A Generalization of the Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

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Abstract

As well known, the standard central limit theorem plays a fundamental role in Boltzmann-Gibbs (BG) statistical mechanics. This important physical theory has been generalized by one of us (CT) in 1988 by using the entropy $S_q = \frac{1-\sum_i p_i^q}{q-1}$ (with $q \in \mathcal{R}$) instead of its particular case $S_1 = S_{BG} = -\sum_i p_i \ln p_i$. The theory which emerges is usually referred to as *nonextensive statistical mechanics* and recovers the standard theory for $q = 1$. During the last two decades, this q-generalized statistical mechanics has been successfully applied to a considerable amount of physically interesting complex phenomena. Conjectures and numerical indications available in the literature were since a few years suggesting the possibility of q-generalizations of the standard central limit theorem by allowing the random variables that are being summed to be correlated in some special manner, the case $q = 1$ corresponding to standard probabilistic independence. This is precisely what we prove in the present paper for some range of q which extends from below to above $q = 1$. The attractor, in the usual sense of a central limit theorem, is given by a distribution of the form $p(x) \propto [1-(1-q)\beta x^2]^{1/(1-q)}$ with $\beta > 0$. These distributions, sometimes referred to as q-Gaussians, are known to make, under appropriate constraints, extremal the functional S_q. Their $q = 1$ and $q = 2$ particular cases recover respectively Gaussian and Cauchy distributions.

1 INTRODUCTION

Limit theorems, in particular, the central limit theorems (CLT), surely are among the most important theorems in probability theory and statistics. They play an essential role in various applied sciences as well, including statistical mechanics. Historically A. de Moivre, P.S. de Laplace, S.D. Poisson and C.F. Gauss have first shown that Gaussian is the attractor of independent systems with a finite second variance. Chebyshev, Markov, Liapounov, Feller, Lindeberg, Levy have contributed essentially to the development of the central limit theorem. Various aspects of this theorem and its links to statistical mechanics and diffusion have been discussed during recent decades as well [1, 2, 3, 4].

It is well known in classical Boltzmann-Gibbs (BG) statistical mechanics that the Gaussian maximizes, under appropriate constraints, the Boltzmann-Gibbs entropy $S_{BG} = -\sum_i p_i \ln p_i$. The q-generalization of the classic entropy introduced in [1] as the basis for generalizing the BG theory,

and denoted by $S_q = \frac{1 - \sum_i p_i^q}{q-1}$ ($q \in \mathcal{R}$; $S_1 = S_{BG}$) reaches its maximum at the distributions usually referred to as q-Gaussian (see [2]). This fact, and a number of conjectures [5] and numerical indications [6] suggest that there should be a q-analog of the CLT as well.

In this paper we prove a generalization of the classical central limit theorem consistent with nonextensive statistical mechanics. Speaking on one particular element of this theorem we note that there is a dual index, q^* connected with q. The first index q defines the region of convergence, while the dual index q^* exhibits existence of q^*-Gaussians corresponding to the limits of sums. The arisen duality, in contrary with the classic CLT, is a specific feature of the q-theory, which comes from the specific definition of q-exponential.

In the general case the suggested generalization of the central limit theorem is represented as a series of theorems, depending on type of correlations. For k, integer, there is a triplet (q_{k-1}, q_k, q_{k+1}) (determined by a given $q \in (1/2, 2)$), which has an important role in description of the system. As we see in Section 3 for systems having correlation identifyed by q_k, the index q_{k-1} determines the q-Gaussian, while the index q_{k+1} indicates the scaling rate. Note if $q = 1$, then all family of theorems reduce to one recovering the classic central limit theorem.

Our analysis is based on the q-mathematics [7, 8, 9, 10]. Recall briefly the basics of q-mathematics. By definition, the q-*sum* of two numbers is defined as $x \oplus_q y = x + y + (1-q)xy$. The q-sum is commutative, associative, recovers the usual summing operation if $q = 1$ (i.e. $x \oplus_1 y = x + y$), and preserves 0 as the neutral element (i.e. $x \oplus_q 0 = x$). By inversion, we can define the q-*subtraction* as $x \ominus_q y = \frac{x-y}{1+(1-q)y}$. The q-*product* for x, y is defined by the binary relation $x \otimes_q y = [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}}$. This operation also commutative, associative, recovers the usual product when $q = 1$, and preserves 1 as the unity. It is defined only when $x^{1-q} + y^{1-q} \geq 1$. Again by inversion, it can be defined the q-*division*: $x \oslash_q y = (x^{1-q} - y^{1-q} + 1)^{\frac{1}{1-q}}$. Note, that $x \otimes_q 0 \neq 0$, and for $q \neq 1$ division by zero is allowed.

The paper is organized as follows. Section 2 we start recalling the definitions of q-*exponential* and q-*logarithm*. Then we introduce the notion of the q-Fourier transform F_q and study its basic properties. Note, that F_q coincides with the classic Fourier transform if $q = 1$. For $q \neq 1$ F_q is not a linear operator. Lemma 2.5 says that F_q is invertible at least in the class of densities. An important property of F_q is that it maps q-Gaussian to q_*-Gaussian, where $q_* \neq q$ if $q \neq 1$. In Section 3 we prove the main result of this paper, q-version of the central limit theorem. It considers q-independent random variables, which recovers the classic notion of independence of random variables only in the case $q = 1$. For $q \neq 1$ the class of q-independent random variables contains globally [2] correlated random variables as well.

2 q-FOURIER TRANSFORM AND ITS PROPERTIES

2.1 q-exponential and q-logarithm

The q-analysis relies essentially on the analogs of exponential and logarithmic functions, which are called q-exponential and q-logarithm [7]. In this paper we introduce and essentially use a new analog of the Fourier transform, which we call q-Fourier transform. The q-Fourier transform is defined based on the q-product and the q-exponential, and, in contrast to the usual Fourier transform, is a nonlinear transform.

