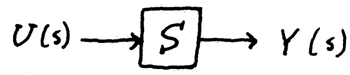
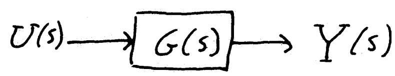
Transfer Functions



The transfer function of a linear system is the ratio of the Laplace Transform of the output to the Laplace Transform of the input, i.e., $Y(s)/U(s)$. Denoting this ratio by $G(s)$, i.e.,

$$G(s) \quad = \quad Y(s)/U(s)$$

we represent the linear system as



In this representation, the output is always the Transfer function times the input

$$Y(s) \quad = \quad G(s)U(s).$$

Example: Suppose a linear system is represented by the differential equation

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y \quad = \quad u$$

$$s^2Y(s) + a_1 sY(s) + a_0 Y(s) \quad = \quad U(s).$$

Taking Laplace Transforms with zero initial conditions,

$$(s^2 + a_1 s + a_0)Y(s) \quad = \quad U(s)$$

we see that the transfer function is

$$G(s) \quad = \quad \frac{Y(s)}{U(s)} \quad = \quad \frac{1}{s^2 + a_1 s + a_0}.$$

Remarks:

– The transfer function is always computed with all initial conditions equal to zero.

– The transfer function is the Laplace Transform of the impulse response function.

[To see this, set $U(s) = 1$.]

The transfer function of any linear system is a *rational function*

$$G(s) \quad = \quad \frac{n(s)}{d(s)} \quad = \quad \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

where $n(s)$, $d(s)$ are the numerator and denominator polynomials of $G(s)$, respectively

– $G(s)$ is proper if

$$m \quad = \quad \deg n(s) \leq n = \deg d(s)$$

– $G(s)$ is strictly proper if $m < n$.

– For a proper or strictly proper rational function, the difference $\alpha = n - m$ is called the *relative degree* of the transfer function.

– The denominator polynomial $d(s)$ is called the *characteristic polynomial.* The roots of $d(s)$ are the *poles* of $G(s)$. The roots of $n(s)$ are the *zeros* of $G(s)$. Assuming that $n(s)$ and $d(s)$ have no common factors then $G(s)$ has n finite poles and m finite zeros in the complex plane.

We will see that the location of the *poles* and *zeros* of $G(s)$ completely determine the behavior of the linear system.

A strictly proper transfer function *always* satisfies $G(s) \to 0$ as $s \to \infty$.

Example:

$$\frac{s}{s^2+1} = \frac{1/s}{1+1/s} \to 0$$

as $s \to \infty$. We say that $G(s)$ has a zero at $s = \infty$. In fact, $G(s)$ has $\alpha = n - m$ "zeros at ∞." In this way a proper transfer function has *exactly* n-poles and n-zeros counting zeros at infinity.

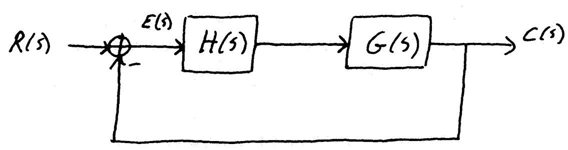
Example:

$$G(s) = \frac{s+1}{s(s+2)(s+3)}.$$

Then $G(s)$ has poles at $s = 0, -2, -3$ and zeros at $s = -1, \infty, \infty$ since the relative degree is $\alpha = 2$.

Standard Forms of Feedback Control Systems

Consider the block diagram shown below:



This is the standard *cascade compensation*, *unity feedback system.*

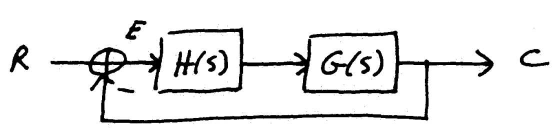
$$\begin{aligned} G(s) &= \text{plant transfer function} \\ H(s) &= \text{compensator transfer function} \\ R(s) &= \text{reference input} \\ C(s) &= \text{output} \\ E(s) = R(s) - C(s) &= \text{error signal or actuating signal} \end{aligned}$$

The product $G(s)H(s)$ is called the *open loop transfer function* or *forward transfer function*

– The plant transfer function is determined from the physical models of the system.

