^{2k-2}.$$

Class **Z**

For singularities $Z^k_{i,0}$ and Z^k_μ $(k>1)$ normal forms are $f=(x+\mathbf{a}y^k)f_2$, where $a_0\neq 0$ and f_2 is given in the following table:

Notation	f_2	Restrictions	Multiplicity, μ	Modules number, m
$Z^k_{i,0}$	$x^3+dx^2y^{k+1}+\mathbf{c}xy^{2k+2i+1}+y^{3k+3i}$	$4d^3+27\neq 0$ $k>1, i\geqq 0$	$12k-3+6i$	$3k+i-2$
$Z^k_{12k+6i-1}$	$x^3+\mathbf{b}xy^{2k+2i+1}+y^{3k+3i+1}$	$k>1, i\geqq 0$	$12k+6i-1$	$3k+i-2$
Z^k_{12k+6i}	$x^3+xy^{2k+2i+1}+\mathbf{b}y^{3k+3i+2}$	$k>1, i\geqq 0$	$12k+6i$	$3k+i-2$
$Z^k_{12k+6i+1}$	$x^3+\mathbf{b}xy^{2k+2i+2}+y^{3k+3i+2}$	$k>1, i\geqq 0$	$12k+6i+1$	$3k+i-2$

Singularity $Z^k_{i,p}$ $(k>1,\ i>0,\ p>0)$ admits the normal form

$$(x^2+\mathbf{a}xy^k+\mathbf{b}y^{2k+i})\cdot(x^2+y^{2k+2i+p}),\quad a_0\neq 0,\ b_0\neq 0,$$

its multiplicity is $\mu=12k+6i+p-3$, the modules number $m=3k+i-2$.

For $k=1$ one modifies the above formulae in the following way:

1. One omits the index $k=1$ in the notations.

2. Singularities $Z_{i,0}, Z_{6i+11}, Z_{6i+12}, Z_{6i+13}$ $(i>0)$ have normal forms $f=yf_2$, where f_2 is given by the table above.

3. $Z_{i,p}$: $y(x^3+x^2y^{i+1}+\mathbf{b}y^{3i+p+3})$, $b_0\neq 0$, $i>0$, $p>0$.

Through this section

$$\mathbf{b}=b_0+\cdots+b_{2k+i-2}y^{2k+i-2},$$

$$\mathbf{c}=c_0+\cdots+c_{2k+i-3}y^{2k+i-3}.$$

Class **W**

Notation	Normal form	Restriction	Multiplicity μ	Modules number m
W_{12k}	$x^4+y^{4k+1}+\mathbf{a}xy^{3k+1}+\mathbf{c}x^2y^{2k+1}$	$k\geqq 1$	$12k$	$3k-2$
W_{12k+1}	$x^4+xy^{3k+1}+\mathbf{a}x^2y^{2k+1}+\mathbf{c}y^{4k+2}$	$k\geqq 1$	$12k+1$	$3k-2$
$W_{k,0}$	$x^4+\mathbf{b}x^2y^{2k+1}+\mathbf{a}xy^{3k+2}+y^{4k+2}$	$k\geqq 1,\ b_0^2\neq 4$	$12k+3$	$3k-1$
$W_{k,i}$	$x^4+\mathbf{a}x^3y^{k+1}+x^2y^{2k+1}+\mathbf{b}y^{4k+2+i}$	$i>0,\ b_0\neq 0$	$12k+3+i$	$3k-1$
$W^{\#}_{k,2q-1}$	$(x^2+y^{2k+1})^2+\mathbf{b}xy^{3k+1+q}+\mathbf{a}y^{4k+2+q}$	$q>0,\ b_0\neq 0$	$12k+2+2q$	$3k-1$
$W^{\#}_{k,2q}$	$(x^2+y^{2k+1})^2+\mathbf{b}x^2y^{2k+1+q}+\mathbf{a}xy^{3k+2+q}$	$q>0,\ b_0\neq 0$	$12k+3+2q$	$3k-1$
W_{12k+5}	$x^4+xy^{3k+2}+\mathbf{a}x^2y^{2k+2}+\mathbf{b}y^{4k+3}$	$k\geqq 1$	$12k+5$	$3k-1$
W_{12k+6}	$x^4+y^{4k+3}+\mathbf{a}xy^{3k+3}+\mathbf{b}x^2y^{2k+2}$	$k\geqq 1$	$12k+6$	$3k-1$

Here $\mathbf{b}=b_0+\cdots+b_{2k-1}y^{2k-1}$, $\mathbf{c}=c_0+\cdots+c_{2k-2}y^{2k-2}$; as always, $\mathbf{a}=a_0+\cdots+a_{k-2}y^{k-2}$, for $k>1$ and $\mathbf{a}=0$ for $k=1$.

III. The Corank 3 Singularities with Reduced 3-Jets

Besides the unimodular singularities T (see $\mathbf{I}_1$), there are 3 infinite series of classes **Q**, **S**, **U** of such singularities.

$\mathbf{III}_1$. **Series Q.** The singularities with the 3-jet x^3+yz^2 form one infinite series of classes:

$$\mathbf{Q}=\leftarrow\mathbf{Q_1}\leftarrow\mathbf{Q_2}\leftarrow\cdots,\qquad \text{where } \mathbf{Q_1}=\leftarrow Q_{10}\leftarrow Q_{11}\leftarrow Q_{12},$$

$$\begin{array}{lll} \mathbf{Q_k}=\leftarrow Q_{k,0}\leftarrow & Q_{k,1} & \leftarrow Q_{6k+4}\leftarrow Q_{6k+5}\leftarrow Q_{6k+6}\leftarrow(\mathbf{Q_{k+1}}) \\ & \uparrow & \\ k>1 & Q_{k,2} & \leftarrow Q_{k,3}\leftarrow\cdots. \end{array}$$

Notation	Normal form	Restrictions	Multiplicity μ	Modules number m
$Q_{k,0}$	$\varphi+\mathbf{b}x^2y^k+xy^{2k}$	$k>1,\ b_0^2\neq 4$	$6k+2$	k
$Q_{k,i}$	$\varphi+x^2y^k+\mathbf{b}y^{3k+i}$	$k>1,\ b_0\neq 0$	$6k+2+i$	k
Q_{6k+4}	$\varphi+y^{3k+1}+\mathbf{b}xy^{2k+1}$	$k\geqq 1$	$6k+4$	k
Q_{6k+5}	$\varphi+xy^{2k+1}+\mathbf{b}y^{3k+2}$	$k\geqq 1$	$6k+5$	k
Q_{6k+6}	$\varphi+y^{3k+2}+\mathbf{b}xy^{2k+2}$	$k\geqq 1$	$6k+6$	k

Here $\varphi=x^3+yz^2$, $\mathbf{b}=b_0+\cdots+b_{k-1}y^{k-1}$.

