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On Zurek's Derivation of the Born Rule

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Recently, W. H. Zurek presented a novel derivation of the Born rule based on a mechanism termed environment-assisted invariance, or "envariance" [W. H. Zurek, Phys. Rev. Lett. **90***(2), 120404 (2003)]. We review this approach and identify fundamental assumptions that have implicitly entered into it, emphasizing issues that any such derivation is likely to face.*

KEY WORDS: Born rule; quantum probabilities; environment-assisted invariance.

1. INTRODUCTION

In standard quantum mechanics, Born's rule[(1)] is simply postulated. A typical formulation of this rule reads:

> If an observable $\widehat{O}$, with eigenstates $\{|o_i\rangle\}$ and spectrum $\{o_i\}$, is measured on a system described by the state vector $|\psi\rangle$, the probability for the measurement to yield the value o_i is given by $p(o_i) = |\langle o_i|\psi\rangle|^2$.

Born's rule is of paramount importance to quantum mechanics as it introduces a probability concept into the otherwise deterministic theory and relates it mathematically to the Hilbert space formalism. No violation of Born's rule has ever been discovered experimentally—which has certainly supported the role of the Born rule as the favorite ingredient of what has been nicknamed the "shut up and calculate" interpretation of quantum mechanics. (Although often attributed to Feynman, it appears that the nickname was actually coined by Mermin.[(2)] For an example of such a stance, see[(3)]).

Replacing the postulate of Born's rule by a derivation would be a highly desirable goal within quantum theory in general. The famous

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theorem of Gleason[4] presented a mathematical motivation for the form of the Born probabilities by showing that if one would like to assign a non-negative real valued function $p(v)$ to every vector v of a vector space $\mathcal{V}$ of dimension greater than two such that for every orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathcal{V}$ the sum of the $p(v_i)$ is equal to one,

$$\sum_i p(v_i) = 1, \tag{1}$$

then the only possible choice is $p(v) = |\langle v|w\rangle|^2$ for all vectors v and an arbitrary but fixed vector w, provided that probabilities are assumed to be non-contextual. The normalization requirement of Eq. (1) for $p(v)$ with respect to any orthonormal basis can be physically motivated by remembering that any orthonormal basis $\{v_1, \dots, v_n\}$ can be viewed as the eigenbasis of observables $\widehat{O} = \sum_i \lambda_i |v_i\rangle\langle v_i|$, and by referring to the fact that in every measurement of such an observable $\widehat{O}$ on a system with state vector w one outcome (represented by the eigenvalue λ_j corresponding to one of the eigenvectors v_j) will occur, such that $p(v_j) = 1$ and $p(v_i) = 0$ for $i \neq j$, and Eq. (1) follows. In spite of its mathematical elegance, Gleason's theorem is usually considered as giving rather little physical insight into the emergence of quantum probabilities and the Born rule.

Other attempts towards a consistent derivation of the Born probabilities have previously been made in particular in the context of relative-state interpretations where both the meaning of probabilities and their relation to Born's rule requires explicit elucidation (see, for example, Refs. 5–10), but the success of these approaches is controversial.[11–13] A widely disputed derivation of the Born rule that is solely based on the non-probabilistic axioms of quantum mechanics and on classical decision theory (and that is more physically motivated than Gleason's argument) has been proposed by Deutsch.[14] It was criticized by Barnum *et al.*[15] but was subsequently defended by Wallace[16] and put into an operational framework by Saunders;[17] no decisive conclusion seems to have been reached on the success of these derivations thus far.

A novel and interesting proposal towards a derivation of Born's rule has recently been put forward by Zurek[18] (see also the follow-ups in Refs. 19, 20). Zurek is a key figure in the development of the decoherence program (for a recent survey of the program and further references, see Refs. 20, 21) that is based on a study of open quantum systems and their interaction with the many degrees of freedom of their environment, leading to explanations for the emergence of the "classical" world of our observation. However, one of the remaining loopholes in a consistent derivation of classicality from decoherence and standard non-collapse quantum mechanics alone has been tied to the fact that the formalism of decoherence and its interpretation rely implicitly on Born's rule, but that decoherence does not yield an independent motivation for the connection between the quantum mechanical state space formalism and probabilities. Any derivation of the Born rule from decoherence[22] is therefore subject to the charge of circularity.[23]

To address this criticism, Zurek has suggested a derivation of Born's rule that is based on the inclusion of the environment—thus matching well the spirit of the decoherence program—but without relying on the key elements of decoherence that presume Born's rule and would thus render the argument circular. Zurek's derivation is of course not only relevant in the context of the decoherence program.

Because we consider Zurek's approach promising, we would like to bring out the assumptions that enter into the derivation but have not been explicitly mentioned in Refs. 18–20. Hopefully such an analysis will help in a careful evaluation of the question to what extent Zurek's derivation can be regarded as fundamental. In fact, after this paper had been posted online as a preprint, two other discussions of Zurek's argument have appeared that also describe variants of the proof.[24,25] Moreover, Zurek himself[26] has recently revised his original derivation in a way that addresses several of the issues raised in this article and that is more explicit about the assumptions (some designated now as "facts") used in his proof. These correspond to what we identify in the following discussion.

To anticipate, we find that Zurek's derivation is based on at least the following assumptions:

(1) The probability for a particular outcome, i.e., for the occurrence of a specific value of a measured physical quantity, is identified with the probability for the eigenstate of the measured observable with eigenvalue corresponding to the measured value—an assumption that would follow from the *eigenvalue–eigenstate link*.

