A Vertex Operator Algebra

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June 26, 2007

Abstract

I introduce the notion of a vertex operator algebra, along with some background information on Lie algebras, and give a specific, well known, example of a vertex operator algebra.

(This is just some background information and me practicing LaTeX, not anything serious, but does provide some information for those un-aquainted with the material.)

1 The (untwisted) Affine Lie Algebra

Let $\mathfrak{g}$ denote be a Lie algebra (for our purposes, over $\mathbb{C}$), and let $\langle \cdot, \cdot \rangle$ be a symmetric invariant bilinear form on $\mathfrak{g}$.

Definition 1.1. *The (untwisted)* ***affine Lie algebra*** *is given by:*

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \tag{1.1}$$

where for $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, *we define the bracket relations to be:*

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b\rangle \delta_{m+n,0} c \tag{1.2}$$

and also require that c *is central:*

$$[\hat{\mathfrak{g}}, c] = 0. \tag{1.3}$$

We equip $\hat{\mathfrak{g}}$ with the following $\mathbb{Z}$-grading:

$$\hat{\mathfrak{g}} = \coprod_{n \in \mathbb{Z}} \hat{\mathfrak{g}}_{(n)} \tag{1.4}$$

where

$$\hat{\mathfrak{g}}_{(0)} = \mathfrak{g} \oplus \mathbb{C}c \;\; and \;\; \hat{\mathfrak{g}}_{(n)} = \mathfrak{g} \otimes t^n \;\; for \;\; n \neq 0. \tag{1.5}$$

The following are also subalgebras of $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}}_{(\pm)} = \coprod_{n>0} \hat{\mathfrak{g}}_{(\pm n)} = \coprod_{n>0} \mathfrak{g} \otimes t^{\pm n} \tag{1.6}$$

and

$$\hat{\mathfrak{g}}_{(\geq 0)} = \coprod_{n \geq 0} \hat{\mathfrak{g}}_{(n)} = \hat{\mathfrak{g}}_{(+)} \oplus \mathfrak{g} \oplus \mathbb{C}c. \tag{1.7}$$

2 The Universal Enveloping Algebra of a Lie Algebra

We now want to discuss an associative algebra that can be constructed from a given Lie algebra.

The modules of this associative algebra are in bijective correspondence with the modules of the Lie algebra upon which it is built, but this new algebra is necessarily infinite dimensional. It is called the universal enveloping algebra.

Let L be a Lie algebra

Definition 2.1. *We define a **Tensor Algebra** as follows: Let:*

$$T^0 = \mathbb{C}$$
$$T^1 = L$$
$$T^n = L \otimes ... \otimes L \quad (n\ times).$$

$$T(L) = \bigoplus_{n \geq 0} T^n$$

We call $T(L)$ the tensor algebra of L.

Now, we take the subspace of $T(L)$ generated by the elements

$$x \otimes y - y \otimes x - [x, y] \tag{2.1}$$

and call it $Q(L)$.

Definition 2.2. *We define the **Universal Enveloping Algebra** of L, denoted $U(L)$, as follows:*

$$U(L) = T(L)/Q(L). \tag{2.2}$$

That is, we quotient the tensor algebra of L by $Q(L)$.

3 Vertex Operator Algebras

Let $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$.

Recall the definition of a vertex operator algebra:

Definition 3.1. *Let V be a vector space over $\mathbb{C}$. We call V a **Vertex Operator Algebra** iff it satisfies the following:*

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}$$

such that

$$dim V_{(n)} < \infty$$

and

$$V_{(n)} = 0 \ for\ n\ sufficiently\ small.$$

There is a linear map

$$
\begin{aligned}
Y(\cdot, x) : V &\to (\mathrm{End}\, V)[[x, x^{-1}]] \\
v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1},
\end{aligned} \tag{3.1}
$$

and vectors $\mathbf{1} \in V_{(0)}$ *(the* vacuum vector*) and* $\omega \in V_{(2)}$ *(the* conformal vector*) s.t. for* $u, v \in V$*:*

$$
u_n v = 0 \quad \text{for } n \text{ sufficiently large,}
$$

$$
Y(\mathbf{1}, x)v = v,
$$

$$
Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)\mathbf{1} = v_{-1}\mathbf{1} = v, \tag{3.2}
$$

$$
\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1) \\
&= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2)
\end{aligned} \tag{3.3}
$$

(the Jacobi identity*).*

Also,

$$
Y(\omega, x) = \sum_{n \in \mathbb{Z}} \omega_n x^{-n-1} = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2}, \tag{3.4}
$$

$$
[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12}\delta_{m+n,0}\, c_V, \tag{3.5}
$$

where $c_V \in \mathbb{C}$ *is called the* central charge *(or* rank*) of* V *and*

$$
Y(L(-1)v, x) = \frac{d}{dx} Y(v, x) \quad \text{for } v \in V, \tag{3.6}
$$

$$
V_{(n)} = \{v \in V \mid L(0)v = nv\} \quad \text{for } n \in \mathbb{Z}. \tag{3.7}
$$

This may seem daunting, but it's really not so bad...