$\mathbf{III}_2$. **Series S.** The singularities with the 3-jet x^2z+yz^2 form one infinite series of classes:

$$\mathbf{S}=\leftarrow\mathbf{S_1}\leftarrow\mathbf{S_2}\leftarrow\cdots,$$

where

$$\begin{tikzcd}[column sep=small]
 & S_{k,1}\leftarrow\cdots \arrow[dl] & \\
\mathbf{S_k}=\leftarrow S_{12k-1}\leftarrow S_{12k}\leftarrow S_{k,0} & & S_{12k+4}\leftarrow S_{12k+5}\leftarrow\mathbf{S^*_{k+1}}\leftarrow(\mathbf{S_{k+1}}) \arrow[ul] \arrow[dl] \\
 & S^{\#}_{k,1}\leftarrow\cdots. \arrow[ul] &
\end{tikzcd}$$

Notation	Normal form	Restrictions	Multiplicity μ	Modules number m
S_{12k-1}	$\varphi+y^{4k}+\mathbf{a}xy^{3k}+\mathbf{c}zy^{2k+1}$	–	$12k-1$	$3k-2$
S_{12k}	$\varphi+xy^{3k}+\mathbf{c}y^{4k+1}+\mathbf{a}zy^{2k+1}$	–	$12k$	$3k-2$
$S_{k,0}$	$\varphi+y^{4k+1}+\mathbf{a}xy^{3k+1}+\mathbf{b}zy^{2k+1}$	$b_0^2\neq 4$	$12k+2$	$3k-1$
$S_{k,i}$	$\varphi+x^2y^{2k}+\mathbf{a}x^3y^k+\mathbf{b}y^{4k+1+i}$	$i>0,\ b_0\neq 0$	$12k+2+i$	$3k-1$
$S^{\#}_{k,2q-1}$	$\varphi+zy^{2k+1}+\mathbf{b}xy^{3k+q-1}+\mathbf{a}y^{4k+q}$	$q>0,\ b_0\neq 0$	$12k+2q+1$	$3k-1$
$S^{\#}_{k,2q}$	$\varphi+zy^{2k+1}+\mathbf{b}x^2y^{2k+q}+\mathbf{a}xy^{3k+q+1}$	$q>0,\ b_0\neq 0$	$12k+2q+2$	$3k-1$
S_{12k+4}	$\varphi+xy^{3k+1}+\mathbf{a}zy^{2k+2}+\mathbf{b}y^{4k+2}$	–	$12k+4$	$3k-1$
S_{12k+5}	$\varphi+y^{4k+2}+\mathbf{a}xy^{3k+2}+\mathbf{b}zy^{2k+2}$	–	$12k+5$	$3k-1$

Here $\varphi=x^2z+yz^2$;

$$\mathbf{a}=a_0+\cdots+a_{k-2}y^{k-2}\quad \text{for } k>1,\ \mathbf{a}=0 \text{ for } k=1;$$
$$\mathbf{b}=b_0+\cdots+b_{2k-1}y^{2k-1};$$
$$\mathbf{c}=c_0+\cdots+c_{2k-2}y^{2k-2}.$$

The $\mathbf{S}_\mathbf{k}^*$ $(k>1)$ singularities are:

$$\begin{tikzcd}[column sep=small] & & & \mathbf{SQ_k} \arrow[dl] & \\ \mathbf{S}_\mathbf{k}^* = \leftarrow & S^*_{k,0} & \mathbf{SP_k} \arrow[l] & & (\mathbf{S_k}) \arrow[ul] \arrow[dl] \\ & & & \mathbf{SR_k} \arrow[ul] & \end{tikzcd}$$

$\mu(S^*_{k,0})=12k-4$, $m(S^*_{k,0},\dots,\mathbf{SR_k})=3k-2$, $\operatorname{codim}\mathbf{S}_\mathbf{k}^*=9k-3$.

III_3. Series U. The singularities with the 3-jet x^3+xz^2 form one infinite series of classes

$$\mathbf{U}=\leftarrow\mathbf{U_1}\leftarrow\mathbf{U_2}\leftarrow\cdots,$$

where

$$\begin{tikzcd}[column sep=small] \mathbf{U_k}=\leftarrow U_{12k} & U_{k,0} \arrow[l] & U_{k,1} \arrow[l] & U_{12k+4} \arrow[l] & \mathbf{U^*_{k+1}} \arrow[l] & (\mathbf{U_{k+1}}) \arrow[l] \\ & & U_{k,2}\leftarrow\cdots. \arrow[u] & & & \end{tikzcd}$$

Notation	Normal form	Restrictions	Multiplicity μ	Modules number m
U_{12k}	$\varphi+y^{3k+1}+\mathbf{a}xy^{2k+1}+\mathbf{b}zy^{2k+1}+\mathbf{d}x^2y^{k+1}$	–	$12k$	$4k-3$
$U_{k,2q}$	$\varphi+xy^{2k+1}+\mathbf{a}x^2y^{k+1}+\mathbf{b}y^{3k+2+q}+\mathbf{c}zy^{2k+1+q}$	$q\geqq 0;\ c_0\neq 0$	$12k+2+2q$	$4k-2$
$U_{k,2q-1}$	$\varphi+xy^{2k+1}+\mathbf{a}x^2y^{k+1}+\mathbf{b}zy^{2k+1+q}+\mathbf{c}z^2y^{k+q}$	$q>0;\ c_0\neq 0$	$12k+1+2q$	$4k-2$
U_{12k+4}	$\varphi+y^{3k+2}+\mathbf{a}xy^{2k+2}+\mathbf{b}zy^{2k+2}+\mathbf{c}x^2y^{k+1}$	–	$12k+4$	$4k-2$

Here $\varphi=x^3+xz^2$; $c_0^2+1\neq 0$ for $q=0$; and trough the table

$$\mathbf{a}=a_0+\cdots+a_{k-2}y^{k-2}\quad \text{for } k>1,\quad \mathbf{a}=0\quad \text{for } k=1,$$
$$\mathbf{b}=b_0+\cdots+b_{k-2}y^{k-2}\quad \text{for } k>1,\quad \mathbf{b}=0\quad \text{for } k=1,$$

$$\mathbf{c}=c_0+\cdots+c_{2k-1}y^{2k-1},$$
$$\mathbf{d}=d_0+\cdots+d_{2k-2}y^{2k-2}.$$

The $\mathbf{U}_k^*$ singularities ($k>1$) are

$$\begin{array}{ccccccc} \mathbf{U}_k^* = \leftarrow U_{k,0}^* \leftarrow & \mathbf{UP_k} & \leftarrow & \mathbf{UR_k} & \leftarrow & \mathbf{UT_k} \\ & \uparrow & & \uparrow & & \uparrow \\ & \mathbf{UQ_k} & \leftarrow & \mathbf{US_k} & \leftarrow & (\mathbf{U_k}), \end{array}$$

$\mu(U_{k,0}^*)=12k-4$, $m(U_{k,0}^*,\ldots,\mathbf{UT_k})=4k-3$, $\operatorname{codim}(\mathbf{U}_k^*)=8k-2$.

IV. Series V. The corank 3 singularities with the 3-jet x^2y belong to the classes

$$\begin{array}{ccccc} & & V_{1,1} \leftarrow V_{1,2} \leftarrow \cdots & & \\ & \swarrow & & \nwarrow & \\ \mathbf{V}=V_{1,0} & & & & \mathbf{V}^* \\ & \nwarrow & & \swarrow & \\ & & V_{1,1}^{\#} \leftarrow V_{1,2}^{\#} \leftarrow \cdots. & & \end{array}$$

Notation	Normal form	Restrictions	Multiplicity μ	Modules number m
$V_{1,0}$	$x^2y+z^4+az^3y+\mathbf{b}z^2y^2+zy^3$	$\Delta(a,b_0)\neq 0$	15	3
$V_{1,p}$	$x^2y+z^4+bz^3y+z^2y^2+\mathbf{a}y^{4+p}$	$b^2\neq 4$, $a_0\neq 0$	$15+p$	3
$V_{1,2q-1}^{\#}$	$x^2y+z^3y+ay^2z^2+y^4+\mathbf{b}xz^{2+q}$	$4a^3+27\neq 0$, $b_0\neq 0$	$15+2q-1$	3
$V_{1,2q}^{\#}$	$x^2y+z^3y+ay^2z^2+y^4+\mathbf{b}z^{4+q}$	$4a^3+27\neq 0$, $b_0\neq 0$	$15+2q$	3

Here $p>0$, $q>0$, $\mathbf{a}=a_0+a_1y$, $\mathbf{b}=b_0+b_1z$. For the $\mathbf{V}^*$ singularities $\mu(\mathbf{V}^*)\geqq 17$, $m(\mathbf{V}^*)\geqq 3$, $c(\mathbf{V}^*)=13$.