Now we recall briefly definitions and some properties of the q-exponential and q-logarithm. These functions are denoted by e_q^x and $ln_q x$ and respectively defined as $e_q^x = [1 + (1-q)x]_+^{\frac{1}{1-q}}$ and $\ln_q x = \frac{x^{1-q}-1}{1-q}$, $(x > 0)$. The symbol $[x]_+$ means that $[x]_+ = x$, if $x \geq 0$, and $[x]_+ = 0$, if $x < 0$. We mention the main properties of these functions, which we will use essentially in this paper. For q-

exponential the relations $e_q^{x \oplus_q y} = e_q^x e_q^y$ and $e_q^{x+y} = e_q^x \otimes_q e_q^y$ hold true. These relations can be written equivalently $\ln_q(x \otimes_q y) = \ln_q x + \ln_q y$ [1] and $\ln_q(xy) = \ln_q x \oplus_q \ln_q y$. q-exponential and q-logarithm have asymptotics $e_q^x = 1 + x + \frac{q}{2}x^2 + o(x^2)$, $x \to 0$ and $\ln_q(1+x) = x - \frac{q}{2}x^2 + o(x^2)$, $x \to 0$. If $q < 1$, then for x reals, $|e_q^{ix}| \geq 1$ and $|e_q^{ix}| \sim (1+x^2)^{\frac{1}{2(1-q)}}$, $x \to \infty$. Similarly, $q > 1$, then $0 < |e_q^{ix}| \leq 1$ and $|e_q^{ix}| \to 0$ if $|x| \to \infty$.

2.2 q-Gaussian

Let β be a positive number. We call the function

$$G_q(\beta; x) = \frac{\sqrt{\beta}}{C_q} e_q^{-\beta x^2}, \tag{1}$$

a q-Gaussian. The constant C_q is the normalizing constant, namely $C_q = \int_{-\infty}^{\infty} e_q^{-x^2} dx$. It is not difficult to verify that

$$C_q = \begin{cases} \frac{2}{\sqrt{1-q}} \int_0^{\pi/2} (\cos t)^{\frac{3-q}{1-q}} dt = \frac{2\sqrt{\pi}\Gamma(\frac{1}{1-q})}{(3-q)\sqrt{1-q}\Gamma(\frac{3-q}{2(1-q)})}, & -\infty < q < 1, \\ \sqrt{\pi}, & q = 1, \\ \frac{2}{\sqrt{q-1}} \int_0^{\infty} (1+y^2)^{\frac{-1}{q-1}} = \frac{\sqrt{\pi}\Gamma(\frac{3-q}{2(q-1)})}{\sqrt{q-1}\Gamma(\frac{1}{q-1})}, & 1 < q < 3. \end{cases} \tag{2}$$

For $q < 1$, the support of $G_q(\beta; x)$ is compact since this density vanishes for $|x| > 1/\sqrt{(1-q)\beta}$. Notice also that, for $q < 5/3$ ($5/3 \leq q < 3$), the variance is finite (diverges). Finally, we can easily check that there are relationships between different values of q. For example, $e_q^{-x^2} = \left[e_{2-\frac{1}{q}}^{-qx^2}\right]^{\frac{1}{q}}$.

The following lemma establishes a general relationship (which contains the previous one as a particular case) between different q-Gaussians.

Lemma 2.1. *For any real q_1, $\beta_1 > 0$ and $\delta > 0$ there exist uniquely determined $q_2 = q_2(q_1, \delta)$ and $\beta_2 = \beta_2(\delta, \beta_1)$, such that*

$$(e_{q_1}^{-\beta_1 x^2})^{\delta} = e_{q_2}^{-\beta_2 x^2}.$$

Moreover, $q_2 = \delta^{-1}(\delta - 1 + q_1)$, $\beta_2 = \delta\beta_1$.

Proof. Let $q_1 \in R^1, \beta_1 > 0$ and $\delta > 0$ be any fixed real numbers. For the equation,

$$(1 - (1-q_1)\beta_1 x^2)^{\frac{\delta}{1-q_1}} = (1 - (1-q_2)\beta_2 x^2)^{\frac{1}{1-q_2}}$$

to be an identity it is needed $(1-q_1)\beta_1 = (1-q_2)\beta_2$, $1 - q_1 = \delta(1 - q_2)$. These equations have a unique solution $q_2 = \delta^{-1}(\delta - 1 + q_1)$, $\beta_2 = \delta\beta_1$.

The set of all q-Gaussians will be denoted by $\mathcal{G}_q$, i.e.,

$$\mathcal{G}_q = \{b\,G_q(\beta, x) : b > 0, \beta > 0\}.$$

2.3 q-Fourier transform and q-characteristic function

Introduce the *q-Fourier transform* for a given function $f(x)$ by the formal formula [2]

$$F_q[f](\xi) = \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)dx. \tag{3}$$

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[1] This property reflects the possible extensivity of S_q in the presence of special correlations [12, 13, 14, 15].

[2] Note, if f has compact support, then integration should be taken over this support, otherwise the integral does not converge.

For discrete functions $f_k, k = 0, \pm 1, ...$, this definition takes the form

$$F_q[f](\xi) = \sum_{k=-\infty}^{\infty} e_q^{ik\xi} \otimes_q f(k). \tag{4}$$

In the future we use the same notation in both cases. We also call (3) or (4) the *q-characteristic function* of a given random variable X with an associated density $f(x)$, using the notations $F_q(X)$ or $F_q(f)$ equivalently. The following lemma establishes the relation of the q-Fourier transform without using the q-product.

Lemma 2.2. *The q-Fourier transform can be written in the form*

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) e_q^{\frac{ix\xi}{(f(x))^{1-q}}} dx. \tag{5}$$

Proof. We have

$$e_q^{ix\xi} \otimes f(x) = [1 + (1-q)ix\xi + f(x)^{1-q} - 1]_+^{\frac{1}{1-q}} =$$

$$f(x)[1 + (1-q)ix\xi f(x)^{q-1}]_+^{\frac{1}{1-q}}. \tag{6}$$

Integrating both sides of Eq. (6) we obtain (5).

