# Technology-assisted discovery of conceptual connections within the cognitive neighborhood of a mathematical topic

**Thierry Dana-Picard**

*Department of Applied Mathematics*
*Jerusalem College of Technology*
*Havaad Haleumi Street 21, POB 16031, Jerusalem 91160 - Israel*
*E-mail: dana @ jct.ac.il*

**Abstract:**

*New technologies provide an efficient tool for broadening the mathematical landscape discovered by the students. Educators should develop compound activities; in order to enhance the epistemic value of the learning process and enlarge the student's knowledge of the internal connections within the cognitive neighborhood of learned topics. Actually three-fold activities are a suitable frame, involving handwork; CAS assisted computations and websurfing.*

## I. Introduction.

Undergraduate courses are traditionally organized around specific topics, Calculus is taught separately from Linear Algebra, a course in Geometry has no ties with any other course, with the little exception of Analytic Geometry which uses some algebraic methods, and so on. For example, if a Calculus teacher tries to show algebraic properties of differentiation, through appropriate exercises, students often complain that it belongs to another course, not to the present one. One notable exception is given by courses in ODEs, which are often linked, firmly to Linear Algebra. Combinatorics is nowhere else than in a course on Discrete Mathematics and a course in Probability.

Suppose that the student has an access to sources of knowledge, beyond the adopted textbooks and lecture notes: the learning process becomes much more comprehensive. The additional sources include, among others:

- suitable websites, providing either ready to learn exposition or interactive activities;
- Computer Algebra Systems, via their computational and graphical features, together with pedagogical indications included in their commands (step-by-step execution of commands in Derive 6, indications in the solution process of ODEs in Maple, etc.).

The exploitation of these sources demands teaching and learning skills beyond the acquisition of notions and needed techniques via direct lecturing and practicing under the supervision of educators, the proposed activities aiming to an enlargement of the students' mathematical world through personal research. According to the student's level, this research is either autonomous or driven by the educator's indications. Nevertheless teacher's intervention will be more rare than in the traditional way.

A few years ago, Cuoco and Goldenberg (1996) wrote:
"New technology poses challenges to mathematics educators. How should the mathematics curriculum change to best make use of this new technology? Often computers are used badly, as a sort of electronic flash card, which does not make good use of the capabilities of either the computer or the learner. However, computers can be used to help students develop mathematical habits of mind and construct mathematical ideas."

It happens that even this level of use is not achieved, for various reasons. Among them:

- Despite the expanding availability of new tools, a great number of teachers still convey Mathematics in a traditional way, with frontal lectures and technical computations. A certain *pragmatic value* of the teaching is obtained, but the *epistemic value* generally not (Artigue, 2002, page 246). For this situation to change, teacher training has to include technological tools; this issue is not discussed here, but see for example Lingefjärd and Holmquist (2002), Baldin (2002) and Kyriasis and Korres (2002).
- Students often manifest a lack of interest for Mathematics, even when they learn a scientific curriculum, and consider Mathematics as a list of techniques for solving problems, i.e. only the pragmatic value, at a low level, seems to them worth of an effort.

Thereafter, Cuoco and Goldenberg (1996) claimed:
"The mathematics curriculum must be restructured to include activities that allow students to experiment and build models to help explain mathematical ideas and concepts. Technology can be used most effectively to help students gather data, and test, modify, and reject or accept conjectures as they think about these mathematical concepts and experience mathematical research."

Among the newly available technological tools, we find CAS and the World-wide-web. Therefore new activities are needed, involving their usage, along with "older" techniques, and aimed at the following achievements:

1. Stimulate students' curiosity for interlaced techniques, using more than one of the newly available technologies.
2. Make Mathematics more attractive, and show them as a living field of knowledge by discovering new tracks.
3. Discover links between apparently different.
4. Last but not least, traditional libraries offer very small appeal to the average student. Searching the WWW makes him/her fonder of looking for documents relevant to his/her learning domain.

For this last point in particular, a suitable search of the WWW leads to new perspectives on old topics and helps to discover on-going research and interactive mathematical processes.  The student is not passive; he/she can influence the teaching process, change the pace of learning, open connections, and discover new horizons.