V. Other Singularities. All the singularities, which normal forms are not given in this article, belong to the following 7 classes:

Notation	Corank	Adjacency	Definition	$c\geqq$	$\mu\geqq$	$m\geqq$	Theorem
N	2	$\mathbf{N}\to W_{13}$	$j_4=0$	12	16	3	**47–49**
$\mathbf{S}^*$	3	$\mathbf{S}_k^*\to S_{12k-9}$	th. **77**	15	20	4	**77–81**
$\mathbf{U}^*$	3	$\mathbf{U}_k^*\to U_{12k-8}$	th. **90**	14	20	5	**90–96**
$\mathbf{V}^*$	3	$\mathbf{V}^*\to V_{1,1}\cup V_{1,1}^{\#}$	th. **98**	13	17	3	**97–102**
$\mathbf{V}'$	3	$\mathbf{V}'\to\mathbf{V}$	$j_3=x^3$	13	18	4	**103**
$\mathbf{V}''$	3	$\mathbf{V}''\to\mathbf{V}'$	$j_3=0$	16	27	10	**104**
O	>3	$\mathbf{O}\to T_{4,4,4}$	corank $\geqq 4$	10	16	5	**105**

Here $k>1$.

Remark. For every arrow $\mathbf{K}\leftarrow\mathbf{L}$ in §1 (say, $\mathbf{J_2}\leftarrow\mathbf{J_3}$), every singularity in class $\mathbf{K}$ has a neighbourhood which does not intersect $\mathbf{L}$ (*e.g* there is no adjacency $J_{2,i}\to J_{3,0}$ for any i).

§ 2. Singularity Determinator

Notations

f	a germ of a holomorphic function at an *isolated* critical point O, with finite multiplicity μ; its Taylor series at O; a formal series in x, y or in x, y, z, with multiplicity μ.
$f \sim g$	germs (or series) f and g at O are equivalent (there exists a germ of a diffeomorphism h (or a formal series h) such that $f = g \circ h$.
$\Rightarrow$	"implies"
$\mapsto$	"see" (references $\mapsto$ **i** are not parts of the theorem statements; they give the number of the theorem containing the classification of the given singularity classes).
$j_k f$	k-jet of f at O (or the Taylor k-polynomial).
$A, \ldots, Z$	stable equivalence classes defined in § 1.
$m(f)$	modules number of the germ of f at O.
$c(f)$	codimension of the μ=const stratum of a function germ f in the space of germs of functions having critical value 0 at the critical point O.
$c(\mathbf{K})$	codimension of the singularity class $\mathbf{K}$ in the same space.
$j\{\mathbf{x}^{\mathbf{m}_i}\}f$	quasijet (or quasi Taylor polynomial of f at O determined by the monomials $\mathbf{x}^{\mathbf{m}_i}$, see the explanation below).
$j\{\mathbf{x}^{\mathbf{m}_i}\}f \approx g$	quasihomogenous equivalence of two quasijets or of two quasi Taylor polynomials, see the explanation below.
j^*, φ	abbreviations used in theorems **58–65**, **66–81**, **82–89**, **98–102**; their meaning is to be found above the first theorem of each group.
Δ	discriminant. In theorems **36**, **37**, **47**, **48**, **98**, **99**, $\Delta = 4(a^3 + b^3) + 27 - a^2 b^2 - 18ab$.

Explanation. A system $\{\mathbf{x}^{\mathbf{m}_i}\}$ of n monomials in n variables $x_1, \ldots, x_n$ with linearly independent exponents $\mathbf{m}_i \in \mathbb{Z}^n \subset \mathbb{R}^n$ defines an hyperplane $\Gamma \subset \mathbb{R}^n$, $\Gamma = \{\mathbf{m}: (\boldsymbol{\alpha}, \mathbf{m}) = 1\}$. If all the components α_i of the vector $\boldsymbol{\alpha}$ are positive, $\boldsymbol{\alpha}$ is called the *quasihomogenity type*. The number $(\boldsymbol{\alpha}, \mathbf{m})$ is then called the *order* of the monomial $\mathbf{x}^{\mathbf{m}}$.

A polynomial $f = \sum f_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ is *quasihomogenous, of degree d and of type $\boldsymbol{\alpha}$* if $(\boldsymbol{\alpha}, \mathbf{m}) = d$ for all $\mathbf{m}$ with $f_{\mathbf{m}} \neq 0$.

The type $\boldsymbol{\alpha}$ defines a graded ring structure in $\mathbb{C}[[x_1, \ldots, x_n]]$ and a ring filtration $\mathscr{E}_0 \supset \ldots, \mathscr{E}_d = \{f: (\boldsymbol{\alpha}, \mathbf{m}) \geq d \, \forall \mathbf{m}: f_{\mathbf{m}} \neq 0\}$.

The factor space $\mathscr{E}_0/\mathscr{E}_d$, $d > 1$ is by definition the *quasijet space defined by the monomials* $\{\mathbf{x}^{\mathbf{m}_i}\}$ (or by the type $\boldsymbol{\alpha}$). For a fixed coordinate system one can indentify quasijets with the polynomials having only monomials of order ≤ 1 (that is, their exponents lie on Γ or the same side of Γ as O).

Quasihomogenous diffeomorphisms are diffomorphisms of $\mathbb{C}^n$ which preserve the degrees in the graded ring $\mathbb{C}[[x_1, \ldots, x_n]]$. The Lie group of the quasihomogenous diffeomorphisms acts on the space of quasijets and on the space of quasihomogenous polynomials. Two quasijets (or two polynomials) are *quasihomogenously equivalent* if they belong to the same orbit of this action.

Determinator

1. $\mu(f)<\infty \Rightarrow$ one of the four possibilities holds:
corank $f \leqq 1 \mapsto$ **2**,
$=2 \mapsto$ **3**,
$=3 \mapsto$ **50**,
$>3 \mapsto$ **105**.

2. corank $f \leqq 1 \Rightarrow f \in A_k$ $(k \geqq 1)$.

Through theorems **3–49**, $f \in \mathbb{C}[[x, y]]$.

3. $j_2 f=0 \Rightarrow$ one of the four possibilities holds:
$j_3 f \approx x^2 y+y^3 \mapsto$ **4**,
$\approx x^2 y \mapsto$ **5**,
$\approx x^3 \mapsto$ $\mathbf{6_1}$,
$=0 \mapsto$ **13**.

4. $j_3 f=x^2 y+y^3 \Rightarrow f \in D_4$.

5. $j_3 f=x^2 y \Rightarrow f \in D_k (k>4)$.

Through theorems **6–9**, $k \geqq 1$.

$\mathbf{6_k}$. $j_{x^3, y^{3k}} f(x, y)=x^3 \Rightarrow$ one of the four possibilities holds:
$j_{x^3, y^{3k+1}} f \approx x^3+y^{3k+1} \mapsto \mathbf{7_k}$,
$j_{x^3, xy^{2k+1}} f \approx x^3+x y^{2k+1} \mapsto \mathbf{8_k}$,
$j_{x^3, y^{3k+2}} f \approx x^3+y^{3k+2} \mapsto \mathbf{9_k}$,
$j_{x^3, y^{3k+2}} f =x^3 \mapsto \mathbf{10_{k+1}}$.

$\mathbf{7_k}$. $j_{x^3, y^{3k+1}} f=x^3+y^{3k+1} \Rightarrow f \in E_{6k}$.

$\mathbf{8_k}$. $j_{x^3, xy^{2k+1}} f=x^3+x y^{2k+1} \Rightarrow f \in E_{6k+1}$.

$\mathbf{9_k}$. $j_{x^3, y^{3k+2}} f=x^3+y^{3k+2} \Rightarrow f \in E_{6k+2}$.

