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Transfer functions of discrete-time nonlinear control systems

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Abstract. The notion of the transfer function of the discrete-time nonlinear control system is defined. The definition is based on a non-commutative twisted polynomial ring, which can be by the Ore condition extended into its quotient ring (field of fractions). Some properties of the transfer function, related to accessibility and observability of the system, are studied and the transfer functions of different composite systems (series, parallel, and feedback connections) are given. The resulting theory is, in principle, similar to that in the linear case, except that the polynomial description relates now the differentials of inputs and outputs, and the resulting polynomial ring is non-commutative.

Key words: non-commutative rings, nonlinear discrete-time systems, twisted polynomials, transfer functions, composite systems.

1. INTRODUCTION

The transfer function plays an important role in the linear theory. In the linear case the transfer function $F(s)$ of a control system is usually expressed as the ratio of two polynomials in the Laplace operator s with real coefficients. Alternatively, in the polynomial systems theory [1], which has many features common with the classical transfer function technique, a ring formed from polynomials in the differentiation operator d/dt, interpreted as a linear mapping between signal spaces, has been used to define the transfer function. The latter approach has been extended to the nonlinear case to study problems like accessibility, irreducibility, and system reduction [2]. The resulting polynomial systems theory is in principle, similar to the linear case. The main differences are as follows:

1. the ring of polynomials is now a non-commutative ring of twisted polynomials, defined over the differential field of meromorphic functions in system variables,
2. the linear mapping is replaced by pseudo-linear mapping, and
3. the polynomial description relates the differentials of inputs and outputs and not just inputs and outputs themselves.

However, this is in agreement with universal algebraic formalism, which is based on the classification of differential one-forms associated with the nonlinear control system, and has proved to be efficient for solving several modelling, analysis, and design problems [3]. The approach is especially useful for checking solvability conditions, but to find the solutions, differential one-forms have to be integrated, which can be a difficult task. Of course, in this algebraic formalism, unlike in the linear case, all computations must be done symbolically, which in practice limits the complexity of the problems to be handled.

In papers [4–6], using the above polynomial approach, the transfer function has been defined for continuous-time nonlinear systems and its properties have been studied. The purpose of this paper is to show that the concept of the transfer function can be extended to discrete-time nonlinear control systems. In the discrete-time case the definition of the transfer function is based on a non-commutative twisted polynomial ring, which is a special case of the skew polynomial ring and can be embedded into its quotient field by the Ore condition. This skew polynomial ring has been used earlier to study accessibility [7], input-output equivalence [8], irreducibility, and reduction [9,10] for discrete-time nonlinear systems.

The paper is organized as follows. In Section 2 we review some of the standard facts of the linear algebraic framework and the twisted polynomial ring, associated with the discrete-time nonlinear control system. We also briefly sketch the construction of the quotient field of twisted polynomials. In Section 3 we introduce the notion of the transfer function of the discrete-time nonlinear system and prove some of its properties. We demonstrate on examples how to calculate the transfer functions both for single and composite systems. Finally, concluding remarks are given in Section 4.

2. TWISTED POLYNOMIAL RING AND ITS QUOTIENT FIELD

In dealing with nonlinear control system properties, we are, similar to [3], interested neither in local nor global, but in generic properties, i.e. in the properties that hold almost everywhere, except on a set of measure zero. That is, we look at ranks (or dimensions) over a field of functions, not over R. Hence, there is no argument either about the points where to evaluate dimension or about constant dimensionality of codistributions. Integrability of codistributions is often characterized by conditions which require that specific functions of system variables vanish. Since there are smooth functions that are neither generically zero nor generically different from zero, the notion of generic property does not make sense, in general, for systems defined by smooth functions. However, the situation is different if we restrict our attention to systems defined by means of analytic (or also meromorphic) functions, and this motivates or choice.

A linear algebraic framework for the analysis of nonlinear discrete-time systems was introduced in [11] and [12]. Consequently, the tools and methods of the algebraic approach were applied to a number of control problems (see, for example, [9,13,14]). In this paper we will follow the usual notations.