– The compensator transfer function is to be designed by us.

The block diagram



represents the equations

$$\begin{aligned} C(s) &= G(s)H(s)E(s) \\ E(s) &= R(s) - C(s) \\ &= R(s) - G(s)H(s)E(s). \end{aligned}$$

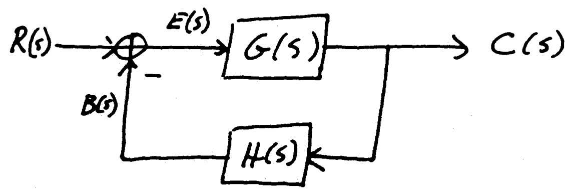
Therefore,

$$E(s) = \frac{1}{1+G(s)H(s)}R(s)$$

and

$$C(s) = \frac{G(s)H(s)}{1+G(s)H(s)}R(s).$$

The transfer function $\frac{G(s)H(s)}{1+G(s)H(s)}$ is called the *closed loop transfer function*. The transfer function $\frac{1}{1+G(s)H(s)}$ is called the system sensitivity function. Typically $R(s)$ represents the 'desired output' and the control design problem is, given $G(s)$, design $H(s)$ so that $C(s) \approx R(s)$, or equivalently so that $E(s) \approx 0$. We see that, to do this we should have $\frac{1}{1+G(s)H(s)} \approx 0$ and $\frac{G(s)H(s)}{1+G(s)H(s)} \approx 1$. We will develop techniques to do this. In some cases the compensator $H(s)$ might be given in the 'feedback path' instead of in cascade with the plant



This forms a non-unity feedback system. In this case the open loop transfer function

$$G(s)H(s) = \frac{B(s)}{E(s)}.$$

The actual signal is

$$\begin{aligned} E(s) &= R(s) - B(s) \\ &= R(s) - H(s)C(s). \end{aligned}$$

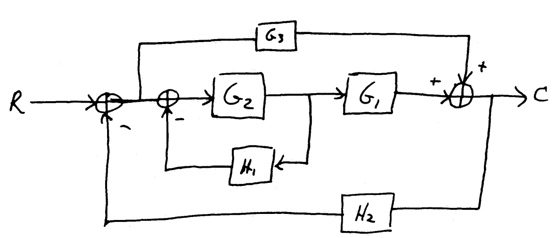
Note in this case that the actuating signal is no longer the same as the error signal, $R(s) - C(s)$. In fact, the error signal is not physically present in the block diagram in the case of non-unity feedback. It is easy to show that the closed loop transfer function now is

$$\frac{G(s)}{1+G(s)H(s)}$$

and the sensitivity function is the same.

Block Diagram Reduction

A general linear control system may be built up from many interconnected subsystems, each of which is a system by itself, and, hence, represented by a transfer function. We need ways to reduce a complicated block diagram to a simpler form, for example, *the linear system*

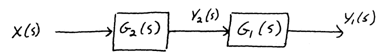


is not in standard form.

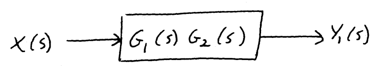
What is the closed loop transfer function $C(s)/R(s)$?

In order to answer this, we need to develop a 'Block Diagram Algebra' to replace complex blocks by equivalent, but simpler blocks.

1) Series (Cascade) Connection



This is equivalent to:



Proof: Note that the original block diagram represents the equations

$$\begin{aligned} Y_1 &= G_1(s)Y_2 \\ Y_2 &= G_2(s)X. \end{aligned}$$

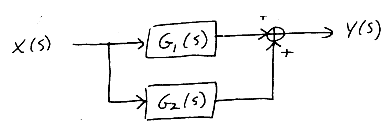
Substituting for Y_2 in the first equation gives

$$Y_1 = G_1(s)G_2(s)X$$

which is represented by the second block diagram.