Through theorems **10–12**, $k>1$.

$\mathbf{10_k}$. $j_{x^3, y^{3k-1}} f=x^3 \Rightarrow$ one of the three possibilities holds:
$j_{x^3, y^{3k}} f \approx x^3+a x^2 y^k+y^{3k}, \quad 4a^3+27 \neq 0 \mapsto \mathbf{11_k}$;
$\approx x^3+x^2 y^k \mapsto \mathbf{12_k}$;
$\approx x^3 \mapsto \mathbf{6_k}$.

$\mathbf{11_k}$. $j_{x^3, y^{3k}} f=x^3+a x^2 y^k+y^{3k}, \quad 4a^3+27 \neq 0 \Rightarrow f \in J_{k, 0}$.

$\mathbf{12_k}$. $j_{x^3, y^{3k}} f=x^3+x^2 y^k \Rightarrow f \in J_{k, p}$ $(p>0)$.

Series X

13. $j_3 f(x, y)=0 \Rightarrow$ one of the six possibilities holds:
$j_4 \approx x^4+a x^2 y^2+y^4, \quad a^2 \neq 4 \mapsto$ **14**;
$\approx x^4+x^2 y^2 \mapsto$ **15**;
$\approx x^2 y^2 \mapsto$ **16**;
$\approx x^3 y \mapsto$ **17**;
$\approx x^4 \mapsto$ **25**;
$=0 \mapsto$ **47**;

14. $j_4 f=x^4+a x^2 y^2+y^4, \quad a^2 \neq 4 \Rightarrow f \in X_9 =X_{1,0}=T_{2,4,4}$.

15. $j_4 f=x^4+x^2 y^2 \Rightarrow f \in X_{1, p}=T_{2,4,4+p}$ $(p>0)$.

16. $j_4 f = x^2 y^2 \qquad \Rightarrow f \in Y^1_{p,q} = T_{2,4+p,4+q}\ (p>0,\ q>0).$

17. $j_4 f = x^3 y \Rightarrow j_{x^3 y, y^4} f = x^3 y \mapsto$ $\mathbf{18_1}$.

Through theorems **18–21**, $p \geqq 1$.

$\mathbf{18_p}$. $j_{x^3 y, y^{3p+1}} f = x^3 y \Rightarrow$ one of the four possibilities holds:

$j_{x^3 y, y^{3p+2}} f \approx x^3 y + y^{3p+2} \mapsto$ $\mathbf{19_p}$,

$j_{x^3 y, xy^{2p+2}} f \approx x^3 y + x y^{2p+2} \mapsto$ $\mathbf{20_p}$,

$j_{x^3 y, y^{3p+3}} f \approx x^3 y + y^{3p+3} \mapsto$ $\mathbf{21_p}$,

$j_{x^3 y, y^{3p+3}} f = x^3 y \mapsto$ $\mathbf{22_{p+1}}$.

$\mathbf{19_p}$. $j_{x^3 y, y^{3p+2}} f = x^3 y + y^{3p+2} \Rightarrow f \in Z_{6p+5}$.

$\mathbf{20_p}$. $j_{x^3 y, xy^{2p+2}} f = x^3 y + x y^{2p+2} \Rightarrow f \in Z_{6p+6}$.

$\mathbf{21_p}$. $j_{x^3 y, y^{3p+3}} f = x^3 y + y^{3p+3} \Rightarrow f \in Z_{6p+7}$.

Through theorems **22–24**, $p > 1$.

$\mathbf{22_p}$. $j_{x^3 y, y^{3p}} f = x^3 y \Rightarrow$ one of the three possibilities holds:

$j_{x^3 y, y^{3p+1}} f = y(x^3 + b x^2 y^p + y^{3p}), \quad 4b^3 + 27 \neq 0 \mapsto$ $\mathbf{23_p}$;

$y(x^3 + x^2 y^p) \mapsto$ $\mathbf{24_p}$;

$x^3 y \mapsto$ $\mathbf{18_p}$.

$\mathbf{23_p}$. $j_{x^3 y, y^{3p+1}} f = y(x^3 + b x^2 y^p + y^{3p}), \quad 4b^3 + 27 \neq 0 \Rightarrow f \in Z_{p-1,0}$.

$\mathbf{24_p}$. $j_{x^3 y, y^{3p+1}} f = y(x^3 + x^2 y^p) \Rightarrow f \in Z_{p-1,r}\ (r>0)$.

Series W

25. $j_4 f(x,y) = x^4 \Rightarrow j_{x^4, y^4} f = x^4 \mapsto$ $\mathbf{26_1}$.

Through theorems **26–34**, $k \geqq 1$.

$\mathbf{26_k}$. $j_{x^4, y^{4k}} f = x^4 \Rightarrow$ one of the three possibilities holds:

$j_{x^4, y^{4k+1}} f \approx x^4 + y^{4k+1} \mapsto$ $\mathbf{27_k}$,

$j_{x^4, xy^{3k+1}} f \approx x^4 + x y^{3k+1} \mapsto$ $\mathbf{28_k}$,

$j_{x^4, xy^{3k+1}} f = x^4 \mapsto$ $\mathbf{29_k}$.

$\mathbf{27_k}$. $j_{x^4, y^{4k+1}} f = x^4 + y^{4k+1} \Rightarrow f \in W_{12k}$.

$\mathbf{28_k}$. $j_{x^4, xy^{3k+1}} f = x^4 + x y^{3k+1} \Rightarrow f \in W_{12k+1}$.

$\mathbf{29_k}$. $j_{x^4, xy^{3k+1}} f = x^4 \Rightarrow$ one of the four possibilities holds:

$j_{x^4, y^{4k+2}} f \approx x^4 + b x^2 y^{2k+1} + y^{4k+2}, \quad b^2 \neq 4 \mapsto$ $\mathbf{30_k}$;

$\approx x^4 + x^2 y^{2k+1} \mapsto$ $\mathbf{31_k}$;

$\approx (x^2 + y^{2k+1})^2 \mapsto$ $\mathbf{32_k}$;

$= x^4 \mapsto$ $\mathbf{33_k}$.

$\mathbf{30_k}$. $j_{x^4, y^{4k+2}} f = x^4 + b x^2 y^{2k+1} + y^{4k+2}, \quad b^2 \neq 4 \Rightarrow f \in W_{k,0}$.

$\mathbf{31_k}$. $j_{x^4, y^{4k+2}} f = x^4 + x^2 y^{2k+1} \Rightarrow f \in W_{k,i,(i>0)}$.

$\mathbf{32_k}$. $j_{x^4, y^{4k+2}} f = (x^2 + y^{2k+1})^2 \Rightarrow f \in W^{\#}_{k,i,(i>0)}$.

$\mathbf{33_k}$. $j_{x^4, y^{4k+2}} f = x^4 \Rightarrow$ one of the three possibilities holds:

$j_{x^4, xy^{3k+2}} f \approx x^4 + x y^{3k+2} \mapsto$ $\mathbf{34_k}$;

$j_{x^4, y^{4k+3}} f \approx x^4 + y^{4k+3} \mapsto$ $\mathbf{35_3}$;

$j_{x^4, y^{4k+3}} f = x^4 \mapsto$ $\mathbf{36_{k+1}}$.

$\mathbf{34_k}$. $j_{x^4, xy^{3k+2}} f = x^4 + x y^{3k+2} \Rightarrow f \in W_{12k+5}$.

35_k. $j_{x^4, y^{4k+3}} f = x^4 + y^{4k+3} \Rightarrow f \in W_{12k+6}$.

Through theorems **36–46**, $k > 1$.