(2) Probabilities of a system $\mathcal{S}$ entangled with another system $\mathcal{E}$ are a function of the *local* properties of $\mathcal{S}$ only, which are exclusively determined by the state vector of the *composite* system $\mathcal{SE}$.

(3) For a composite state in the Schmidt form $|\psi_{\mathcal{SE}}\rangle = \sum_k \lambda_k |s_k\rangle|e_k\rangle$, the probability for $|s_k\rangle$ is *equal* to the probability for $|e_k\rangle$.

(4) Probabilities associated with a system $\mathcal{S}$ entangled with another system $\mathcal{E}$ remain *unchanged* when certain transformations (namely, Zurek's "envariant transformations") are applied that only act on $\mathcal{E}$ (and similarly for $\mathcal{S}$ and $\mathcal{E}$ interchanged).

Our paper is organized as follows. First, we review Zurek's derivation of the Born rule as given in his original papers,[18–20] and also include a line of reasoning presented in his recent follow-up[26] (that in turn takes issues raised in the following discussion into account). We then elucidate and discuss step-by-step the assumptions that we believe have entered into Zurek's approach. In the final section, we summarize our main points.

2. REVIEW OF ZUREK'S DERIVATION

(I) Zurek suggests a derivation of Born's rule for the following pure state that describes an entanglement between a system $\mathcal{S}$, described by a Hilbert space $\mathcal{H}_\mathcal{S}$, and its environment $\mathcal{E}$, represented by a Hilbert space $\mathcal{H}_\mathcal{E}$:

$$|\psi_{\mathcal{S}\mathcal{E}}\rangle = \sum_k \lambda_k |s_k\rangle|e_k\rangle, \tag{2}$$

where $\{|s_k\rangle\}$ and $\{|s_k\rangle\}$ are orthonormal bases of $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_\mathcal{E}$, respectively. Zurek holds that after the $\mathcal{S}\mathcal{E}$ correlation has been established, the system no longer interacts with the environment (Ref. 19, p. 10), i.e., that $\mathcal{E}$ is "dynamically decoupled" [Ref. 18, p. 120404-1] and thus "causally disconnected" [Ref. 20, p. 754] from $\mathcal{S}$.

For the sake of clarity and simplicity, we shall in the following restrict ourselves to the case of coefficients of equal magnitude and to two-dimensional state spaces $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_\mathcal{E}$, i.e., we consider the state

$$|\psi_{\mathcal{S}\mathcal{E}}\rangle = \frac{1}{\sqrt{2}}\left(e^{i\alpha_1}|s_1\rangle|e_1\rangle + e^{i\alpha_2}|s_2\rangle|e_2\rangle\right). \tag{3}$$

Once a valid derivation of Born's rule is accomplished for this situation, the case of non-equal probabilities and of state spaces of more than two dimensions can be treated by means of a relatively straightforward counting argument[18] (at least for probabilities that are rational numbers). What Zurek's derivation now aims to establish is the result that for an observer of $\mathcal{S}$, the probabilities for $|s_1\rangle$ and $|s_2\rangle$ will be equal. That claim is the focus of our analysis.

(II) Zurek considers pairs of unitary transformations $\widehat{U}_\mathcal{S} = \widehat{u}_\mathcal{S} \otimes \widehat{I}_\mathcal{E}$ and $\widehat{U}_\mathcal{E} = \widehat{I}_\mathcal{S} \otimes \widehat{u}_\mathcal{E}$. Here $\widehat{u}_\mathcal{S}$ acts only on the Hilbert state space $\mathcal{H}_\mathcal{S}$ of $\mathcal{S}$, and $\widehat{I}_\mathcal{E}$ is the identity operator in $\mathcal{H}_\mathcal{E}$. Similarly $\widehat{u}_\mathcal{E}$ acts only on the Hilbert state space $\mathcal{H}_\mathcal{E}$ of $\mathcal{E}$, and $\widehat{I}_\mathcal{S}$ is the identity operator in $\mathcal{H}_\mathcal{S}$.

If the composite state $|\psi_{\mathcal{S}\mathcal{E}}\rangle$ is invariant under the combined application of $\widehat{U}_\mathcal{S}$ and $\widehat{U}_\mathcal{E}$,

$$\widehat{U}_\mathcal{E}(\widehat{U}_\mathcal{S}|\psi_{\mathcal{S}\mathcal{E}}\rangle) = |\psi_{\mathcal{S}\mathcal{E}}\rangle, \tag{4}$$

the composite state is called *envariant under* $\widehat{u}_{\mathcal{S}}$. (The word "envariant" stems from the abbreviation "envariance" of the term "environment-assisted invariance," an expression coined by Zurek). Zurek gives the following interpretation of envariance (Ref. 18, p. 120404-1):

> When the transformed property of the system can be so "untransformed" by acting only on the environment, it is not the property of $\mathcal{S}$. Hence, when $\mathcal{SE}$ is in the state $|\psi_{\mathcal{SE}}\rangle$ with this characteristic, it follows that the envariant properties of $\mathcal{S}$ must be completely unknown.

It is difficult to understand just what the term "property" refers to here, since it is the composite state that is transformed and untransformed, and so the "properties" involved would seem to be features of the state, not of the system. It seems that envariance under $\widehat{u}_{\mathcal{S}}$ is taken to imply that an observer who "in the spirit of decoherence" (Ref. 19, p. 10) only has access to $\mathcal{S}$ will not be able to determine features of the combined state that are affected by $\widehat{u}_{\mathcal{S}}$ (or, more properly, by $\widehat{U}_{\mathcal{S}}$). For such an observer a local description of $\mathcal{S}$ will be independent of these features, which may depend on a particular decomposition. While this general description is far from precise, the uses to which Zurek puts envariance are clear enough.