This construction of a ``compound cognitive process'' fits Artigue's point of view: the paper and pencil part of the work, together with a possible CAS assisted part provide both efficient mathematical practice and conceptual insight into the mathematics involved in the problem under consideration (Artigue (2002), page 246). A web-search can provide an added value to the solution, and generally makes the learning process more efficient, with a broader perspective on the problem, its solution and what we will call *the cognitive neighborhood* of the pair problem-solution. This neighborhood includes domains in Mathematics related to the problem under consideration; the relation can be either obvious from start or be discovered during the student's autonomous work. The example developed in section II shows combinatorial identities belonging to the cognitive neighborhood of a parametric integral, two topics generally taught in independent courses.

Sequences of definite integrals are often the central topic of exercises leading to induction formulas and/or closed formulas, as exposed by Glaister (2003) and Dana-Picard (2004a). After a finite number of iterations, various properties of the sequence of integrals are discovered, such as a closed expression for the general term of the sequence, which appears often to be of a combinatorial nature. Searching the web reveals various unexpected items; among them:

- a concrete meaning for the combinatorial properties of the given sequence of integrals;
- the history of the mathematical works having produced these combinatorial expressions;
- ``real-world'' situations with the same mathematical translation.

Such a task is generally built by the teacher, i.e. in this context, the teacher is active and creative; the student reproduces the teacher's working steps. After that, incite the students to search for related material, in particular using the WWW.  Links to neighboring mathematical topics can be discovered. The student becomes more autonomous and develops more initiative. Actually, both the educator and the student are creative.

## II. An infinite sequence of integrals.

For any positive integer $n$, we define the definite integral

$$I_n = \int_0^{\pi/2} \sin^n x dx .$$

Using integration by parts, the following recurrence relation appears:

$$I_n = \frac{n-1}{n} I_{n-2} .$$

The sequence of integral splits naturally into two distinct subsequences respectively formed by the terms with even indices and by the terms with odd indices. Consider the first subsequence; by a telescoping process, a closed form is obtained for the general term of this subsequence (for details, see (Dana-Picard 2004a)):

$$I_{2p} = \frac{(2p)!\pi}{2^{2p+1}(p!)^2}$$

Now denote the rational coefficient by $F_p$, i.e. $F_p = \frac{(2p)!}{2^{2p+1}(p!)^2}$.

The first terms of the sequence are

$$F_1 = \frac{1}{4}, F_2 = \frac{3}{16}, F_3 = \frac{5}{32}, F_4 = \frac{35}{256}, F_5 = \frac{63}{512}, F_6 = \frac{231}{2048}, \ldots .$$

A search in the database named On-Line Encyclopedia of Integer Sequences (2004) provides a combinatorial interpretation for the sequence of numerators, but no interpretation for the sequence of denominators. Look at the sequence of denominators: with a slight modification, it can appear as the sequence of successive powers of 4; the first terms of the sequence $(F_p)$ are equal to

$$F_1 = \frac{1}{4}, F_2 = \frac{3}{4^2}, F_3 = \frac{10}{4^3}, F_4 = \frac{35}{4^4}, F_5 = \frac{126}{4^5}, F_6 = \frac{462}{4^6}, \ldots$$

A new search in the database leads to the following interpretation of the sequence of numerators: for any positive integer $p$,

$$F_p = \frac{1}{4^p}\binom{2p-1}{p}.$$

Actually, only a few terms at the beginning of the sequence are entered for performing the search; the database provides many other terms which can be compared to the values of $F_p$ for greater $p$, thus obtaining a firm conviction that the interpretation proposed by the database fits the given sequence of integrals. Note that in this specific example, an index translation has to be performed for the above closed formula to be established. Moreover, the database proposes "real-world" interpretations, such as the number of walks of length $p$ on a square lattice, starting at the origin, staying in the first and second quadrants, or the number of leaves on all ordered trees with $p+1$ edges, and so on.

In conclusion, the following integral-combinatorial relation has been obtained:

$$\int_0^{\pi/2} \sin^{2p} x dx = \frac{\pi}{4^p}\binom{2p-1}{p}.$$

Another connection can be discovered: the sequence 1, 3 10, 35, 126, 462, … is described in the database as a convolution from the sequence of *Catalan numbers*. Real-world meanings for Catalan numbers are available in the On-Line Encyclopedia of Integer Sequences (2004), in Dickau (1996) and many other sources. A biography of Catalan, with a description of his mathematical work, is to be found in (Mac Tutor 2003). . Catalan numbers have also integral interpretations, as parametric definite integrals; one of them is given in the database, another one has been studied by Dana-Picard (2004b). We should mention that for this sequence, no modification of the immediate output has been needed in order to discover the nature of the sequence via the websearch.