Remark 2.3. *It should be noted that if the q-Fourier transform of a given function $f(x)$ defined by the formal definition in (3) exists, then it coincides with the expression in (5). The q-Fourier transform determined by the formula (5) has an advantage to compare to the formal definition: it does not use the q-product, which is, in general, restrictive in use. From now on we refer to (5) when we speak about the q-Fourier transform.*

Corollary 2.4. *The q-Fourier transform exists for any $f \in L_1(R)$ if $q \geq 1$. For $q < 1$ the q-Fourier transform exists if f additionally satisfies the condition $|f| \sim \frac{1}{|x|^\gamma}, \gamma > \frac{2-q}{1-q}$. Moreover, $|F_q[f](\xi)| \leq \|f\|_{L_1}$ [3], for $q \geq 1$, and $|F_q[f](\xi)| \leq \|f(x)(1 + |x|)^{\frac{1}{1-q}}\|_{L_1}$ for $q < 1$.*

Proof. This is a simple implication of Lemma 2.2 and of the asymptotics of e_q^{ix} for large $|x|$ mentioned above.

Lemma 2.5. *Assume $f_1(x) \geq 0$, $f_2(x) \geq 0$, $x \in R$ and $F_q[f_1](\xi) = F_q[f_2]$ for all $\xi \in R$. Then $f_1(x) = f_2(x)$ for almost all $x \in R$.*

Proof. Denote $\mathcal{F}(\rho, x, \xi; f) = f(x)(1 + \rho ix\xi f^\rho(x))^{1/\rho}$ and $\mathcal{H}(\rho, x, \xi; f) = \rho^{-1} \tan(\rho x\xi f^\rho)$. Then the equation $F_q[f_1](\xi) = F_q[f_2]$ can be written in the form

$$\int_{-\infty}^{\infty} (\mathcal{F}(q-1, x, \xi; f_1) - \mathcal{F}(q-1, x, \xi; f_2) e^{i\mathcal{H}(1-q, x, \xi; f_1)} dx+$$

$$\int_{-\infty}^{\infty} \mathcal{F}(q-1, x, \xi; f_2)(e^{i\mathcal{H}(1-q, x, \xi; f_1)} - e^{i\mathcal{H}(1-q, x, \xi; f_2)}) dx = 0, \tag{7}$$

for all $\xi \in R$. The fact that the system $\{e^{i\rho^{-1} \tan(\rho x\xi f^\rho)}\}_{\xi \in R}$ can be approximated by the system $\{e^{ix\xi f^\rho}\}_{\xi \in R}$ with any desired accuracy, yields the completeness of the first system. Hence Eq. (7) is equivalent to

$$\mathcal{F}(q-1, x, \xi; f_1) - \mathcal{F}(q-1, x, \xi; f_2) = 0,$$

$$e^{i\mathcal{H}(1-q, x, \xi; f_1)} - e^{i\mathcal{H}(1-q, x, \xi; f_2)} = 0,$$

for all ξ. The first equation and positiveness of f_1 and f_2 yields $f_1(x) = f_2(x)$ for almost all $x \in R$.

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[3]Here, and elsewhere, $\|f\|_{L_1} = \int_R f(x) dx$, and L_1 is the space of absolutely integrable functions.

Remark 2.6. *Taking $f_2 = 0$ in Lemma 2.5 we get the statement: if $f \geq 0$ and $F_q[f](\xi) = 0$ for all $\xi \in R$, then $f(x) = 0$ a.e.*

Lemma 2.7. *Let $q < 3$. For the q-Fourier transform of the q-Gaussian the following formula holds:*

$$F_q[G_q(\beta; x)](\xi) = \left(e_q^{-\frac{\xi^2}{4\beta^{2-q}C_q^{2(q-1)}}} \right)^{\frac{3-q}{2}}. \tag{8}$$

Proof. Denote $a = \frac{\sqrt{\beta}}{C_q}$ and write

$$F_q[a\, e_q^{-\beta x^2}](\xi) = \int_{-\infty}^{\infty} (a\, e_q^{-\beta x^2}) \otimes_q (e_q^{ix\xi}) dx$$

using the property $e_q^{x+y} = e_q^x \otimes_q e_q^y$ of the q-exponential, in the form

$$F_q[ae_q^{-\beta x^2}](\xi) = a \int_{-\infty}^{\infty} e_q^{-\beta x^2 + ia^{q-1}x\xi} dx = a \int_{-\infty}^{\infty} e_q^{-(\sqrt{\beta}x - \frac{ia^{q-1}\xi}{2\sqrt{\beta}})^2 - \frac{a^{2(q-1)}\xi^2}{4\beta}} dx =$$

$$a \int_{-\infty}^{\infty} e_q^{-(\sqrt{\beta}x - \frac{ia^{q-1}\xi}{2\sqrt{\beta}})^2} \otimes_q e_q^{-\frac{a^{2(q-1)}\xi^2}{4\beta}} dx.$$

The substitution $y = \sqrt{\beta}x - \frac{ia^{q-1}\xi}{2\sqrt{\beta}}$ yields the equation

$$F_q[ae_q^{-\beta x^2}](\xi) = \frac{a}{\sqrt{\beta}} \int_{-\infty+i\eta}^{\infty+i\eta} e_q^{-y^2} \otimes_q e_q^{-\frac{a^{2(q-1)}\xi^2}{4\beta}} dy,$$

where $\eta = \frac{\xi a^{q-1}}{2\sqrt{\beta}}$. Further using the Cauchy theorem on integrals over closed curves, which is applicable because of a power law decay of q-exponential for any $q < 3$, we can transfer the integration from $R + i\eta$ to R. Hence, applying again Lemma 2.2, we have

$$F_q[G_q(\beta; x)](\xi) = \frac{ae_q^{-\frac{a^{2(q-1)}}{4\beta}\xi^2}}{\sqrt{\beta}} \int_{-\infty}^{\infty} e_q^{-y^2 \left(e_q^{-\frac{a^{2(q-1)}}{4\beta}\xi^2} \right)^{q-1}} dy =$$

$$\frac{aC_q}{\sqrt{\beta}} \left(e_q^{-\frac{a^{2(q-1)}\xi^2}{4\beta}} \right)^{1-\frac{q-1}{2}}.$$