(IIa) The first type of an envariant transformation that Zurek considers is the pair

$$\widehat{u}_{\mathcal{S}}^{(\beta_1,\beta_2)} = e^{i\beta_1}|s_1\rangle\langle s_1| + e^{i\beta_2}|s_2\rangle\langle s_2|, \quad \widehat{u}_{\mathcal{E}}^{(\beta_1,\beta_2)} = e^{-i\beta_1}|e_1\rangle\langle e_1| + e^{-i\beta_2}|e_2\rangle\langle e_2|. \tag{5}$$

The effect of the first transformation $\widehat{U}_{\mathcal{S}} = \widehat{u}_{\mathcal{S}}^{(\beta_1,\beta_2)} \otimes \widehat{I}_{\mathcal{E}}$ is to change the phases associated with the terms in the Schmidt state, Eq. (3), that is,

$$\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle = \frac{1}{\sqrt{2}}\left(e^{i(\alpha_1+\beta_1)}|s_1\rangle|e_1\rangle + e^{i(\alpha_2+\beta_2)}|s_2\rangle|e_2\rangle\right). \tag{6}$$

It is easy to see that if one subsequently acts on this state with $\widehat{U}_{\mathcal{E}} = \widehat{I}_{\mathcal{S}} \otimes \widehat{u}_{\mathcal{E}}^{(\beta_1,\beta_2)}$, the original $|\psi_{\mathcal{SE}}\rangle$ will be restored. Thus, $|\psi_{\mathcal{SE}}\rangle$, and in particular the phases associated with the states in the Schmidt decomposition of $|\psi_{\mathcal{SE}}\rangle$, are envariant under the phase transformation $\widehat{u}_{\mathcal{S}}^{(\beta_1,\beta_2)}$ given by Eq. (5).

In the spirit of Zurek's interpretation of envariance stated above, this implies that the phases of the Schmidt coefficients are not a property of $\mathcal{S}$ alone, so that a local description of $\mathcal{S}$ cannot depend on the phases α_1 and α_2 in the composite state $|\psi_{\mathcal{SE}}\rangle$ of Eq. (3). This leads Zurek to the conclusion that also the probabilities associated with $\mathcal{S}$ must be independent of these phases, and that it thus suffices to show that equal likelihoods arise for the state

$$|\psi_{\mathcal{S}\mathcal{E}}\rangle = \frac{1}{\sqrt{2}}\Big(|s_1\rangle|e_1\rangle + |s_2\rangle|e_2\rangle\Big). \tag{7}$$

We shall therefore use this state in the rest of the argument.

(IIb) Another type of envariant transformations relevant to Zurek's derivation are the so-called "swaps,"

$$\widehat{u}_{\mathcal{S}}^{(1\leftrightarrow 2)} = |s_1\rangle\langle s_2| + |s_2\rangle\langle s_1|, \tag{8}$$

$$\widehat{u}_{\mathcal{E}}^{(1\leftrightarrow 2)} = |e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|. \tag{9}$$

Application of $\widehat{U}_{\mathcal{S}} = \widehat{u}_{\mathcal{S}}^{(1\leftrightarrow 2)} \otimes \widehat{I}_{\mathcal{E}}$, with $\widehat{u}_{\mathcal{S}}^{(1\leftrightarrow 2)}$ from Eq. (8), to the state $|\psi_{\mathcal{S}\mathcal{E}}\rangle$ in Eq. (7) yields

$$\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{S}\mathcal{E}}\rangle = \frac{1}{\sqrt{2}}\Big(|s_2\rangle|e_1\rangle + |s_1\rangle|e_2\rangle\Big), \tag{10}$$

i.e., the states of the environment $\mathcal{E}$ correlated with the states of the system $\mathcal{S}$ have been interchanged. This swap can obviously be undone by a "counterswap" $\widehat{U}_{\mathcal{E}} = \widehat{I}_{\mathcal{S}} \otimes \widehat{u}_{\mathcal{E}}^{(1\leftrightarrow 2)}$, with $\widehat{u}_{\mathcal{E}}^{(1\leftrightarrow 2)}$ from Eq. (9), applied to the state $\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{S}\mathcal{E}}\rangle$ in Eq. (10). Thus, the composite state $|\psi_{\mathcal{S}\mathcal{E}}\rangle$, Eq. (7), is envariant under swaps. The invariant property is then "$|s_k\rangle$ is correlated with $|e_l\rangle$." On the basis of the interpretation of envariance quoted above, this implies that a local description of $\mathcal{S}$ must be independent of which particular environmental state $|e_l\rangle$ is correlated with a given $|s_k\rangle$, i.e., that swapping of the states of the system cannot be detected by a local observation of $\mathcal{S}$ alone.

(IIIa) To make the connection between envariance of $|\psi_{\mathcal{S}\mathcal{E}}\rangle$ under swaps with quantum probabilities and Born's rule, Zurek states (Ref. 18, p. 120404-2):

> Let us now make a rather general (and a bit pedantic) assumption about the measuring process: When the states are swapped, the corresponding probabilities get relabeled ($i \leftrightarrow j$). This leads us to conclude that the probabilities for any two envariantly swappable $|s_k\rangle$ are equal.

This argument assumes that the swapping transformation (that interchanges the correlations between the states of the system and the environment) also swaps the probabilities associated with the states of the system.