Series X_k

36_k. $j_{x^4, y^{4k-1}} f = x^4 \Rightarrow$ one of the five possibilities holds:

$j_{x^4, y^{4k}} f \approx x^4 + b x^3 y^k + a x^2 y^{2k} + x y^{3k}$, $\Delta \neq 0$, $ab \neq 9 \mapsto$ **37_k**;

$\approx x^2 (x^2 + a x y^k + y^{2k})$, $a^2 \neq 4$ $\mapsto$ **38_k**;

$\approx x^2 (x + y^k)^2$ $\mapsto$ **39_k**;

$\approx x^3 (x + y^k)$ $\mapsto$ **40_k**;

$\approx x^4$ $\mapsto$ **26_k**.

37_k. $j_{x^4, y^{4k}} f = x^4 + b x^3 y^k + a x^2 y^{2k} + x y^{3k}$, $\Delta \neq 0 \Rightarrow f \in X_{k,0}$.

38_k. $j_{x^4, y^{4k}} f = x^2 (x^2 + a x y^k + y^{2k})$, $a^2 \neq 4 \Rightarrow f \in X_{k,p}\,(p > 0)$.

39_k. $j_{x^4, y^{4k}} f = x^2 (x^2 + y^k)^2 \Rightarrow f \in Y_{r,s}\,(1 \leqq s \leqq r)$.

40_k. $j_{x^4, y^{4k}} f = (x + y^k)\, x^3 \Rightarrow f = f_1 f_2$, where $j_{x, y^k} f_1 \approx x + y^k$;

$j_{x^3, y^{3k}} f_2 = x^3 \mapsto$ **41_k**.

Through theorems **41–44**, $i \geqq 0$, $p > 0$.

41_k. $j_{x^3, y^{3k}} f_2 = x^3 \Rightarrow$ one of the five possibilities holds:

$f_2 \in E_{6(k+i)} \mapsto$ **$42_{k,i}$**;

$f_2 \in E_{6(k+i)+1} \mapsto$ **$43_{k,i}$**;

$f_2 \in E_{6(k+i)+2} \mapsto$ **$44_{k,i}$**;

$f_2 \in J_{k+i+1,0} \mapsto$ **$45_{k,i+1}$**;

$f_2 \in J_{k+i+1,p} \mapsto$ **$46_{k,i+1,p}$**.

Through theorems **42–46**, $f(x, y) = f_1 f_2$, where

$j_{x, y^k} f_1 \approx x + y^k$, $j_{x^3, y^{3k}} f_2 = x^3$.

$42_{k,i}$. $f_2 \in E_{6(k+i)} \Rightarrow f \in Z^k_{12k+6i-1}$.

$43_{k,i}$. $f_2 \in E_{6(k+i)+1} \Rightarrow f \in Z^k_{12k+6i}$.

$44_{k,i}$. $f_2 \in E_{6(k+i)+2} \Rightarrow f \in Z^k_{12k+6i+1}$.

Through theorems **45–46**, $i \geqq 1$, $p > 0$.

$45_{k,i}$. $f_2 \in J_{k+i,0} \Rightarrow f \in Z^k_{i,0}$.

$46_{k,i,p}$. $f_2 \in J_{k+i,p} \Rightarrow f \in Z^k_{i,p}$.

47. $j_4 f = 0 \Rightarrow$ one of the two possibilities holds:

$j_5 f \approx x^4 y + a x^3 y^2 + b x^2 y^3 + x y^4$, $\Delta \neq 0$, $ab \neq 9 \mapsto$ **48**;

$j_5 f$ is degenerate $\mapsto$ **49**.

48. $j_5 f = x^4 y + a x^3 y^2 + b x^2 y^3 + x y^4$, $\Delta \neq 0 \Rightarrow f \in N_{16}$, i.e.

$f \sim x^4 y + a x^3 y^2 + b x^2 y^3 + x y^4 + c x^3 y^3$, $ab \neq 9$, $\Delta(a, b) \neq 0$;

$\mu(f) = 16$, $m(f) = 3$, $c(f) = 12$.

49. $j_5 f$ is degenerate $\Rightarrow \mu(f) > 16$, $m(f) > 2$, $c(f) > 12$.

Corank 3 Singularities

Through theorems **50–104**, $f \in \mathbb{C}[[x, y, z]]$.

50. $j_2 f(x, y, z) = 0 \Rightarrow$ one of the ten possibilities holds.

$j_3 f \approx x^3 + y^3 + z^3 + a x y z$, $a^3 + 27 \neq 0 \mapsto$ **51**;

$$\begin{array}{ll}
\approx x^3+y^3+xyz & \mapsto \mathbf{52} \text{ (series } \mathbf{P});\\
\approx x^3+xyz & \mapsto \mathbf{54} \text{ (series } \mathbf{R}=T_{3,p,q});\\
\approx xyz & \mapsto \mathbf{56} \text{ (series } \mathbf{T}),\\
\approx x^3+yz^2 & \mapsto \mathbf{58} \text{ (series } \mathbf{Q});\\
\approx x^2z+yz^2 & \mapsto \mathbf{66} \text{ (series } \mathbf{S});\\
\approx x^3+xz^2 & \mapsto \mathbf{82} \text{ (series } \mathbf{U});\\
\approx x^2y & \mapsto \mathbf{97} \text{ (class } \mathbf{V});\\
\approx x^3 & \mapsto \mathbf{103};\\
=0 & \mapsto \mathbf{104}.
\end{array}$$

Series T

51. $j_3 f = x^3+y^3+z^3+axyz, \quad a^3+27 \neq 0 \Rightarrow f \in P_8 = T_{3,3,3}$.

52. $j_3 f = x^3+y^3+xyz \Rightarrow f \sim x^3+y^3+xyz+\alpha(z), \quad j_3\alpha=0, \quad \mapsto$ **53.**

53. $f = x^3+y^3+xyz+\alpha(z), \quad j_3\alpha=0 \Rightarrow f \in P_{p+5} = T_{3,3,p} (p>3)$.

54. $j_3 f = x^3+xyz \Rightarrow f = x^3+xyz+\alpha(y)+\beta(z), \quad j_3\alpha=j_3\beta=0, \quad \mapsto$ **55.**

55. $f = x^3+xyz+\alpha(y)+\beta(z), \quad j_3(\alpha,\beta)=0 \Rightarrow f \in R_{p,q} = T_{3,p,q} \quad (q \geqq p>3)$.

56. $j_3 f = xyz \Rightarrow f \sim xyz+\alpha(x)+\beta(y)+\gamma(z), \quad j_3(\alpha,\beta,\gamma)=0, \quad \mapsto$ **57.**

57. $f = xyz+\alpha(x)+\beta(y)+\gamma(z), \quad j_3(\alpha,\beta,\gamma)=0 \Rightarrow f \in T_{p,q,r} \quad (r \geqq q \geqq p>3)$.

Series Q

Through theorems **58–65** $\varphi = x^3+yz^2$, $j_\lambda^* = j_{yz^2,x^3,\lambda}$ (λ is a monomial).

58. $j_3 f = \varphi \Rightarrow f \sim \varphi+\alpha(y)+x\beta(y), \quad j_3(\alpha, x\beta)=0, \quad \mapsto$ **59**$_{\mathbf{1}}$.

Through theorems **59–62**, $k \geqq 1$.

59$_{\mathbf{k}}$. $f = \varphi+\alpha(y)+x\beta(y), \quad j_{y^{3k}}^* f = \varphi \Rightarrow$ one of the four possibilities holds:

$$\begin{array}{ll}
j_{y^{3k+1}}^* f \approx \varphi+y^{3k+1} & \mapsto \mathbf{60_k};\\
j_{xy^{2k+1}}^* f \approx \varphi+xy^{2k+1} & \mapsto \mathbf{61_k};\\
j_{y^{3k+2}}^* f \approx \varphi+y^{3k+2} & \mapsto \mathbf{62_k};\\
j_{y^{3k+2}}^* f = \varphi & \mapsto \mathbf{63_{k+1}}.
\end{array}$$

60$_{\mathbf{k}}$. $j_{y^{3k+1}}^* f = \varphi+y^{3k+1} \Rightarrow f \in Q_{6k+4}$.