Simplifying the last expression, we arrive at (8). Introduce the function $z(s) = \frac{1+s}{3-s}$ for $s \in (-\infty, 3)$, and denote its inverse $z^{-1}(t)$, $t \in (-1, \infty)$. It can be easily verified that $z(\frac{1}{z(s)}) = \frac{1}{s}$ and $z(\frac{1}{s}) = \frac{1}{z^{-1}(s)}$. Let $q_1 = z(q)$ and $q_{-1} = z^{-1}(q)$. It follows from the mentioned properties of $z(q)$ that

$$z(\frac{1}{q_1}) = \frac{1}{q} \quad \text{and} \quad z(\frac{1}{q}) = \frac{1}{q_{-1}}. \tag{9}$$

The function $z(s)$ also possess the following two important properties

$$z(s)\, z(2 - s) = 1 \quad \text{and} \quad z(2 - s) + z^{-1}(s) = 2. \tag{10}$$

It follows from these properties that $q_{-1} + \frac{1}{q_1} = 2$.

Corollary 2.8. *For q-Gaussians the following q-Fourier transforms hold*

$$F_q[G_q(\beta; x)](\xi) = e_{q_1}^{-\beta_*(q)\xi^2}, \quad q_1 = z(q), \ q < 3; \tag{11}$$

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$$F_{q-1}[G_{q-1}(\beta; x)](\xi) = e_q^{-\beta_*(q-1)\xi^2}, \ q_{-1} = z^{-1}(q), \ q > -1, \tag{12}$$

where $\beta_*(s) = \frac{3-s}{8\beta^{2-s}C_s^{2(s-1)}}$.

Remark 2.9. *Note that $\beta_*(s) > 0$ if $s < 3$.*

Corollary 2.10. *The following mappings*

$$F_q : \mathcal{G}_q \to \mathcal{G}_{q_1}, q_1 = z(q), \ q < 3,$$

$$F_{q-1} : \mathcal{G}_{q-1} \to \mathcal{G}_q, \ q_{-1} = z^{-1}(q), \ q > -1,$$

hold and they are injective.

Corollary 2.11. *There exist the following inverse q-Fourier transforms*

$$F_q^{-1} : \mathcal{G}_{q_1} \to \mathcal{G}_q, \ q_1 = z(q), \ q < 3,$$

$$F_{q-1}^{-1} : \mathcal{G}_q \to \mathcal{G}_{q-1}, q_{-1} = z^{-1}(q), \ q > -1.$$

Lemma 2.12. *The following mappings*

$$F_{\frac{1}{q_1}} : \mathcal{G}_{\frac{1}{q_1}} \to \mathcal{G}_{\frac{1}{q}}, q_1 = z(q), \ q < 3,$$

$$F_{\frac{1}{q}} : \mathcal{G}_{\frac{1}{q}} \to \mathcal{G}_{\frac{1}{q-1}}, \ q_{-1} = z^{-1}(q), \ q > -1.$$

hold.

Proof. The assertion of this lemma follows from Corollary 2.10 if we take into account the properties (9).

Introduce the sequence $q_n = z_n(q) = z(z_{n-1}(q)), n = 1, 2, ...,$ with a given $q = z_0(q), q < 3$. We can extend the sequence q_n for negative integers $n = -1, -2, ...$ as well putting $q_{-n} = z_{-n}(q) = z^{-1}(z_{1-n}(q)), n = 1, 2, ...$ It is not hard to verify that [4]

$$q_n = \frac{2q + n(1-q)}{2 + n(1-q)}, \ n = 0, \pm 1, \pm 2, ... \tag{13}$$

In Equation (13) we require $q < 1 + \frac{2}{n}$ for $n > 0$ and $q > 1 + \frac{2}{n}$ for $n < 0$. Note $q_n \equiv 1$ for all $n = 0, \pm 1, \pm 2, ...,$ if $q = 1$ and $lim_{n \to \pm\infty} z_n(q) = 1$ for all $q \neq 1$. Let us note also that the definition of the sequence q_n can be given through the series of mappings

Definition 2.13.

$$z : \ \to q_{-2} \to q_{-1} \to q_0 = q \to q_1 \to q_2 \to ... \tag{14}$$

$$z^{-1} : \ \leftarrow q_{-2} \leftarrow q_{-1} \leftarrow q_0 = q \leftarrow q_1 \leftarrow q_2 \leftarrow ... \tag{15}$$

Further, let $F_q^n(f) = F_q[F_q^{n-1}[f]], n = 1, 2, ..., F_q^0(f) = f$. Summarizing the above mentioned relationships related to $z_n(q)$, we obtain the following assertions.

Lemma 2.14. *There holds the following duality relations*

$$q_{n-1} + \frac{1}{q_{n+1}} = 2, \ n = 0, \pm 1, \pm 2, ... \tag{16}$$

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[4]Essentially the same mathematical structure has already appeared in a quite different, though possibly related, context: see Footnote of page 15378 of [12].