To motivate this assumption, the following line of reasoning has been described to us by Zurek in private communication and has subsequently also appeared in published form in Refs. 24, 26. Let $p(|s_1\rangle; |\psi_{\mathcal{S}\mathcal{E}}\rangle)$ denote the probability for $|s_1\rangle$ when the $\mathcal{SE}$ combination is in the composite state $|\psi_{\mathcal{SE}}\rangle$, and similarly for $|s_2\rangle$, $|e_1\rangle$ and $|e_2\rangle$. Before the first swap, Zurek states that

$$\begin{aligned} p(|s_1\rangle; |\psi_{\mathcal{SE}}\rangle) &= p(|e_1\rangle; |\psi_{\mathcal{SE}}\rangle), \\ p(|s_2\rangle; |\psi_{\mathcal{SE}}\rangle) &= p(|e_2\rangle; |\psi_{\mathcal{SE}}\rangle), \end{aligned} \tag{11}$$

by referring to the direct connection between the states of $\mathcal{S}$ and $\mathcal{E}$ in the state vector expansion Eq. (3). After the first swap (acting on $\mathcal{S}$),

$$\begin{aligned} p(|s_1\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|e_2\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle), \\ p(|s_2\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|e_1\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle), \end{aligned} \tag{12}$$

where we have used $\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle$ instead of $|\psi_{\mathcal{SE}}\rangle$ as the second argument of the probability function to take into account the transformation of the state of $\mathcal{SE}$. Zurek now holds that under a swap, properties of the environment cannot have been affected by the first swap acting on the system $\mathcal{S}$ only, so that we must have

$$\begin{aligned} p(|e_1\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|e_1\rangle; |\psi_{\mathcal{SE}}\rangle), \\ p(|e_2\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|e_2\rangle; |\psi_{\mathcal{SE}}\rangle). \end{aligned} \tag{13}$$

After the application of the counterswap, we get

$$\begin{aligned} p(|s_1\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|e_1\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle), \\ p(|s_2\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|e_2\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle), \end{aligned} \tag{14}$$

where Zurek has that

$$\begin{aligned} p(|s_1\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|s_1\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle), \\ p(|s_2\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|s_2\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle), \end{aligned} \tag{15}$$

since the counterswap only acted on $\mathcal{E}$. Moreover, since after the counterswap the final state vector will be identical to the initial state vector, Zurek concludes that

$$\begin{aligned} p(|s_1\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|s_1\rangle; |\psi_{\mathcal{SE}}\rangle), \\ p(|s_2\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|s_2\rangle; |\psi_{\mathcal{SE}}\rangle), \\ p(|e_1\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|e_1\rangle; |\psi_{\mathcal{SE}}\rangle), \\ p(|e_2\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) &= p(|e_2\rangle; |\psi_{\mathcal{SE}}\rangle). \end{aligned} \tag{16}$$

This implies, from the above Eqs. (11)–(16), that

$$p(|s_1\rangle; |\psi_{\mathcal{SE}}\rangle) = p(|s_1\rangle; \widehat{U}_{\mathcal{E}}\widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) = p(|s_1\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) = p(|e_2\rangle; |\psi_{\mathcal{SE}}\rangle)$$
$$= p(|s_2\rangle; |\psi_{\mathcal{SE}}\rangle), \quad (17)$$

which establishes the desired result $p(|s_1\rangle; |\psi_{\mathcal{SE}}\rangle) = p(|s_2\rangle; |\psi_{\mathcal{SE}}\rangle)$.

(IIIb) Since the connection between envariance under swaps and equal probabilities is the crucial step in the derivation, we would like to mention another line of argument found in Zurek's papers(19,20) that more explicitly connects envariance with ignorance and the information available to a local observer. Here, Zurek considers a von Neumann measurement carried out on the composite state vector $|\psi_{\mathcal{SE}}\rangle$ by an observer, described by "memory states" $|\mu_0\rangle$ (the premeasurement memory state) and $|\mu_1\rangle$, $|\mu_2\rangle$ (the post-measurement memory states corresponding to the perception of the "outcomes" $|s_1\rangle$ and $|s_2\rangle$, respectively):

$$|\mu_0\rangle|\psi_{\mathcal{SE}}\rangle \propto |\mu_0\rangle\big(|s_1\rangle|e_1\rangle + |s_2\rangle|e_2\rangle\big) \longrightarrow |\mu_1\rangle|s_1\rangle|e_1\rangle + |\mu_2\rangle|s_2\rangle|e_2\rangle. \quad (18)$$

Zurek then states (Ref. 20, p. 755):

> [Envariance of $|\psi_{\mathcal{SE}}\rangle$ under swaps] allows the observer (who knows the joint state of $\mathcal{SE}$ exactly) to conclude that the probabilities of all the envariantly swappable outcomes must be the same. The observer cannot predict his memory state after the measurement of $\mathcal{S}$ because he knows too much: the exact combined state of $\mathcal{SE}$. (...) Probabilities refer to the guess the observer makes on the basis of his information before the measurement about the state of his memory—the future outcome—after the measurement. Since the left-hand side of Eq. (18) is envariant under swaps of the system states, the probabilities of all the states must be equal.

In a different paper, Zurek argues (Ref. 19, p. 12):

> When the state of the observer's memory is not correlated with the system, and the absolute values of the coefficients in the Schmidt decomposition of the entangled state describing $\mathcal{SE}$ are all equal, and $\mathcal{E}$ cannot be accessed, the resulting state of $\mathcal{S}$ is *objectively invariant* under all *local* measure-preserving transformations. Thus, with no need for further excuses, probabilities of events $\{|s_k\rangle\}$ must be—prior to measurement—equal.