## III. Computer assisted activities.

Examples of parametric integrals can be found where computation by hand of the induction relation and of the closed form for the given integrals is beyond the abilities of an average student. Usage of a CAS can help.

First, the general form of the parametric integral $I_n$ is entered. In most cases, the immediate output is identical to the input and no pattern appears. Then the student substitutes special values for the parameter and computes $I_n$ for small values of the parameter $n$. Suppose that the successive substitutions give answers without a visible general pattern. A web-search, through ad-hoc interactive sites like those mentioned previously, provides sometimes a remedy to this problem, by enabling the student to find either previous work on the same topic, or a pathway into further inquiry. The following frame can be accurate:

a. Using a CAS, compute $I_n$, for $n$ equal to $0,1,2,\cdots,10$.
b. Look for a simple pattern in the output.
c. Connect to the On-Line Encyclopedia of Integer Sequences (2004); enter the sequence of numbers obtained during the first step (eventually, decompose a sequence of fractions into two distinct sequences, for numerators and denominators). This should provide a conjecture for a general formula for $I_n$. In the example above, the web-answer is unique. In other examples, there can be multiple propositions; further exploration is then needed in order to make a decision
d. With the CAS, check the conjecture for greater values of the parameter. Of course, such a process does not provide a *proof* of an explicit formula, only some kind of conviction is afforded. This is an example of Trouche's *théorème-en-actes* (Trouche, 2004).

With some CAS, a pattern appears immediately for the general term of the sequence. In the example above, it involves the Gamma function (this function generalizes the factorial to non integer positive numbers; see (Thomas's Calculus, 2002) page 605; it is generally taught only in an advanced course. The output is:

$$\int_0^{\pi/2} \sin^n x dx = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1+n}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)}$$

In such a case, the hope to bypass the lack of knowledge using the CAS is deceived: the student replaced his/her problem by a problem still worse from his/her point of view: he/she cannot understand the actual meaning of the output, the CAS is used as a blackbox and the pedagogical aspect of the work is lost. It is still possible to make substitutions into the obtained "strange" formula, for the sequence of numbers to appear and to be studied as described previously, but no conceptual understanding is afforded from a study of the general formula on display. Dana-Picard and Steiner

(2004) point out the fact that the usage of such “high level” commands (here “high level function” could be more appropriate) does not help to building conceptual understanding. Or maybe the educator can catch this opportunity to reverse the trend, by giving a definition of the Gamma function and showing its first properties; this is part of the educator’s building of the theoretical discourse accompanying the technique. The “bad problem” becomes a motivating example for further discovery.

Nevertheless, the educator must pay attention to the danger inherent to the multiplication of the goals of an activity: maybe none of them is totally achieved. Moreover, the student is sometimes mislead and thinks that a lack of conceptual understanding can be bypassed by multiplying technicalities with the computer. The usage of commands whose output involves the Gamma function should be postponed to a later task, after the present one has been fully performed. In other words, the cognitive neighborhood of a given topic can be too large for the student to be able to find reasonable pathways for an exhaustive exploration. The same remark is valid for the convolution mentioned with respect to the connection between our example and Catalan numbers. The educator must make the appropriate choices: which topic is at a “reasonable distance” within the cognitive neighborhood from the main topic under consideration, and which one is too far away at that time ?

At this point, we should emphasize the fact, already mentioned in the previous paragraph, that the theoretical discourse for instrumented techniques (Artigue, 2002) is intimately connected to the choice of the CAS: in this example, shall we explain the Gamma function (Mathematica’s output uses the Gamma function) or not (Derive’s display does not include the Gamma function)? Moreover, in the proposed activity model, the discourse has to include a subdiscourse aimed to master ways of web searching to broaden mathematical horizons, and not getting lost in this huge amount of more or less relevant sources of information.

## IV. Three-fold activities and exploration of a cognitive neighborhood.

### 1. Diagram presentations for the cognitive neighborhood.

The frame of the author's courses is fixed by the institution where he teaches; the added value of extra tasks, not officially present in the syllabus but given as pilots, is received by most students as a “plus” in their education. After some adaptation process, their reaction is very positive and, for example, the best results of their explorations are dispatched among their peers, generally via the electronic forum of the class. Moreover, some students use another CAS than the teacher; the comparison between the methods and their results is very enriching. This takes place generally in “private” conversations, not during plenary lectures or exercises sessions.