61$_{\mathbf{k}}$. $j_{xy^{2k+1}}^* f = \varphi+xy^{2k+1} \Rightarrow f \in Q_{6k+5}$.

62$_{\mathbf{k}}$. $j_{y^{3k+2}}^* f = \varphi+y^{3k+2} \Rightarrow f \in Q_{6k+6}$.

Through theorems **63–65**, $k>1$.

63$_{\mathbf{k}}$. $f = \varphi+\alpha(y)+x\beta(y), \quad j_{y^{3k-1}}^* f = \varphi \Rightarrow$ one of the three possibilities holds:

$$\begin{array}{ll}
j_{y^{3k}}^* f \approx \varphi+ax^2y^k+xy^{2k}, \quad a^2 \neq 4 & \mapsto \mathbf{64_k};\\
\approx \varphi+x^2y^k & \mapsto \mathbf{65_k};\\
= \varphi & \mapsto \mathbf{59_k}.
\end{array}$$

64$_{\mathbf{k}}$. $j_{y^{3k}}^* f = \varphi+ax^2y^k+xy^{2k}, \quad a^2 \neq 4 \Rightarrow f \in Q_{k,0}$.

65$_{\mathbf{k}}$. $j_{y^{3k}}^* f = \varphi+x^2y^k \Rightarrow f \in Q_{k,i} \quad (i>0)$.

Series S

Through theorems **66–81** $\varphi = x^2z+yz^2$, $j^* = j_{x^2z,yz^2,\lambda}$ (λ is a monomial).

66. $j_3 f = \varphi \Rightarrow f = \varphi+\alpha(y)+x\beta(y)+z\gamma(y), \quad j_3(\alpha, x\beta, z\gamma)=0, \quad \mapsto$ **67**$_{\mathbf{1}}$.

Through theorems **67–76**, $k \geqq 1$.

67_k. $f=\varphi+\alpha(y)+x\beta(y)+z\gamma(y)$,
$j^*_{y^{4k-1}}f=\varphi \Rightarrow$ one of the three possibilities holds:
$j^*_{y^{4k}}f \approx \varphi+y^{4k} \mapsto$ **68_k**;
$j^*_{xy^{3k}}f \approx \varphi+xy^{3k} \mapsto$ **69_k**;
$j^*_{xy^{3k}}f = \varphi \mapsto$ **70_k**.

68_k. $j^*_{y^{4k}}f=\varphi+y^{4k} \Rightarrow f\in S_{12k-1}$.

69_k. $j^*_{xy^{3k}}f=\varphi+xy^{3k} \Rightarrow f\in S_{12k}$.

70_k. $f=\varphi+\alpha(y)+x\beta(y)+z\gamma(y)$, $j^*_{xy^{3k}}f=\varphi \Rightarrow$ one of the four possibilities holds:
$j^*_{y^{4k+1}}f \approx \varphi+y^{4k+1}+bzy^{2k+1}$, $b^2 \neq 4 \mapsto$ **71_k**;
$\approx \varphi+x^2y^{2k} \mapsto$ **72_k**;
$\approx \varphi+zy^{2k+1} \mapsto$ **73_k**;
$=\varphi \mapsto$ **74_k**.

71_k. $j^*_{y^{4k+1}}f=\varphi+y^{4k+1}+bzy^{2k+1}$, $b^2 \neq 4 \Rightarrow f\in S_{k,0}$.

72_k. $j^*_{y^{4k+1}}f=\varphi+x^2y^{2k} \Rightarrow f\in S_{k,i}$ $(i>0)$.

73_k. $j^*_{y^{4k+1}}f=\varphi+zy^{2k+1} \Rightarrow f\in S^{\#}_{k,i}$ $(i>0)$.

74_k. $f=\varphi+\alpha(y)+x\beta(y)+z\gamma(y)$,
$j^*_{y^{4k+1}}f=\varphi \Rightarrow$ one of the three possibilities holds:
$j^*_{xy^{3k+1}}f \approx \varphi+xy^{3k+1} \mapsto$ **75_k**;
$j^*_{y^{4k+2}}f \approx \varphi+y^{4k+2} \mapsto$ **76_k**;
$j^*_{y^{4k+2}}f = \varphi \mapsto$ **77_{k+1}**.

75_k. $j^*_{xy^{3k+1}}f=\varphi+xy^{3k+1} \Rightarrow f\in S_{12k+4}$.

76_k. $j^*_{y^{4k+2}}f=\varphi+y^{4k+2} \Rightarrow f\in S_{12k+5}$.

Through theorems **77–81**, $k>1$.

77_k. $f=\varphi+\alpha(y)+x\beta(y)+z\gamma(y)$, $j^*_{y^{4k-2}}f=\varphi \Rightarrow$ one of the five possibilities holds:
$j^*_{y^{4k-1}}f \approx \varphi+ax^2y^{k-1}+bxy^kz+xy^{3k-1}$, $\Delta \neq 0 \mapsto$ **78_k**.
$\approx \varphi+xy^kz+ax^3y^{k-1}$, $a^2 \neq a \mapsto$ **79_k**.
$\approx \varphi+x^3y^{k-1} \mapsto$ **80_k**.
$\approx \varphi+xy^kz \mapsto$ **81_k**.
$=\varphi \mapsto$ **67_k**.

78_k. $j^*_{y^{4k-1}}f=ax^2y^{k-1}+bxy^kz+xy^{3k-1}$, $\Delta \neq 0 \Rightarrow f\in S^*_{k,0}$, $\mu(f)=12k-4$,
$m(f)=3k-2$, $c(S^*_{k,0})=9k-3$.

79_k. $j^*_{y^{4k-1}}f=\varphi+xy^kz+ax^3y^{k-1}$, $a^2 \neq a \Rightarrow f\in \mathbf{SP}_k$, $\mu(f)\geqq 12k-3$,
$m(f)=3k-2$, $c(\mathbf{SP}_k)=9k-2$.

80_k. $j^*_{y^{4k-1}}f=\varphi+x^3y^{k-1} \Rightarrow f\in \mathbf{SQ}_k$, $\mu(f)\geqq 12k-2$, $m(f)=3k-2$,
$c(\mathbf{SQ}_k)=9k-1$.

81_k. $j^*_{y^{4k-1}}f=\varphi+xy^kz \Rightarrow f\in \mathbf{SR}_k$, $\mu(f)\geqq 12k-2$, $m(f)=3k-2$,
$c(\mathbf{SR}_k)=9k-1$.

Series U

Through theorems **82–89**, $\varphi=x^3+xz^2$, $j^*=j_{x^3,z^3,\lambda}$ (λ is a monomial).

82. $j_3f=\varphi \Rightarrow f\sim\varphi+\alpha(y)+x\beta(y)+z\gamma(y)+x^2\delta(y)$, $j_3(\alpha, x\beta, z\gamma, x^2\delta)=0 \mapsto$ **83_1**.
Through theorems **83–89**, $k\geqq 1$.

$\mathbf{83_k}$. $f=\varphi+\alpha(y)+x\beta(y)+z\gamma(y)+x^2\delta(y)$, $j^*_{y^{3k}}f=\varphi \Rightarrow$ one of the two possibilities holds:
$j^*_{y^{3k+1}}f \approx \varphi+y^{3k+1} \mapsto \mathbf{84_k}$,
$\quad = \varphi \mapsto \mathbf{85_k}$.

$\mathbf{84_k}$. $j^*_{y^{3k+1}}f=\varphi+y^{3k+1} \Rightarrow f\in U_{12k}$.