Obviously, these arguments appeal to a rather different explanation for the emergence of equal likelihoods from the envariance of $|\psi_{\mathcal{SE}}\rangle$ under swaps than the previously quoted argument. Now, probabilities are introduced from the point of view of the observer to account for his lack of knowledge of the individual state of $\mathcal{S}$, since he has perfect knowledge of the composite state of $\mathcal{SE}$. Then, so goes Zurek's claim, since the observer cannot detect the swapping of the possible outcome states $|s_1\rangle$ and $|s_2\rangle$ of $\mathcal{S}$ before the measurement, he will regard them as "equivalent" and therefore attach equal likelihoods to them.

3. DISCUSSION

(A) First of all, Zurek's derivation intrinsically requires the split of the total Hilbert space $\mathcal{H}$ into at least two subspaces $\mathcal{H}_{\mathcal{S}}$ and $\mathcal{H}_{\mathcal{E}}$ which are identified with a system $\mathcal{S}$ and its environment $\mathcal{E}$, where $\mathcal{S}$ is presumed to have interacted with $\mathcal{E}$ at some point in the past. The environment is then responsible for the emergence of probabilities within the system, similar to the spirit of decoherence where the environment is responsible for the emergence of subjective classicality within the system.[20,21]

The obvious question is then to what extent the necessity to include the environment constitutes a restriction of generality. Apart from the problem of how to do cosmology, we might take a pragmatic point of view here by stating that any observation of the events to which we wish to assign probabilities will always require a measurement-like context that involves an open system interacting with an external observer, and that therefore the inability of Zurek's approach to derive probabilities for a closed, undivided system should not be considered as a shortcoming of the derivation.

(B) Secondly, we might wonder whether the choice of the entangled pure Schmidt state, Eq. (2), implies a lack of generality in the derivation. Any two-system composite pure state can be diagonalized in the Schmidt form above, so the particular form of the expansion of $|\psi_{\mathcal{SE}}\rangle$ implies no loss of generality. Furthermore, if $\rho_{\mathcal{SE}}$ were non-pure, it could be made pure simply by enlarging the space $\mathcal{H}_{\mathcal{E}}$, which cannot influence probabilities of $\mathcal{S}$ since $\mathcal{E}$ is assumed to be dynamically decoupled from $\mathcal{S}$ after the initial interaction that established the entanglement between $\mathcal{S}$ and $\mathcal{E}$.[19] We thus conclude that once the requirement for openness is acknowledged, the consideration of the state $|\psi_{\mathcal{SE}}\rangle$, Eq. (2), will suffice for a general derivation of the Born rule.

(C) Before introducing any probability concept into quantum theory, we need to define what these probabilities are supposed to be assigned to. Clearly, from the point of view of observations and measurements, we would like to assign probabilities to the occurrence of the specific values of the observable O that has been measured, i.e., to the "outcomes". The eigenvalue–eigenstate link of quantum mechanics postulates that a system has a value for an observable if and only if the state of the system is an eigenstate characteristic of that value (or a proper mixture of those eigenstates). If we consider only a measurement situation, one way of getting this link is first to assume that the only possible values are outcomes of measurements and that those are restricted to the eigenvalues o_i of an operator $\widehat{O}$ that represents the measured observable O. If one then assumes the collapse or projection postulate, that after the measurement the state of the system will be in an eigenstate $|o_i\rangle$ of $\widehat{O}$, it follows that in the non-degenerate case (i.e., when a certain eigenvalue corresponds only to a single eigenvector of the operator observable) an outcome o_i (the value of a physical quantity that appears in a measurement) can be directly related to the eigenstate $|o_i\rangle$ of the measured operator $\widehat{O}$, as the eigenvalue–eigenstate link requires, and we can talk equivalently about the probability for a certain outcome, or eigenvalue, or eigenstate.

The basis states $\{|s_1\rangle, |s_2\rangle\}$ and $\{|e_1\rangle, |e_2\rangle\}$ appearing in the composite Schmidt state $|\psi_{\mathcal{SE}}\rangle$ of Eq. (3) may then be thought of as the eigenstates of operator observables $\widehat{O}_{\mathcal{S}}$ and $\widehat{O}_{\mathcal{E}}$. In this sense, Zurek's derivation tries to establish that for the state $|\psi_{\mathcal{SE}}\rangle$ of Eq. (3), the outcomes represented by the eigenvalues s_1 and s_2 corresponding to the eigenstates $|s_1\rangle$ and $|s_2\rangle$ of an operator observable $\widehat{O}_{\mathcal{S}} = s_1|s_1\rangle\langle s_1| + s_2|s_2\rangle\langle s_2|$ are equally likely. However, in the context of the relative-state view that Zurek promotes, he never explicitly talks about observables and instead directly speaks of determining the "probabilities of events $\{|s_k\rangle\}$" (Ref. 19, p. 12). This identifies the probability for the occurrence of a specific value of a measured physical quantity with the probability for an eigenstate of the measured observable with an eigenvalue equal to the measured value. That assumption would be justified by the eigenvalue–eigenstate link, although it does not require it.

(D) Zurek furthermore assumes that the probabilities of the outcomes associated with the *individual* states $\{|s_1\rangle, |s_2\rangle\}$ of $\mathcal{S}$ and $\{|e_1\rangle, |e_2\rangle\}$ of $\mathcal{E}$ are functions of the *composite* state vector $|\psi_{\mathcal{SE}}\rangle$ only. (Zurek spells out the assumption that the derivation will be based on the composite state vector but without direct reference to probabilities: "Given the state of the combined $\mathcal{SE}$ expressed in the Schmidt form (...) what sort of invariant *quantum facts* can be known about $\mathcal{S}$?" (Ref. 18, p. 120404-1).) This assumption about the functional dependence of the probabilities is certainly reasonable, especially since Zurek's aim is clearly to derive Born's rule from within standard quantum mechanics, where the state vector is assumed to provide a complete description of the physical system. The assumption, of course, might well be questioned in a hidden variable or modal interpretation.