The solution of an old problem with new techniques has always a great mathematical value and a pedagogical interest. Therefore the introduction into the curriculum of compound activities, including traditional ways of doing mathematics together with the most up-to-date technologies, is important. As already claimed, the widening of the mathematical landscape provided by new technological tools reinforces the students' will for a deeper understanding of what they learn and stimulates them to further learning. Some time ago, a former student, whose name is Dor, came to the author's office, asking for extra mathematical material on a certain topic and for personal help. He said: “I learnt this material, but I still want more profound insight into what these objects are”. In Dor’s words, this means: it is not sufficient in my eyes to know only what has been taught, I wish to understand more profoundly the nature of the mathematical objects under study, and the connections between them. In our words, Dor wishes to explore the cognitive neighborhood of his topic.

Consider the cognitive neighborhood of a given mathematical topic as included in a kind of space, which could be called ``Mathematical Knowledge". The topics within the neighborhood are related by ``connections’’, which can be represented by a diagram, as in Figure 1. Actually, even if we represent mathematical notions as vertices, we shall not represent the connections as edges, thus not obtaining a graph in the ordinary sense. Because of their non-uniqueness, the connections should rather be displayed as ``clouds".

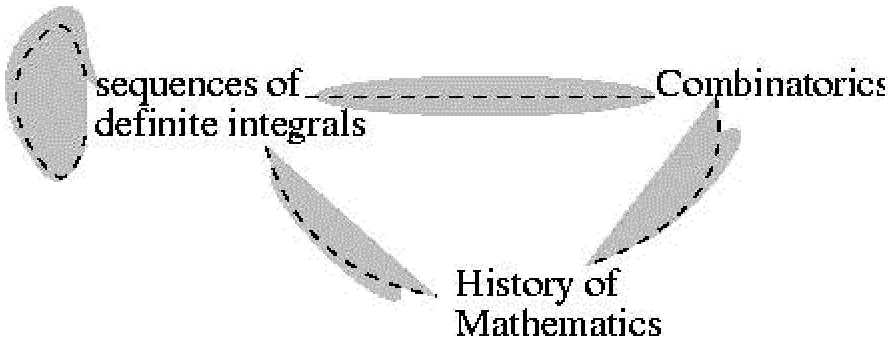


Figure 1: Restricted cognitive neighborhood.

In this diagram, a line crosses the clouds, to show the fact that a specific connection has been established during the proposed activity; other connections can exist, and actually do exist. The convolution mentioned in the example is an example of (unexpected?) connection among sequences of definite integrals; therefore we added a connecting cloud from this field to itself. The field “History of Mathematics” represents here all the general knowledge surrounding the Mathematics under study, such as whom is Catalan, how Catalan numbers appeared for the first time, and so on.

In fact, the cognitive process at work when learning Mathematics is not only composed of mathematical topics. In the three-fold activities described above, *tools* are used and the student’s mathematical thinking moves in two reversed directions, performing an *instrumentalisation process*, together with an *instrumentation process,* as shown by Trouche (2004). Figure 2 presents a diagram for what we will call *an extended cognitive neighborhood,* where not only the connections between mathematical notions and topics are on display, but also the tools which are to be used. The lower level in the diagram lies within the “Mathematical Knowledge” space; it appeared in section II that some of the internal connections are discovered and explored with the help of the diagram’s upper level techniques, and would have been quite impossible to discover without, in particular, the websearch (remember that the sequences’ database provided a lot of information from which even internal connections between different integrals were made possible.