$\mathbf{85_k}$. $f=\varphi+\alpha(y)+x\beta(y)+z\gamma(y)+x^2\delta(y)$, $j^*_{y^{3k+1}}f=\varphi \Rightarrow$ one of the three possibilities holds:
$j^*_{xy^{2k+1}}f \approx \varphi+xy^{2k+1}+czy^{2k+1}$, $c(c^2+1)\neq 0 \mapsto \mathbf{86_k}$;
$\quad \approx \varphi+xy^{2k+1} \mapsto \mathbf{87_k}$;
$\quad = \varphi \mapsto \mathbf{88_k}$.

$\mathbf{86_k}$. $j^*_{xy^{2k+1}}f=\varphi+xy^{2k+1}+czy^{2k+1}$, $c(c^2+1)\neq 0 \Rightarrow f\in U_{k,0}$.

$\mathbf{87_k}$. $j^*_{xy^{2k+1}}f=\varphi+xy^{2k+1} \Rightarrow f\in U_{k,p}$ $(p>0)$.

$\mathbf{88_k}$. $f=\varphi+\alpha(y)+x\beta(y)+z\gamma(y)+x^2\delta(y)$, $j^*_{xy^{2k+1}}f=\varphi \Rightarrow$ one of the two possibilities holds:
$j^*_{y^{3k+2}}f \approx \varphi+y^{3k+2} \mapsto \mathbf{89_k}$;
$\quad = \varphi \mapsto \mathbf{90_{k+1}}$.

$\mathbf{89_k}$. $j^*_{y^{3k+2}}f=\varphi+y^{3k+2} \Rightarrow f\in U_{12k+4}$.
Through theorems **90–96**, $k \geqq 2$;
$\varphi=x^2z+xz^2$; $j^*=j_{x^3,z^3,\lambda}$ (λ is a monomial).

$\mathbf{90_k}$. $f=x^3+xz^2+\alpha(y)+x\beta(y)+z\gamma(y)+x^2\delta(y)$, $j^*_{y^{3k-1}}f=x^3+xz^2 \Rightarrow$ one of the seven possibilities holds:
$j^*_{y^{3k}}f \approx \varphi+ax^2y^k+bxy^kz+y^kz^2+cxy^{2k}$, $\Delta\neq 0 \mapsto \mathbf{91_k}$; $\mapsto \mathbf{91_k}$;
$\quad \approx \varphi+ax^2y^k+bxy^kz+y^kz^2$, $4a\neq b^2$, $a(a+1-b)\neq 0 \mapsto \mathbf{92_k}$;
$\quad \approx x^3+ax^2z+xz^2+z^2y^k$, $a^2\neq 4 \mapsto \mathbf{93_k}$;
$\quad \approx \varphi+x^2y^k+axy^kz$, $a^2\neq a \mapsto \mathbf{94_k}$;
$\quad \approx \varphi+x^2y^k \mapsto \mathbf{95_k}$;
$\quad \approx \varphi+xy^kz \mapsto \mathbf{96_k}$;
$\quad = \varphi \mapsto \mathbf{83_k}$.

$\mathbf{91_k}$. $j^*_{y^{3k}}f=\varphi+ax^2y^k+bxy^kz+y^kz^2+cxy^{2k}$, $\Delta\neq 0 \Rightarrow f\in U^*_{k,0}$,
$\mu(f)=12k-4$, $m(f)=4k-3$, $c(U^*_{k,0})=8k-2$.

$\mathbf{92_k}$. $j^*_{y^{3k}}f=\varphi+ax^2y^k+bxy^kz+y^kz^2$, $4a\neq b^2$, $a(a+1-b)\neq 0 \Rightarrow f\in \mathbf{UP_k}$,
$\mu(f)\geqq 12k-3$, $m(f)=4k-3$, $c(\mathbf{UP_k})=8k-1$.

$\mathbf{93_k}$. $j^*_{y^{3k}}f=x^3+ax^2z+xz^2+z^2y^k$, $a^2\neq 4 \Rightarrow f\in \mathbf{UQ_k}$, $\mu(f)\geqq 12k-2$,
$m(f)=4k-3$, $c(\mathbf{UQ_k})=8k$.

$\mathbf{94_k}$. $j^*_{y^{3k}}f=\varphi+x^2y^k+axy^kz$, $a^2\neq a \Rightarrow f\in \mathbf{UR_k}$, $\mu(f)\geqq 12k-2$,
$m(f)=4k-3$, $c(\mathbf{UR_k})=8k$.

$\mathbf{95_k}$. $j^*_{y^{3k}}f=\varphi+x^2y^k \Rightarrow f\in \mathbf{US_k}$, $\mu(f)\geqq 12k-1$, $m(f)=4k-3$,
$c(\mathbf{US_k})=8k+1$.

$\mathbf{96_k}$. $j^*_{y^{3k}}f=\varphi+xy^kz \Rightarrow f\in \mathbf{UT_k}$, $\mu(f)\geqq 12k-1$, $m(f)=4k-3$,
$c(\mathbf{UT_k})=8k+1$.

Class V

97. $j_3 f(x,y,z)=x^2y \Rightarrow f \sim x^2y+\alpha(y,z)+x\beta(z) \mapsto$ **98.**

In theorems **98** and **102** φ is one of the 10 polynomials:
$z^4+z^3y,\ z^3y+z^2y^2,\ z^2y^2+zy^3,\ z^4+z^2y^2,\ z^4,\ z^3y,\ z^2y^2,\ zy^3,\ y^4,\ 0.$

98. $f=x^2y+\alpha(y,z)+x\beta(z),\quad j_3f=x^2y \Rightarrow$ one of the four possibilities holds:

$$\begin{aligned} j_{x^2y,y^4,z^4}f &\approx x^2y+z^4+az^3y+bz^2y^2+zy^3, & \Delta\neq 0, \quad ab\neq 9 &\mapsto \mathbf{99};\\ &\approx x^2y+z^4+bz^3y+z^2y^2, & b^2\neq 4 &\mapsto \mathbf{100};\\ &\approx x^2y+z^3y+az^2y^2+y^4, & 4a^3+27\neq 0 &\mapsto \mathbf{101};\\ &\approx x^2y+\varphi & &\mapsto \mathbf{102}. \end{aligned}$$

99. $j_{x^2y,y^4,z^4}f=x^2y+z^4+az^3y+bz^2y^2+zy^3,\quad \Delta\neq 0 \Rightarrow f\in V_{1,0}$.

100. $j_{x^2y,y^4,z^4}f=x^2y+z^4+bz^3y+z^2y^2,\quad b^2\neq 4 \Rightarrow f\in V_{1,p},\quad p>0.$

101. $j_{x^2y,y^4,z^4}f=x^2y+z^3y+az^2y^2+y^4,\quad 4a^3+27\neq 0 \Rightarrow f\in V^{\#}_{1,p},\quad p>0.$

102. $j_{x^2y,y^4,z^4}f=x^2y+\varphi \Rightarrow \mu(f)\geqq 17,\quad m(f)\geqq 3,\quad c(f)\geqq 13.$

103. $j_3f(x,y,z)=x^3 \Rightarrow \mu(f)\geqq 18,\quad m(f)\geqq 4,\quad c(f)\geqq 13.$

104. $j_3f(x,y,z)=0 \Rightarrow \mu(f)\geqq 27,\quad m(f)\geqq 10,\quad c(f)\geqq 16.$

105. Corank $f>3 \Rightarrow \mu(f)\geqq 16,\quad m(f)\geqq 5,\quad c(f)\geqq 10.$

§ 3. The Proofs

Theorems **1**, **17** and **25** are obvious. Theorems $\mathbf{12_1}$, **15**, **16** are proved in [8]. The proofs of theorems $\mathbf{12_k}$ ($k>1$), **24, 31, 32, 38–46, 65, 72, 73, 79, 80, 81, 87, 92–96, 100, 101** are based on new techniques as compared with [8] (some spectral sequence) and are not given here.