But Zurek's argument requires actually a more detailed assumption than stated so far. Obviously, since the $\mathcal{SE}$ composition is in an entangled pure state, there is no individual state vector of $\mathcal{S}$ alone. But Zurek *infers* the properties of $\mathcal{S}$ from the *composite* state vector $|\psi_{\mathcal{SE}}\rangle$ by studying its properties under envariant transformations. The idea is to use envariance to deduce statements about $\mathcal{S}$ alone. The assumption is now that probabilities are *local* in the sense that the probabilities that an observer of $\mathcal{S}$ alone can associate with the "events" $|s_k\rangle$—following Zurek's identification of outcomes with eigenstates, cf. our discussion in (C)—only depend on the *local* properties of $\mathcal{S}$ (i.e., those properties that cannot be affected by envariant transformations). An analogous assumption must also be invoked with respect to $\mathcal{E}$ for Zurek's argument to go through: probabilities for the states $|e_k\rangle$ of $\mathcal{E}$ are only dependent on the local properties of $\mathcal{E}$. This is used to infer the crucial conclusion that probabilities of $\mathcal{S}$ and $\mathcal{E}$ must be independent of the envariant properties of $\mathcal{SE}$ (i.e., properties of $\mathcal{SE}$ that do not belong to $\mathcal{S}$ or $\mathcal{E}$ individually).

This locality of probabilities can be related to the decomposition of the total Hilbert space into the state space of the system and the state space of the environment, together with the focus of any observation on the system alone, on which also the whole definition of envariance relies. But without having explicitly connected the Hilbert state space description with the functional dependence of the probabilities on the state, affirming that probabilities of $\mathcal{S}$ and $\mathcal{E}$ can only depend on the local properties of $\mathcal{S}$ and $\mathcal{E}$, respectively, must be counted as an important additional assumption.

(E) We would also like to point out that Zurek holds that his argument does not require a causality or locality assumption, but that reference to envariance suffices (see, for example, Ref. 18, p. 120404-2). Zurek suggests that one could alternatively argue for the independence of probabilities from an envariant property of the entangled $\mathcal{SE}$ combination directly if causality and the impossibility of faster-than-light signaling is assumed. He claims that a measurable property associated with $\mathcal{S}$ cannot depend on an envariant property of the entangled $\mathcal{SE}$ state, since otherwise one could influence measurable properties of $\mathcal{S}$ by acting on a "distant" environment $\mathcal{E}$, and superluminal communication would be possible. We do not find this argument compelling, since influencing measurable properties of a system entangled with a distant partner by locally acting on the partner does not necessarily require the effect to be instantaneously transmitted to the system. Even if it were, it is not clear that this would entail any violation of relativistic no-signaling requirements (witness the Bohm theory!).

Of course, even if the assumption of causality and the impossibility of faster-than-light signaling indeed justified the conclusion that envariant properties of $\mathcal{SE}$ cannot influence locally measurable physical quantities of $\mathcal{S}$, any necessity for an appeal to causality to justify Eq. (13) would be rather undesired, since the goal is to derive the Born rule from quantum theory alone which, strictly speaking, does not entail the impossibility of superluminal communication. Zurek is clearly aware of this point by stating that causality is "more potent" (Ref. 20, p. 754) and "more 'costly' (and not entirely quantum)" (Ref. 18, p. 120404-2) than envariance. He consequently holds that his derivation only requires envariance, although he sometimes seems to implicitly refer to causality, for instance in arguing that "only the absolute values of the coefficients can matter since phases [of the coefficients] can be altered by acting on $\mathcal{E}$ alone, and $\mathcal{E}$ is causally disconnected from $\mathcal{S}$" (Ref. 19, p. 10).

(F) Let us now turn to the chain of relations between the probabilities as established in Eqs. (11)–(17).

(F1) The first relations, Eqs. (11), infer equal probabilities for the outcomes represented by $|s_1\rangle$ and $|e_1\rangle$ from their correlation in the direct product $|s_1\rangle|e_1\rangle$ as appearing in the composite state vector $|\psi_{\mathcal{SE}}\rangle$ in the Schmidt decomposition, Eq. (7). From a point of view that *presupposes* Born's rule, this assumption is of course trivially fulfilled, since a simple projection yields

$$|\underbrace{\langle e_1|\langle s_1|s_1\rangle|e_1\rangle}_{=1} + \underbrace{\langle e_2|\langle s_1|s_1\rangle|e_1\rangle}_{=0}|^2 = |\underbrace{\langle e_1|\langle s_1|s_1\rangle|e_1\rangle}_{=1} + \underbrace{\langle e_1|\langle s_2|s_1\rangle|e_1\rangle}_{=0}|^2, \tag{19}$$

due to orthonormality of the Schmidt basis states $\{|s_1\rangle, |s_2\rangle\}$ and $\{|e_1\rangle, |e_2\rangle\}$. But without this (obviously undesired) presupposition, the relations in Eqs. (11) represent an additional assumption about the connection between state and probabilities which does not follow from the assumption (D) that probabilities are a function of the state vector only.