Proof. Making use the properties (10), we obtain

$$q_{n-1} = z^{-1}(q_n) = 2 - z(2 - q_n) = 2 - \frac{1}{z(q_n)} = 2 - \frac{1}{q_{n+1}}.$$

Lemma 2.15. *The following mappings hold:*

$$F_q^k : \mathcal{G}_{q_n} \to \mathcal{G}_{q_{k+n}}, \quad k, n = 0, \pm 1, \pm 2, \dots$$

$$\lim_{k \to \pm \infty} F_q^k \mathcal{G}_q = \mathcal{G}$$

where $\mathcal{G}$ is the set of classic Gaussians.

Lemma 2.16. *The series of mappings hold:*

$$F_q : \to \mathcal{G}_{q_{-2}} \to \mathcal{G}_{q_{-1}} \to \mathcal{G}_q \to \mathcal{G}_{q_1} \to \mathcal{G}_{q_2} \to \dots \tag{17}$$

$$F_q^{-1} : \leftarrow \mathcal{G}_{q_{-2}} \leftarrow \mathcal{G}_{q_{-1}} \leftarrow \mathcal{G}_q \leftarrow \mathcal{G}_{q_1} \leftarrow \mathcal{G}_{q_2} \leftarrow \dots \tag{18}$$

3 MAIN RESULTS

3.1 q-independent random variables

In this section we establish a q-generalization of the classical CLT. First we introduce some notions necessary to formulate the corresponding results. Let X be a random variable and $f(x)$ be an associated density. Denote

$$f_q(x) = \frac{[f(x)]^q}{\nu_q(f)},$$

where $\nu_q(f) = \int_{-\infty}^{\infty} [f(x)]^q \, dx$. The density $f_q(x)$ is commonly referred to as *escort density* [11]. Further, introduce for X the notions *q-mean*, $\mu_q = \mu_q(X) = \int_{-\infty}^{\infty} x f_q(x) \, dx$, and *$q$-variance* $\sigma_q^2 = \sigma_q^2(X - \mu_q) = \int_{-\infty}^{\infty} (x - \mu_q)^2 f_q(x) \, dx$, and *$q$-moment of order k*, $M_{q,k} = M_{q,k}(X) = \int_{-\infty}^{\infty} x^k f_q(x) \, dx$, subject to all integrals used in these definitions to converge. Analogously, for N random variables $X_1, X_2, \dots, X_N$ with joint density $f(x_1, \dots, x_N)$ we introduce the corresponding *escort joint density*

$$f_q(x_1, \dots, x_N) = \frac{[f(x_1, \dots, x_N)]^q}{\int_{R^N} [f(x_1, \dots, x_N)]^q \, dx_1 \dots dx_N}.$$

and

$$\mu_q(g(X_1, \dots, X_N)) = \int_{R^N} g(x_1, \dots, x_N) f_q(x_1, \dots, x_N) dx_1 \dots dx_N.$$

Analogously to the case of one random variable, we can introduce the q-mean of the sum $X_1 + \dots + X_N$ and q-moments of $X_1, \dots, X_N$. We also use the notation $\nu_{q,k} = \int_{\omega_N} (x_1 + \dots + x_N)^k f_q(x_1, \dots, x_N) dx_1 \dots dx_N$, where ω_N is the support, i.e., $\omega_N = supp \, f_q(x_1, \dots, x_N)$.

Lemma 3.1. *The following formulas hold true*

1. $\mu_q(aX) = a\mu_q(X)$;

2. $\mu_q(X - \mu_q(X)) = 0$;

3. $\sigma_q^2(aX) = a^2 \sigma_q^2(X)$;

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4. $\mu_q(X_1 + ... + X_N) = \sum_{i=1}^{N} \mu_q(X_i)$;

Further, we introduce the notions of *q-independence*, *q-convergence* and *q-normality*.

Definition 3.2. *Two random variables X and Y are called to be q-independent, if*

$$F_q[X + Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi). \tag{19}$$

Note that, for $q = 1$, this coincides with the classical independence of random variables. The relation (19) can be rewritten as follows. Let f and g be densities of X and Y respectively, and $H(x, y)$ be their joint density. Then

$$\int_{R^2} e_q^{i(x+y)\xi} \otimes_q H(x, y) dx dy = F_q[f](\xi) \otimes_q F_q[g](\xi). \tag{20}$$

Definition 3.3. *A sequence of random variables X_N is said to be q-convergent if $\lim_{N\to\infty} F_q[X_N](\xi) \in \mathcal{G}_q$ locally uniformly by ξ for some $q < 3$. Further, we will say that q-limit of the sequence X_N is q_*-normal, if there are some $q_* < 3$ and $\beta > 0$ such that $\lim_{N\to\infty} F_q(X_N) = F_{q_*}(G_{q_*}(\beta; x))$.*

Remark 3.4. *In other words the q-limit of a sequence X_N is q_*-normal, if for some $q_* < 3$ and $\beta > 0$, $\lim_{N\to\infty} X_N \in F_q^{-1} \circ F_{q_*}(G_{q_*}(\beta; x))$.*

We will study limits of sums

$$Z_N = \frac{1}{D_N(q)} (X_1 + ... + X_N - N\mu_q), N = 1, 2, ...$$

where $D_N(q)$, $N = 1, 2, ...$, are some reals (scaling parameter), in the sense of Definition 3.3, when $N \to \infty$. Namely, the question we are interested in: *Is there a q-normal distribution that attracts the sequence Z_N? If yes, what is the admissible range of values of q?* For $q = 1$ the answer is well known and it is the content of the classical central limit theorem.

The *q*-generalization of the central limit theorem, we are suggesting in the present paper, is formulated as follows.

Theorem 1. *Assume a sequence $\{..., q_{-2}, q_{-1}, q_0, q_1, q_2, ...\}$ is given as (14) with $q_0 = q \in (1/2, 2)$. Let $X_1, ..., X_N, ...$ be a sequence of q_k-independent for some $k \in \mathcal{Z}$ and identically distributed random variables with a finite q_k-mean μ_{q_k} and a finite second $(2q_k - 1)$-moment $\sigma_{2q_k-1}^2$.*

Then $Z_N = \frac{X_1+...+X_N-N\mu_{q_k}}{D_N(q)}$, with $D_N(q) = \sqrt{N}\sigma_{2q_k-1}\nu_{2q_k-1}$, is q_k-convergent to a q_{k-1}-normal distribution as $N \to \infty$.