4 A Construction

Now, we construct an example of a vertex operator algebra (but do not prove the above properties here).

Assume that $\mathfrak{g}$ is finite dimensional of dimension d, and that our symmetric invariant bilinear form on $\mathfrak{g}$ is nondegenerate. Let $\{u_{(i)} | 1 \le i \le d\}$ be an orthonormal basis for $\mathfrak{g}$, and let $\{u^{(i)} | 1 \le i \le d\}$ be dual to it.

We start by contructing a module for the Lie algebra $\hat{\mathfrak{g}}$. Let l be a complex number. Let $\hat{\mathfrak{g}}_{(+)}$ and $\mathfrak{g}$ act trivially on $\mathbb{C}$. Let c act as the scalar l. Now, we have that $\mathbb{C}$ is a $\hat{\mathfrak{g}}_{(\ge 0)}$-module.

We want a $\hat{\mathfrak{g}}$ module, so we create the induced module:

$$
V_{\hat{\mathfrak{g}}}(l, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{(\ge 0)})} \mathbb{C}_l \tag{4.1}
$$

By the PBW Theorem, this module is (linearly) isomorphic to $U(\hat{\mathfrak{g}}_{(-)})$.

We now let $a(n)$ denote the action of $a \otimes t^n$ on $V_{\hat{\mathfrak{g}}}(l, 0)$, and put a vertex operator algebra structure on $V_{\hat{\mathfrak{g}}}(l, 0)$.

First, we choose our vaccum vector $\mathbf{1}$ to be $1 \in V_{\hat{\mathfrak{g}}}(l,0)$.
Now, we pick a grading for $V_{\hat{\mathfrak{g}}}(l,0)$:

$$V_{\hat{\mathfrak{g}}}(l,0) = \coprod_{n\geq 0} V_{\hat{\mathfrak{g}}}(l,0)_{(n)}, \tag{4.2}$$

where $V_{\hat{\mathfrak{g}}}(l,0)_{(n)}$ is spanned by the vectors

$$a_{(1)}(-m_{(1)})...a_{(j)}(-m_{(j)})\mathbf{1} \tag{4.3}$$

where $\jmath \geq 0$, $a_{(i)} \in \mathfrak{g}$, $m_{(i)} \geq 1$, and $n = m_{(1)} + ... + m_{(j)}$.
Clearly, $\mathbf{1} \in V_{\hat{\mathfrak{g}}}(l,0)_{(0)}$

We now define $\omega \in V_{\hat{\mathfrak{g}}}(l,0)_{(2)}$ to be

$$\omega = \frac{1}{2(l+h)} \sum_{i=0}^{d} u^{(i)}(-1)u^{(i)}(-1)\mathbf{1}, \tag{4.4}$$

where h is the dual-Coxeter number (Take note now that we must exclude the case of $V_{\hat{\mathfrak{g}}}(l,0)$ where $l = -h$ for this construction).

We now define a vertex operator Y, and state that this gives $V_{\hat{\mathfrak{g}}}(l,0)$ the structure of a vertex operator algebra:

$$Y(\mathbf{1},x) = Id_{V_{\hat{\mathfrak{g}}}(l,0)}, \tag{4.5}$$

the identity map,

$$Y(a(-1)\mathbf{1},x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1} \tag{4.6}$$

$$Y(a(-m)\mathbf{1},x) = \frac{d^{m-1}}{dx^{m-1}} \sum_{n\in\mathbb{Z}} a(n)x^{-n-1} \tag{4.7}$$

$$\begin{aligned} Y(a(-m)u,x_2) = {} & Res_{x_1}(x_1-x_2)^{-m}Y(a(-1)\mathbf{1},x_1)Y(u,x_2) - \\ & Res_{x_1}(-x_2+x_1)^{-m}Y(u,x_2)Y(a(-1),x_1) \end{aligned} \tag{4.8}$$

This gives us a vertex operator algebra structure on $V_{\hat{\mathfrak{g}}}(l,0)$.

For more information on vertex operator algebras, and to see this construction in more detail, but from a slightly different angle, please see [LL], specifically section (6.2). It is essentially the same as this one and uses the general theory to prove that $V_{\hat{\mathfrak{g}}}(l,0)$ is indeed a vertex operator algebra.

References

[LL] J. Lepowsky and H.-S. Li, *Introduction to Vertex Operator Algebras and Their Representation Theory*, Progress in Math., **Vol. 227**, Birkhäuser, Boston, 2004.