Of course Eqs. (11) may seem innocuous because most of us are accustomed to thinking in terms of state space projections, and we make an intuitive connection to probabilities from such projections. But it seems important in evaluating a derivation of the quantum probability concept and Born's rule to be aware of where such presupposed conceptions enter, as assumptions, into the derivation.

(F2) Yet another important assumption appears to be contained in Eqs. (13). We recall that Zurek justified the relations $p(|e_1\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) = p(|e_1\rangle; |\psi_{\mathcal{SE}}\rangle)$ and $p(|e_2\rangle; |\psi_{\mathcal{SE}}\rangle) = p(|e_2\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle)$ by saying that the probabilities associated with the environment $\mathcal{E}$ cannot change as a result of the envariant swap acting on $\mathcal{S}$ since this swap cannot affect properties of $\mathcal{E}$. An analogous statement is made in justifying the relations of Eqs. (15).

But this argument requires the assumption that the probabilities behave similar to the envariant property that the transformation refers to; i.e., that the behavior of probabilities under envariant transformations, in particular swaps, is somehow known. This knowledge, however, is not established by Zurek's derivation, and we do not see how it could automatically follow from envariance (as suggested by Zurek). To illustrate this point, consider the following two statements regarding the implications derived from envariance in the course of Zurek's argument:

(i) Phase envariance implies that the probabilities of $\mathcal{S}$ must be independent of the phases of the Schmidt coefficients.

(ii) Envariance under swaps implies that the probabilities of $\mathcal{S}$ cannot be influenced by a swap acting on $\mathcal{E}$ alone.

The important difference between (i) and (ii) is that (i) aims at demonstrating the independence of the probabilities from an envariant property, whereas (ii) claims invariance of the probabilities under an envariant transformation. Statement (i) only requires assumption (D) to hold: phase envariance implies that a local description of $\mathcal{S}$ cannot depend on the phase factors of the Schmidt coefficients, so if we assume that the probabilities are a function of the local properties ascribed to $\mathcal{S}$ on the basis of the entangled state vector only (and a study of its envariant transformations), (i) follows. But (ii) requires more: Employing a reasoning analogous to (i), envariance of the composite state under swaps solely means that probabilities of $\mathcal{S}$ will not depend on whether $|s_1\rangle$ is entangled with $|e_1\rangle$ or with $|e_2\rangle$, since this "property" of a specific correlation is not property of $\mathcal{S}$ alone; but we have said nothing about whether the application of the swap operation itself to $\mathcal{E}$ might disturb the probabilities associated with $\mathcal{S}$.

We might reinforce this concern by drawing attention to the physical interpretation of the swap operation. A swap applied to $\mathcal{S}$ implies that the existing correlations $|s_1\rangle|e_1\rangle$ and $|s_2\rangle|e_2\rangle$ between the system $\mathcal{S}$ and the environment $\mathcal{E}$ need to be "undone," and new correlations of the form $|s_1\rangle|e_2\rangle$ and $|s_2\rangle|e_1\rangle$ between $\mathcal{S}$ and $\mathcal{E}$ have to be created. From the form of the swap transformations, $\widehat{U}_\mathcal{S} = \widehat{u}_\mathcal{S} \otimes \widehat{I}_\mathcal{E}$ and $\widehat{U}_\mathcal{E} = \widehat{I}_\mathcal{S} \otimes \widehat{u}_\mathcal{E}$, it is clear that swaps can be induced by *local* interactions. But we do not see why shifting features of $\mathcal{E}$, that is, doing something to the environment, should not alter the "guess" (to use Zurek's expression (Ref. 20, p. 755); cf. the quote in (IIIb) above) an observer of $\mathcal{S}$ would make concerning $\mathcal{S}$-outcomes. Here, if possible, one would like to see some further argument (or motivation) for why the probabilities of one system should be immune to swaps among the basis states of the other system.

(G) Let us finally discuss Zurek's alternative argument based on the ignorance of an observer of $\mathcal{S}$ with respect to the individual state of the system $\mathcal{S}$.

In his derivation, Zurek takes the entangled Schmidt state $|\psi_{\mathcal{SE}}\rangle$ describing the correlation between $\mathcal{S}$ and $\mathcal{E}$ as the given starting point and assumes that the observer somehow knows this state exactly already *before* any measurement has taken place. According to Zurek this knowledge seems to imply that the observer is aware of the "menu" of possible outcomes (but cannot attribute a particular outcome state to $\mathcal{S}$ before the measurement). But since the observer has only access to $\mathcal{S}$, how is this knowledge established in the first place?

In the case of the composite state $|\psi_{\mathcal{SE}}\rangle$ with coefficients of equal magnitude, Eq. (3), one can choose *any* other orthonormal basis for $\mathcal{H_S}$ and always find a corresponding orthonormal basis of $\mathcal{H_E}$ such that the composite state $|\psi_{\mathcal{SE}}\rangle$ has again the diagonal Schmidt form of Eq. (3). Therefore, no preferred basis of $\mathcal{H_S}$ or $\mathcal{H_E}$ has been singled out. On one hand, this implies that Zurek's argument does not require any *a priori* knowledge of the environmental states $|e_k\rangle$ for the observer of $\mathcal{S}$. On the other hand, however, this also means that there is nothing that would tell the observer of $\mathcal{S}$ which possible "events" $|s_k\rangle$ he is dealing with. (Decoherence provides a mechanism, termed environment-induced superselection, in which the interaction of $\mathcal{S}$ with $\mathcal{E}$ singles out a preferred basis in $\mathcal{H_S}$[20]; however, a fundamental derivation of the Born rule must of course be independent of decoherence to avoid circularity of the argument.) Even if one holds that a choice of a particular set of basis vectors is irrelevant to the derivation since the aim is solely to demonstrate the emergence of equal likelihoods for *any* orthonormal basis $\{|s_k\rangle\}$, one is still left with the question how the observer of $\mathcal{S}$ establishes the knowledge that the composite state must be described by coefficients of equal magnitude.