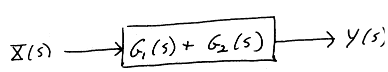
Remarks: Notice that the intermediate signal $Y_2(s)$ is lost, i.e., it does not appear in the equivalent block diagram.

2) Parallel Connection



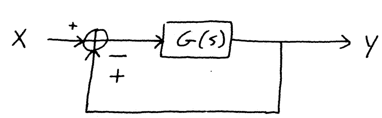
$$\begin{aligned} Y(s) &= G_1(s)x(s) + G_2(s)x(s) \\ &= (G_1(s) + G_2(s))x(s). \end{aligned}$$

Therefore the equivalent block diagram representation is

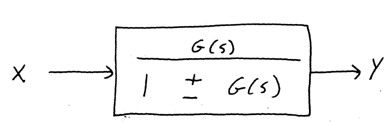


3) Feedback Connection

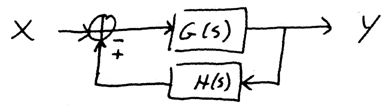
a) *unity feedback*



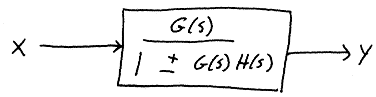
is equivalent to



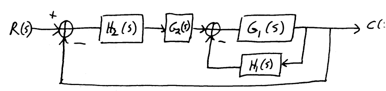
b non-unity feedback



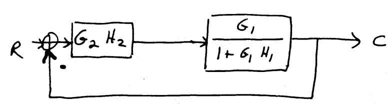
is equivalent to



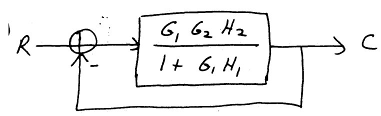
Example: Find the closed transfer function $\frac{C(s)}{R(s)}$



Step 1)



Step 2)

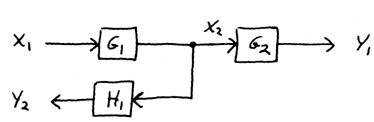


Finally

$$C(s)/R(s) = \frac{\frac{G_1G_2H_2}{1+G_1H_1}}{1+\frac{G_1G_2H_2}{1+G_1H_1}} = \frac{G_1G_2H_2}{1+G_1H_1+G_1G_2H_2}$$

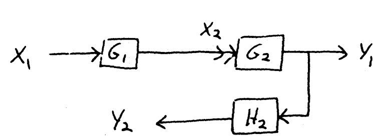
4) Moving a Pick-Off Point

a) ahead of a block



(I)

is equivalent to



(II)

provided $H_2 = H_1/G_2$.

Proof: In diagram (I) we have

$$Y_2 = H_1 X_2$$

In diagram (II) we have

$$Y_2 = H_2 Y_1 = H_2 G_2 X_2$$

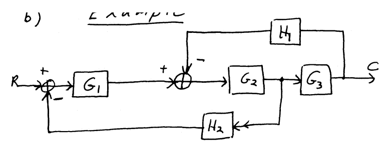
Therefore

$$H_1 = H_2 G_2$$

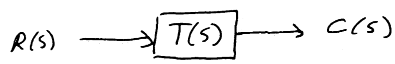
or

$$H_2 = H_1 / G_2$$

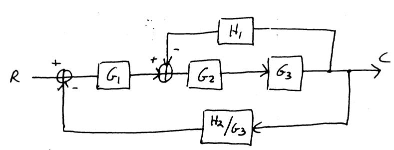
b)



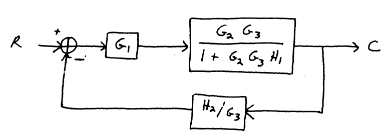
Find $C(s)/R(s) = T(s)$



Step 1: Move pick-off point ahead of G_3.