Remark 3.5. *Note the corresponding attractor is $G_{q_{k-1}}(\beta_k; x)$, where*

$$\beta_k = \left(\frac{3 - q_{k-1}}{4q_k C_{q_{k-1}}^{2q_{k-1}-2}}\right)^{\frac{1}{2-q_{k-1}}}. \tag{21}$$

The proof of this theorem follows from Theorem 2 proved below and Lemma 2.16. Theorem 2 represents one element ($k = 0$) in the series of assertions contained in Theorem 1.

Theorem 2. *Assume $1/2 < q \leq 2$, or equivalently $1/3 < q^* < 5/3$, $q^* = z^{-1}(q)$. Let $X_1, ..., X_N, ...$ be a sequence of q-independent and identically distributed random variables with a finite q-mean μ_q and a finite second $(2q - 1)$-moment σ_{2q-1}^2.*

Then $Z_N = \frac{X_1+...+X_N-N\mu_{2q-1}}{D_N(q)}$, with $D_N(q) = \sqrt{N}\sigma_{2q-1}\nu_{2q-1}$, is q-convergent to a q_{-1}-normal distribution as $N \to \infty$. The corresponding q_{-1}-Gaussian is $G_{q_{-1}}(\beta; x)$, with $\beta = \left(\frac{3-q_{-1}}{4qC_{q_{-1}}^{2q_{-1}-2}}\right)^{\frac{1}{2-q_{-1}}}$.

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Proof. Let f be the density associated with $X_1 - \mu_q$. First we evaluate $F_q(X_1 - \mu_q) = F_q(f(x))$. Using Lemma 2.2 we have

$$F_q[f](\xi) = \int_{-\infty}^{\infty} (e_q^{ix\xi}) \otimes_q f(x)dx = \int_{-\infty}^{\infty} f(x)e_q^{\frac{ix\xi}{[f(x)]^{1-q}}} dx. \tag{22}$$

Making use of the asymptotic expansion $e_q^x = 1 + x + \frac{q}{2}x^2 + o(x^2)$, $x \to 0$, we can rewrite the right hand side of (22) in the form

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x)\left(1 + \frac{ix\xi}{[f(x)]^{1-q}} - q/2\frac{x^2\xi^2}{[f(x)]^{2(1-q)}} + o(\frac{x^2\xi^2}{[f(x)]^{2(1-q)}})\right) dx =$$

$$1 + i\xi\mu_q\nu_q - (q/2)\xi^2\sigma_{2q-1}^2\nu_{2q-1} + o(\xi^2), \; \xi \to 0. \tag{23}$$

In accordance with the condition of the theorem and Lemma 3.1, $\mu_q = \mu_q(X_1 - \mu_q) = 0$. Denote $Y_j = D_N(q)^{-1}(X_j - \mu_q), j = 1, 2, ...$ Then $Z_N = Y_1 + ... + Y_N$. Further, it is readily seen that, for a given random variable X and real $a > 0$, there holds $F_q[aX](\xi) = F_q[X](a\xi)$. It follows from this relation that $F_q(Y_1) = F_q[f](\frac{\xi}{\sqrt{N}\sigma_{2q-1}\nu_{2q-1}})$. Moreover, it follows from the q-independence of $Y_1, Y_2, ...$ (which is an obvious consequence of the q-independence of $X_1, X_2, ...$) and the associativity of the q-product that

$$F_q[Z_N](\xi) = F_q[f](\frac{\xi}{\sqrt{N}\sigma_{2q-1}\nu_{2q-1}}) \otimes_q...\otimes_q F_q[f](\frac{\xi}{\sqrt{N}\sigma_{2q-1}\nu_{2q-1}}) \; (N \text{ factors}). \tag{24}$$

Hence, making use of properties of the q-logarithm, from (24) we obtain

$$\ln_q F_q[Z_N](\xi) = N \ln_q F_q[f](\frac{\xi}{\sqrt{N}\sigma_{2q-1}\nu_{2q-1}}) = N \ln_q(1 - \frac{q}{2}\frac{\xi^2}{N} + o(\frac{\xi^2}{N})) =$$

$$-\frac{q}{2}\xi^2 + o(1), \; N \to \infty, \tag{25}$$

locally uniformly by ξ.

Consequently, locally uniformly by ξ,

$$\lim_{N\to\infty} F_q(Z_N) = e_q^{-(q/2)\xi^2} \in \mathcal{G}_q. \tag{26}$$

Thus, Z_N is q-convergent.

In accordance with Corollary 2.8 for q_{-1} and some β we have $F_{q_{-1}}(G_{q_{-1}}(\beta; x)) = e_q^{-(q/2)\xi^2}$. Now we find β. It follows from Corollary 2.8 (see (12)) that $\beta_*(q_{-1}) = q/2$. Solving this equation with respect to β we obtain

$$\beta = \left(\frac{3 - q_{-1}}{4qC_{q_{-1}}^{2(q_{-1}-1)}}\right)^{\frac{1}{2-q_{-1}}}, \tag{27}$$

where $q = z(q_{-1})$. The explicit form of the corresponding q_{-1}-Gaussian reads as

$$G_{q_{-1}}(\beta; x) = C_{q_{-1}}^{-1}\left(\frac{3 - q_{-1}}{2C_{q_{-1}}^{q_{-1}-1}\sqrt{1 + q_{-1}}}\right)^{\frac{1}{2-q_{-1}}} e_{q_{-1}}^{-\left(\frac{(3-q_{-1})^2}{4(1+q_{-1})C_{q_{-1}}^{2(q_{-1}-1)}}\right)^{\frac{1}{2-q_{-1}}} x^2}.$$

3.2 Generalization of the previous theorem

Obviously, Theorem 1 is true if a sequence $X_1, X_2, ..., X_N$ is *asymptotically q-independent*, i.e, if they are mutually q-independent for all $N > N_0$ starting from a number $N_0 > 1$ [5]. We shall now extend the domain of validity of Theorem 1 by showing that, although the hypothesis used in Theorem 1 are sufficient, they are not necessary. We can somewhat relax them and the attractors still remain the same. In what follows, the particular case $\rho = 0$ (see definition just below) of Theorem 3 recovers Theorem 1. Note that in this section we use $q^* = z^{-1}(q)$ instead of q_{-1}.