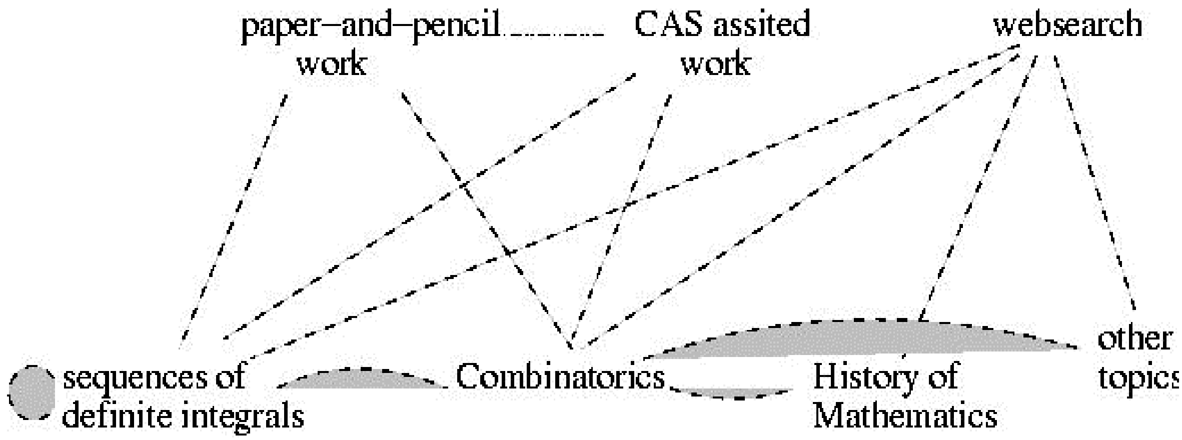


Figure 2: Extended cognitive neighborhood.

**2. Advantages and disadvantages of compound activities.**

Another advantage of this teaching-learning process is the student's self-teaching, at least part of the time. The learning process is composed of a synchronous part (as in a traditional process) and an asynchronous part (mostly the exploration of the problem's cognitive neighborhood), the importance of this asynchronous work being emphasized.

Finally, the author wishes to thank the referee for the following remark: the world wide web is not a “tool” by itself but gives a overwhelming amount of very different tools (including CAS) and information. It can help to explore the cognitive neighbourhood and offers a more profound understanding of the mathematical objects, but there is also a danger that a lot of superficial information is collected and students could loose the focus on the mathematical topic.

For such reasons (and others), too big an enthusiasm for this kind of mixed learning process must yet be tempered. The example of a compound mathematical activity that we described here shows an application of Lagrange’s (2000, page 27) claim. The coordination of new techniques with the traditional ones will not change in a miraculous way the learning process. With new technological tools, some results will be obtained more quickly, but such a compound activity demands profound reflection from the educator, and ``demands from the students time and efforts for their passage towards theory’’. ``The difficulties encountered when implementing new praxeologies should not be underestimated".

## References

Artigue, M.: 2002, Learning Mathematics in a CAS Environment: The Genesis of a Reflection about Instrumentation and the Dialectics between Technical and Conceptual Work, *International Journal of Computers for Mathematical Learning* 7, 245-274.

Baldin, Y.: 2002, On some important aspects in preparing teachers to teach Mathematics with technology, *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*, University of Crete, Greece, available: http://www.math.uoc.gr/~ictm2/Proceedings/pap325.pdf

Cuoco, A. A. and Goldenberg, E. P.: 1996, A role for technology in mathematics education, *Journal of Education* 178 (2), 15-32.

Dana-Picard, Th.: 2004a, Explicit closed forms for parametric integrals, *International Journal of Mathematical Education in Science and Technology* 35 (3), 456-467.

Dana-Picard, Th.: 2004b, Parametric integrals and Catalan numbers, to appear in *International Journal of Mathematical Education in Science and Technology*.

Dana-Picard, Th. and Steiner, J.: 2004, The importance of ``low level'' CAS commands in teaching Engineering Mathematics, *European Journal of Engineering Education* 29 (1), 139-146.

Dickau, R.: 1996, Catalan numbers, available:
http://mathforum.org/advanced/robertd/catalan.html

P. Glaister, P.: 2003, Factorial sums, *International Journal of Mathematical Education in Science and Technology* 34 (2), 250-257.

Finney, R., Weir, M. and Giordano, F.: 2001, Thomas' Calculus 10/e, Reading, Massachussets: Addison-Wesley.

Lagrange, J.B.: 2000, L'intégration d'instruments informatiques dans l'enseignement: une approche par les techniques, *Educational Studies in Mathematics* 43, 1-30.

Kyriasis, A. and Korres, K.: 2002, Pre-service and in-service teacher of Mathematics' training in teaching with the use of computers, *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level*), University of Crete, Greece, available:
http://www.math.uoc.gr/~ictm2/Proceedings/pap178.pdf

On-Line Encyclopedia: 2003, N.J.A. Sloane: The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/

The MacTutor History of Mathematics archive: 2003, www-gap.dcs.st-and.ac.uk/~history/ Miscellaneous/CatalanNumbers/catalan.html

Trouche, L.: 2004, Environnements informatisés: quels usages pour quels apprentissages? *Educational Studies in Mathematics* 55, 181-197.