Zurek then goes on to claim that (i) because all possible outcome states $|s_k\rangle$ are envariantly swappable, these states appear as "equivalent" to the observer of $\mathcal{S}$, and (ii) that this "equivalence of outcomes" translates into an attribution of equal likelihoods for each of these outcomes. With respect to part (i) of the argument, perfect knowledge of the pure composite state implies that the observer (before the measurement) cannot know the individual state of $\mathcal{S}$, which adds in an ignorance-based probability concept, but without having established equal likelihoods. Now, as mentioned before, envariance under swaps simply means that the question of which $|e_l\rangle$ of $\mathcal{E}$ is correlated with a particular $|s_k\rangle$ of $\mathcal{S}$ is irrelevant to a complete local description of $\mathcal{S}$. But we do not see how this state of affairs forces the observer of $\mathcal{S}$ to conclude that all the $|s_k\rangle$ are "equivalent." For part (ii), we note that even if the previous argument did establish an "equivalence of outcomes," this epistemic indifference about the occurrence of a particular outcome among a set of possible outcomes would not necessarily, from a general point of view of probability theory, force out the implication of equal likelihoods; this conclusion would be particularly questionable when dealing with a set of continuous cardinality.

4. CONCLUDING REMARKS

To summarize, we have pointed out four important assumptions in Zurek's derivation about the connection between the state vector and probabilities, and about the behavior of probabilities under envariant transformations of the state vector:

(1) The probability for a particular outcome in a measurement is directly identified with the probability for an eigenstate of the measured observable with an eigenvalue equal to the value of the measured physical quantity, an assumption that would follow from the eigenvalue–eigenstate link.

(2) For two entangled systems $\mathcal{S}$ and $\mathcal{E}$ described by the Schmidt state $|\psi_{\mathcal{SE}}\rangle = \sum_k \lambda_k |s_k\rangle|e_k\rangle$, probabilities associated with the "outcome states" $|s_k\rangle$ and $|e_k\rangle$ of each individual system are a function of the local properties of the systems only; these properties are exclusively determined by the state vector $|\psi_{\mathcal{SE}}\rangle$ of the composite system.

(3) In an entangled Schmidt state of the form $|\psi_{\mathcal{SE}}\rangle = \sum_k \lambda_k |s_k\rangle|e_k\rangle$, the "outcome states" $|s_k\rangle$ and $|e_k\rangle$ are equally likely: $p(|s_k\rangle; |\psi_{\mathcal{SE}}\rangle) = p(|e_k\rangle; |\psi_{\mathcal{SE}}\rangle)$.

(4) Probabilities associated with the Schmidt states $|s_k\rangle$ of a system $\mathcal{S}$ entangled with another system $\mathcal{E}$ remain unchanged under the application of an envariant transformation $\widehat{U}_{\mathcal{E}} = \widehat{I}_{\mathcal{S}} \otimes \widehat{u}_{\mathcal{E}}$ that only acts on $\mathcal{E}$ (and similarly for $\mathcal{S}$ and $\mathcal{E}$ symmetrically exchanged): $p(|s_k\rangle; \widehat{U}_{\mathcal{E}}|\psi_{\mathcal{SE}}\rangle) = p(|s_k\rangle; |\psi_{\mathcal{SE}}\rangle)$ and $p(|e_k\rangle; \widehat{U}_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) = p(|e_k\rangle; |\psi_{\mathcal{SE}}\rangle)$.

The necessity for an assumption like (3) in the derivation of the Born rule can be traced back to a fundamental statement about any probabilistic theory: we cannot derive probabilities from a theory that does not already contain some probabilistic concept; at some stage, we need to "put probabilities in to get probabilities out." Our analysis suggests that this has been done via assumption (3) above.

We have pointed out that assumption (4) is necessary to have the argument that is contained in the chain of relations in Eqs. (11)–(17) between transformed and untransformed probabilities go through, but we claim that this assumption neither follows from envariance alone nor from assumption (2). We have also questioned whether this assumption is physically plausible.

Furthermore, we have expressed doubts that Zurek's alternative approach that appeals to the information available to a local observer is capable of leading to a derivation of the Born rule. It is neither clear to us how exact knowledge of the composite state is established "within" the local observer before the measurement, nor how envariance under swaps leads the observer to conclude that the possible outcomes must be equally likely.

We hope the questions we raise here will not downplay the interest of Zurek's derivation in the mind of the reader. To the contrary, because we regard it as significant, we aimed at facilitating a balanced and careful evaluation of Zurek's approach by bringing out central assumptions implicit in his derivation. We note that Zurek uses both "derivation" and "motivation" to describe his treatment of the emergence of the Born rule. Once the critical assumptions are made explicit, however, as they are here and now in his, Ref. 26, the former term seems more appropriate. Moreover, any derivation of quantum probabilities and Born's rule will require some set of assumptions that put probabilities into the theory. In the era of the "Copenhagen hegemony," to use Jim Cushing's apt phrase, probabilities were put in by positing an "uncontrollable disturbance" between object and apparatus leading to a brute quantum "individuality" that was taken not to be capable of further analysis. Certainly Zurek's approach improves our understanding of the probabilistic character of quantum theory over that sort of proposal by at least one quantum leap.

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