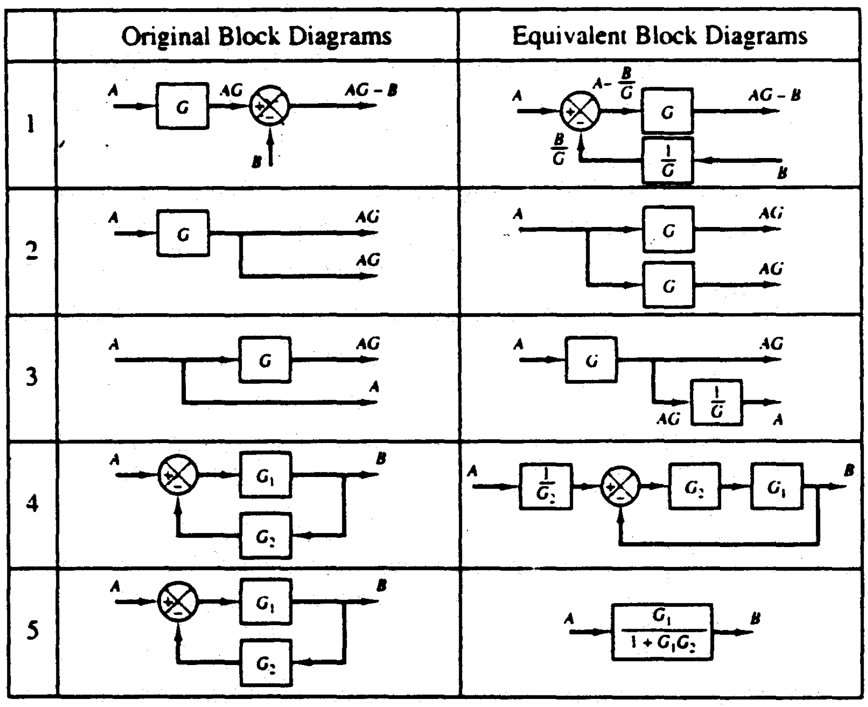
Step 2: Combine cascade of G_2 and G_3 and eliminate the upper feedback loop.



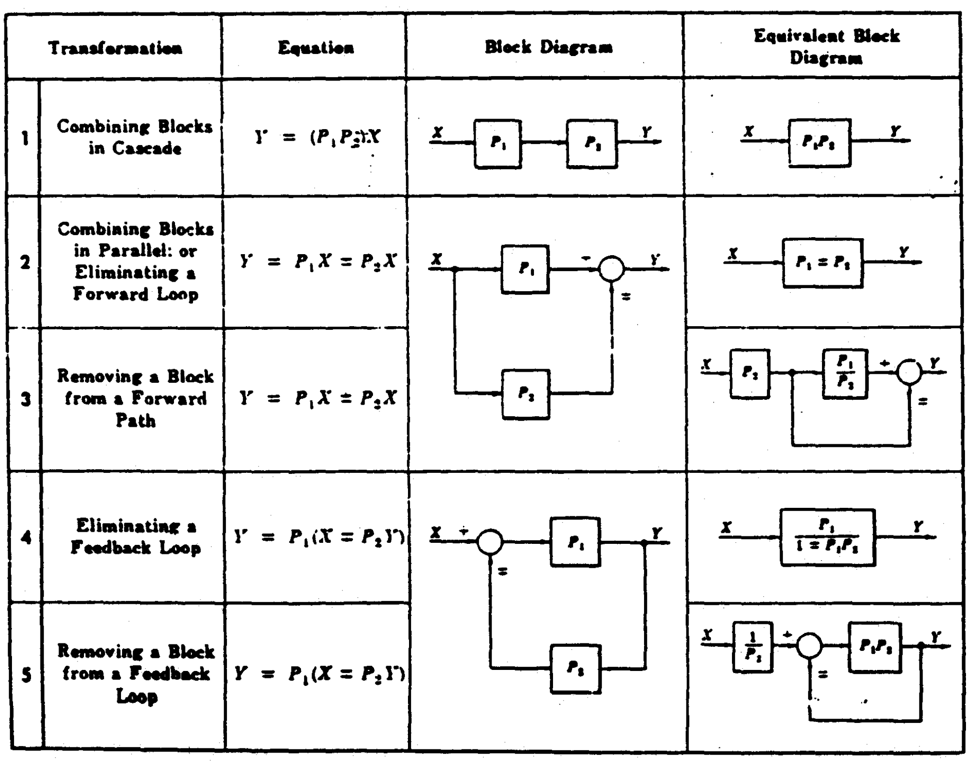
Step 3: Simplify cascade connection and eliminate the final non-unity feedback loop.