The proofs of the classification theorems for unimodular singularities do not depend on these theorems.

The classification of unimodular singularities is obtained from the following theorems:

1–5, $\mathbf{6_{1,2}}$–$\mathbf{9_{1,2}}$, $\mathbf{10_2}$, $\mathbf{11_2}$, **13–17**, $\mathbf{18_1}$–$\mathbf{21_1}$, $\mathbf{22_2}$, $\mathbf{23_2}$, **25**, $\mathbf{26_1}$–$\mathbf{30_1}$, $\mathbf{36_1}$, $\mathbf{37_1}$, **47**, **48**, **50–58**, $\mathbf{59_1}$–$\mathbf{62_1}$, $\mathbf{63_2}$, $\mathbf{64_2}$, **66**, $\mathbf{67_1}$–$\mathbf{71_1}$, **82**, $\mathbf{83_1}$–$\mathbf{86_1}$, **97**, **98**, **105**.

Here one needs only the simplest cases (the first one) of theorems $\mathbf{10_2}$, $\mathbf{22_2}$, $\mathbf{29_1}$, **36**, **47**, $\mathbf{63_2}$, $\mathbf{70_1}$, $\mathbf{85_1}$, **98**.

To classify all bimodular singularities one needs in addition theorems:

$\mathbf{6_3}$–$\mathbf{11_3}$, $\mathbf{12_2}$, $\mathbf{18_2}$–$\mathbf{21_2}$, $\mathbf{22_3}$, $\mathbf{23_3}$, $\mathbf{24_2}$, $\mathbf{31_1}$–$\mathbf{35_1}$, $\mathbf{36_2}$, $\mathbf{37_2}$, $\mathbf{59_2}$–$\mathbf{62_2}$, $\mathbf{63_3}$, $\mathbf{64_3}$, $\mathbf{65_2}$, $\mathbf{72_1}$–$\mathbf{76_1}$, $\mathbf{77_2}$, $\mathbf{78_2}$, $\mathbf{87_1}$–$\mathbf{89_1}$, $\mathbf{90_2}$, $\mathbf{91_2}$, **99**.

(In fact, only the simplest cases of theorems $\mathbf{10_3}$, $\mathbf{22_3}$, $\mathbf{36_2}$, $\mathbf{63_3}$, $\mathbf{77_2}$, $\mathbf{90_2}$ are needed.)

Theorems **6, 18, 26, 33, 59, 67, 74, 83, 88** are proved by the Newton method [9] of a moving ruler (line, plane). This method reduces the proof to the counting of the integer points in triangles resp. polyhedrones on the exponent plane (resp. in the space).

The proofs of theorems **3, 10, 13, 22, 29, 36, 47, 50, 52, 54, 56, 58, 63, 66, 70, 77, 82, 85, 90, 97, 98** can be reduced to the classifications of orbits of the actions of some quasihomogenous diffeomorphism groups on the spaces of quasihomenous polynomials.

Some quasihomogenous Lie algebra "roots" technique reduces all these classification problems to geometrical problems as shown in the table below.

Theorem	Series	Geometrical problem
3	D	Linear classification of the 3-forms in $\mathbb{C}^2$.
10, 22	J, Z	Affine classifications of the triples of points in $\mathbb{C}^1$.
13	X	Linear classification of the 4-forms in $\mathbb{C}^2$.
29	W	Linear classification of the couples of points in $\mathbb{C}^1$.
36	X	Affine classification of the quadruples of points in $\mathbb{C}^1$.
47	N	Linear classification of the 5-forms in $\mathbb{C}^2$.
50	P	Linear classification of the 3-forms in $\mathbb{C}^3$.
63	Q	Affine classification of the cubics with a finite cusp point in $\mathbb{C}^2$.
70	S	Affine classification of the cubics with at least 2 finite nodes in $\mathbb{C}^2$.
77	S*	Affine classification of the cubics in $\mathbb{C}^2$, having the infinity line a simple tangent.
85	U	Affine classification of the cubics in $\mathbb{C}^2$, admitting a centre of symmetry and having exactly 3 points at infinity.
90	U*	Affine classification of the cubics in $\mathbb{C}^2$, having exactly 3 points at infinity.
98	V	Affine classification of the polynomials of degree $\leqq 4$ in 1 variable in $\mathbb{C}^1$.

Theorems on the normal forms of the semiquasihomogenous functions (theorems **2**, **4**, **5**, **7**, **8**, **9**, **11**, **14**, **19**, **20**, **21**, **23**, **27**, **28**, **30**, **34**, **35**, **37**, **48**, **51**, **60**, **61**, **62**, **64**, **68**, **69**, **71**, **75**, **78**, **84**, **86**, **89**, **91**, **99**) and theorems **52**, **54**, **56**, **58**, **66**, **82**, **97** follow from section **7.3** in [8]. Theorems **49** and **102** are corollaries of **48** and **100**, **101**.

To prove theorems **53**, **55**, **57** we use the following lemmata (the terminology is that of [8]).

Lemma 1. *Let* $f_0 = a x_1 x_2 x_3 + x_1^{p_1} + x_2^{p_2} + x_3^{p_3}$, *where* $3 \leqq p_1 \leqq p_2 \leqq p_3 > 3$, $a \neq 0$. *Then*

1) *at* O f_0 *has an isolated critical point with* $\mu = p_1 + p_2 + p_3 - 1$;

2) *monomials* 1, $x_1 x_2 x_3$, $x_i^{s_i}$ $(0 < s_i < p_i,\ i = 1, 2, 3)$ *define a regular basis of the local ring*;

3) *condition A holds for the Newton filtration, defined by the 4 monomials of* f_0.

To prove this lemma we use the crossword technique (see [8], section **9.7**). We find the following 2 geometrical facts:

Lemma 2. *All the cycles formed by the admissible segments for* f_0, *are trivial.*

This follows from the linear independence of the 3 main segments; this independence holds when p_i satisfies the above stated restrictions.

Lemma 3. *The maximal admissible chains are:*

1) *every point* $x_i^{s_i}$ $(0 \leqq s_i \leqq p_i - 2)$ *is a trivial chain formed by this point*;

2) *there exist 4 finite maximal chains* $x_1 x_2 - x_3^{p_3 - 1}$, $x_2 x_3 - x_1^{p_1 - 1}$, $x_3 x_1 - x_2^{p_2 - 1}$, $x_1 x_2 x_3 - x_i^{p_i}$;

3) *all other points have infinite admissible chains.*

Lemma 1 is an easy corollary of lemmata 2 and 3.

The proofs of theorems **53**, **55**, **57**.

These theorems can be stated as follows.

Theorem. *Let f have a critical point O of finite multiplicity and*

$$f = xyz + \alpha(x) + \beta(y) + \gamma(z), \qquad j_2(\alpha, \beta) = 0, \qquad j_3(\gamma) = 0.$$

Then in some neighbourhood of O

$$f \sim f_0 = axyz + x^p + y^q + z^r, \qquad a \neq 0, \ 3 \leqq p \leqq q \leqq r > 3.$$

Proof. Let the first nonzero terms of the Taylor series for α, β, γ have exponents p, q, r. By changing if nesseserly the notations, we obtain $3 \leqq p \leqq q \leqq r > 3$. By dilatations of coordinates we reduce f to the form $f_0 + f_1$, where all the exponents of the monomials of f_1 are above the Newton polyhedron of f_0.

According to lemma 1, all the monomials of f_1 belong to the ideal $(\partial f_0/\partial x_i)$, and f_0 satisfies the condition A of [8].

By theorem **9.5** of [8] we have $f \sim f_0$; theorems **53**, **55**, **57** are thus proved.

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