Theorem 3. *Assume $1/2 < q \leq 2$ or equivalently $1/3 < q^* < 5/3$, $q^* = z^{-1}(q)$. Let $X_1, ..., X_N, ...$ be a sequence of identically distributed and totally correlated random variables satisfying the conditions*

1. $\mu_q = 0$ *and* $\nu_{2q-1,2} \sim N^{1+\rho}$, $0 \leq \rho < 1$;

2. $\nu_{3q-2,3} \sim N^{\gamma}$, *where* $\gamma < \frac{3(1+\rho)}{2}$.

Then $Z_N = \frac{X_1+...+X_N}{D_N(q)}$, with a scaling parameter $D_N(q) \sim N^{\frac{1+\rho}{2}}$, is q-convergent to a q^*-normal distribution as $n \to \infty$. The corresponding q^*-Gaussian is $G_{q^*}(\beta; x)$, with $\beta = \left(\frac{3-q^*}{4qC_{q^*}^{2q^*-2}}\right)^{\frac{1}{2-q^*}}$.

Proof. Let $f_N(x_1, ..., x_N)$ and $g_N(x_1, ..., x_N)$ be the joint 'escort' density functions of the sums $X_1 + ... + X_N$, and $Y_1 + ... + Y_N$, where again $Y_j = D_N(q)^{-1}(X_j - \mu_q)$, $j = 1, ...N$, respectively. Evaluate $F_q[g_N]$. Using Lemma 2.2 we have

$$F_q[g_N](\xi) = \int_{R^N} (e_q^{i(x_1+...+x_N)\xi}) \otimes_q g_N(x_1, ..., x_N) dx =$$

$$\int_{R^N} g_N(x_1, ..., x_N) e_q^{\frac{i(x_1+...+x_N)\xi}{(g_N(x_1,...,x_N))^{1-q}}} dx_1...dx_N. \tag{28}$$

Again using the asymptotic expansion $e_q^x = 1 + x + \frac{q}{2}x^2 + o(x^2)$, $x \to 0$, we obtain

$$F_q[g_N](\xi) = \int_{R^N} g_N(x_1, ..., x_N)\{1 + \frac{i(x_1 + ... + x_N)\xi}{(g_N)^{1-q}} -$$

$$q/2\frac{(x_1 + ... + x_N)^2\xi^2}{(g_N)^{2(1-q)}} + O(\frac{(x_1 + ...x_N)^3\xi^3}{(g_N)^{2(1-q)}})\}dx_1...x_N =$$

$$= 1 - (q/2)\xi^2 \nu_{2q-1,2}(g_N) + O(\nu_{3q-2,3}(g_N)), \ N \to \infty. \tag{29}$$

Taking into account the relationship between X_j and Y_j, we have

$$F_q[Z_N](\xi) = 1 - (q/2)\xi^2 + \frac{O(\nu_{3q-2,3})}{N^{3/2(\rho+1)}}, \ N \to \infty. \tag{30}$$

It follows from (30) and the condition of the Theorem, that

$$F_q[Z_N](\xi) = 1 - (q/2)\xi^2 + o(1), \ N \to \infty, \tag{31}$$

locally uniformly by ξ. Further, taking q-logarithm of both sides of (31), we have

$$\lim_{N\to\infty} \ln_q(F_q(Z_N)) = -(q/2)\xi^2. \tag{32}$$

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[5]Such a strong cutoff might be relaxed into a softer one.

Consequently, locally uniformly by ξ,

$$\lim_{N \to \infty} F_q(Z_N) = e_q^{-(q/2)\xi^2} \in \mathcal{G}_q. \tag{33}$$

Thus, Z_N is q-convergent.

The rest of the proof follows in exactly the same way as in Theorem 1.

Remark 3.6. *Let us note that the conditions $\rho < 1$ and $\gamma < \frac{3(1+\rho)}{2}$ are necessary to guarantee that the appropriately scaled escort third moment of the sum $X_1 + ... + X_N$ vanishes in the $N \to \infty$ limit.*

Corollary 3.7. *Let $X_1, ..., X_N, ...$ be a given sequence of q-independent and identically distributed random variables with a q-mean μ_q and a finite second moment σ^2_{2q-1}. Then $Z_N = D_N^{-1}(q)(X_1 + ... + X_N - N\mu_q)$ is q-convergent to a q-normal distribution if and only if $q = 1$, that is, in the classic case.*