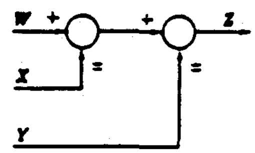
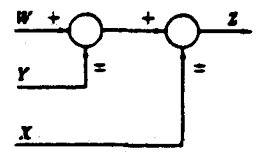
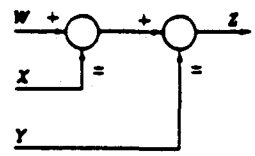
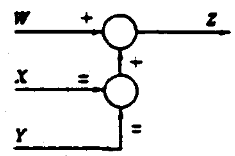
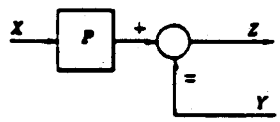
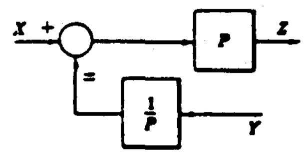
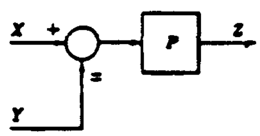
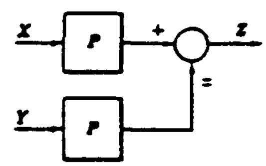
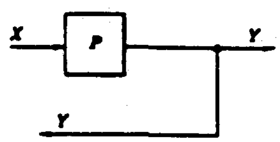
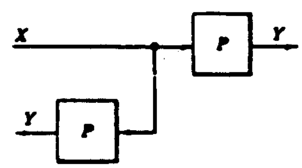
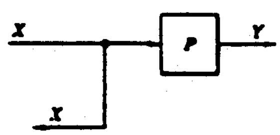
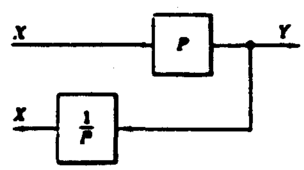
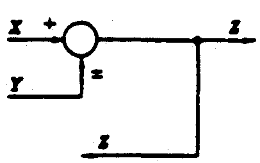
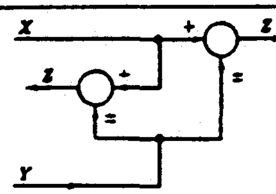
$$\begin{aligned} & \frac{\frac{G_1 G_2 G_3}{1+G_2 G_3 H_1}}{1 + \frac{G_1 G_2 G_3}{1+G_2 G_3 H_1} \cdot \frac{H_2}{G_3}} \\ = \; & \frac{G_1 G_2 G_3}{1 + G_2 G_3 H_1 + G_1 G_2 H_2} \\ = \; & T(s). \end{aligned}$$

There are other rules for moving summing junctions, etc. The important point to remember is that the block diagram is a representation of algebraic equation so the equivalent block diagram can always be reproduced from the equations.



Summary of Block Diagram Algebra



	Transformation	Equation	Block Diagram	Equivalent Block Diagram
6a	Rearranging Summing Points	$Z = W \pm X \pm Y$		
6b	Rearranging Summing Points	$Z = W \pm X \pm Y$		
7	Moving a Summing Point Ahead of a Block	$Z = PX \pm Y$		
8	Moving a Summing Point Beyond a Block	$Z = P[X \pm Y]$		
9	Moving a Takeoff Point Ahead of a Block	$Y = PX$		
10	Moving a Takeoff Point Beyond a Block	$Y = PX$		
11	Moving a Takeoff Point Ahead of a Summing Point	$Z = X \pm Y$		
12	Moving a Takeoff Point Beyond a Summing Point	$Z = X \pm Y$