4 CONCLUSION

We studied here a possible q-generalization of the classic central limit theorem. As is known, q-Gaussians extremize, under appropriate constraints, the entropy S_q. The classic analog of this fact is that the usual Gaussian maximizes the classic Boltzmann-Gibbs-Shannon entropy. Following this correspondence, it is expected that there exists an entire class of q- central limit theorems. In other words, normalized sums of sequences of identically distributed random variables with a finite q-variance must converge to q-Gaussians. Theorem 1 represents one of the possible generalizations of the classic central limit theorem for a sequence of q-independent random variables. The notion of q-independence coincides with the classic notion of independence if $q = 1$, and characterizes a specific type of global correlations otherwise. Theorem 3 considers more general sequences of correlated random variables, which are nevertheless attracted by the same q-Gaussians. The conditions of this theorem indicate that the escort joint density of random variables has to have a specific support (e.g., an hierarchical structure with zero Lebesgue measure).
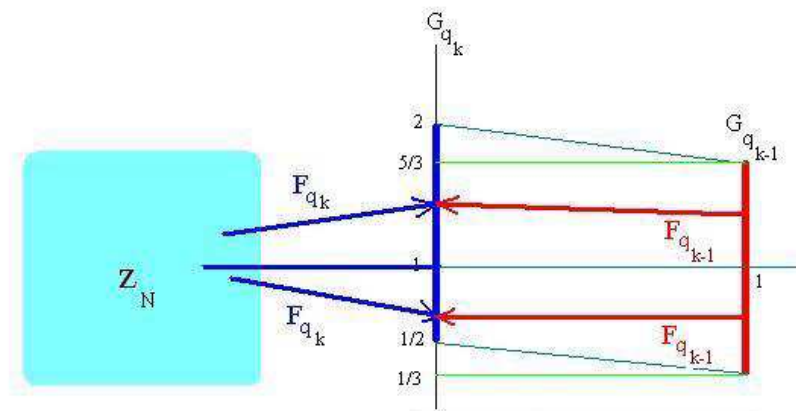

Figure 1: Schematic representation of the q-CLT: Z_N represents the set of rescaled sums of all q_k-independent random variables. The q_k-Fourier transforms of these sums belong to $\mathcal{G}_{q_k}$, which in turn is the $F_{q_{k-1}}$ image of $\mathcal{G}_{q_{k-1}}$. The process described in this scheme reflects the q_k-convergence of Z_N to a q_{k-1}-Gaussian. These transformations admit only one fixed point, namely $q_k = 1$, corresponding to the classical CLT (represented here as a horizontal straight line).

At the same time the corresponding q-normal distribution is described exactly by the $z^{-1}(q)$-Gaussian, see Figure 1. In the classic case q- or $z^{-1}(q)$-Gaussians do not differ. So, Corollary 3.7 says that such duality is a specific feature of the statistical q-theory, which comes from the specific definition of q-exponential.

We conclude the papr by making an important remark. The classical CLT may in principle be generalized in various manners, each of them referring to global correlations of specific kinds. A first example is the model numerically discussed in [6]. The correlations were introduced, in a scale-invariant manner, through a q-product in the space of the joint probabilities of N binary variables, with $0 \leq q \leq 1$. It was numerically shown that the attractors are (double-branched) Q-Gaussians, with $Q = 2 - \frac{1}{q} \in (-\infty, 1]$, and that the model is superdiffusive [15] (i.e., space x scales with time t, for large values of t, as $x \propto t^{\delta/2}$, with $1 \leq \delta \leq 2$). The relation $Q = 2 - \frac{1}{q}$ corresponds to the particular case $k = -1$ of the present Theorem 1. It comes from Lemma 2.14, with $q_{-2} = 2 - \frac{1}{q_0} = 2 - \frac{1}{q}$, which holds when k=-1. Notice, however, that these two models differ in other aspects. Indeed, although they share the relation $Q = 2 - \frac{1}{q}$, there δ's are different. In the model introduced in [6] only superdiffusion occurs, with δ monotonically decreasing from 2 to 1 when Q increases from $-\infty$ to 1 [15]. In contrast, in the $k = -1$ model associated with the present Theorem 1, we have $\delta = q = \frac{1}{2-Q}$, with $q \in (1/2, 2)$, hence $\delta \in (1/2, 2)$.

A second example is suggested by the exact stable solutions of a nonlinear Fokker-Planck equation [4]. The correlations are introduced through a $q = 2 - Q$ exponent in the spatial member of the equation (the second derivative term). The solutions are Q-Gaussians with $Q \in (-\infty, 3)$, and $\delta = 2/(3 - Q) \in [0, \infty]$, hence both superdiffusion and subdiffusion can exist in addition to normal diffusion. This model is particularly interesting because the scaling $\delta = 2/(3 - Q)$ was conjectured in [16], and it was verified in various experimental and computational studies [17, 18, 19].

A third example is the family of models presented here. The correlations are introduced through q_k-products of q_k-Fourier transforms, where $q_k = \frac{2q + k(1-q)}{2 + k(1-q)}$, $q \in (1/2, 2)$. The attractors are q_{k-1}-Gaussians and $\delta = 1/(2 - q_{k-1})$, as can be seen from Eq. (21). Applying Lemma 2.14 we obtain $2 - q_{k-1} = 1/q_{k+1}$, henceforce $\delta = q_{k+1}$. Thus the triplet (q_{k-1}, q_k, q_{k+1}) characterises features of the system under study identifying the type of correlation, the corresponding attractor, and the scaling rate.

In the particular case, $k = 1$, we have $\delta = 1/(2 - q)$. This coincides with the nonlinear Fokker-Planck equation mentioned above. Indeed, in our theorem we required the finiteness of $(2q - 1)$-variance. Denoting $2q - 1 = Q$, we get $\delta = 1/(2 - q) = 2/(3 - Q)$. Notice, however, that this example differs from the nonlinear Fokker-Planck above. Indeed, although we do obtain, from the finiteness of the second momentum, the same expression for δ, the attractor is not a Q-Gaussian, but rather a q-Gaussian, with $q = (Q + 1)/2$.

Summarizing, the present Theorems 1 and 3 suggest a quite general and rich structure at the basis of nonextensive statistical mechanics. Moreover, they recover, as particular instances, central relations emerging in the above first and second examples. The structure we have presently shown might pave a deep understanding of the so-called q-triplet (q_s, q_r, q_{ss}), where q_s means q-sensitivity, q_r - q-relaxation and q_{ss} - q-stationary state, [20, 21] in nonextensive statistics, but this remains as a challenge at the present stage.

Acknowledgments

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