Consider the nonlinear discrete-time system described by a set of first-order difference equations of the form

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \\ y(t) &= g(x(t), u(t)), \end{aligned} \tag{1}$$

where the entries of f and g are meromorphic functions, which we think of as elements of the quotient field of the ring of analytic functions, and $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, and $y(t) \in \mathbf{R}^p$ denote, respectively, the state, the input, and the output of the system.

Since in our paper we apply the algebraic formalism, based on differential one-forms, associated with the control system, we emphasize that from this moment on, instead of studying the system description in terms of difference equations (**??**), it is sufficient to study the linearized system description in terms of differential one-forms

$$\begin{aligned} \mathrm{d}x(t+1) &= A\mathrm{d}x(t) + B\mathrm{d}u(t), \\ \mathrm{d}y(t) &= C\mathrm{d}x(t) + D\mathrm{d}u(t), \end{aligned} \tag{2}$$

which have been obtained by taking the total differentials of equations in (1) and where $A = \partial f/\partial x$, $B = \partial f/\partial u$, $C = \partial g/\partial x$, and $D = \partial g/\partial u$. The advantage of using one-forms lies in the fact that this allows extending the concept of the transfer function to the nonlinear case in a manner that largely resembles the linear case.

Let $\mathcal{K}$ denote the field of meromorphic functions of $\{x(0), u(t); t \geq 0\}$. The field $\mathcal{K}$ can be endowed with a difference structure determined by system (1). Hence, a forward-shift operator δ is defined as follows:

$$\delta\varphi(x(0), u(0), \ldots, u(N)) = \varphi(f(x(0), u(0)), u(1), \ldots, u(N+1)). \tag{3}$$

It is important for δ to be an automorphism on $\mathcal{K}$, in which case we think of $(\mathcal{K}, \delta)$ as a difference field. Therefore, system (1) has to be generically submersive, that is

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial f(\cdot)}{\partial(x(t), u(t))} = n.$$

The submersivity assumption is not restrictive. First, it is the necessary condition for accessibility, and second, in the case of shift-invariant systems it means that the backward shift operator can be applied a sufficient number of times. Under the submersivity assumption, there exist, up to an isomorphism, a unique difference field $(\mathcal{K}^*, \delta^*)$, called the inversive closure of $(\mathcal{K}, \delta)$. For an explicit construction of the latter, see [12]. Here we assume that the inversive closure is given and by abuse of notation we use the same notation $(\mathcal{K}, \delta)$ for both.

Define a vector space of one-forms spanned over $\mathcal{K}$ by differentials of elements of $\mathcal{K}$, namely $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi; \xi \in \mathcal{K}\}$. Any element in $\mathcal{E}$ is a vector of the form $v = \sum_i \alpha_i \mathrm{d}\xi_i$, where all $\alpha_i \in \mathcal{K}$.

The operator $\delta : \mathcal{K} \to \mathcal{K}$ induces a forward-shift operator $\delta : \mathcal{E} \to \mathcal{E}$ by $\delta v = \sum_i (\delta\alpha_i)(\delta \mathrm{d}\xi_i)$, where $v \in \mathcal{E}$.

Our aim is now to extend this algebraic point of view by introducing twisted polynomials which would act as shift operators on the vector space $\mathcal{E}$ and to end up with quotients of twisted polynomials which will enable us to define transfer functions for nonlinear discrete-time systems. We basically follow [6], but this time for discrete-time nonlinear systems.

The difference field $\mathcal{K}$ and the shift operator δ induce a polynomial ring $\mathcal{K}[\delta]$ with usual addition and non-commutative multiplication given by the commutation rule

$$\delta \cdot \varphi = \delta(\varphi) \cdot \delta. \tag{4}$$

If the multiplication is defined in the above way, the non-commutative ring $\mathcal{K}[\delta]$ is called the twisted polynomial ring [15], and is proved to satisfy the Ore condition, i.e. to be the Ore ring.

Remark 1. There exists an area of mathematics known as pseudo-linear algebra, alternatively called Ore algebra, which deals with common properties of linear differential and difference operators, see [16]. Basic objects of its study are pseudo-derivations, skew polynomials, and pseudo-linear operators. In pseudo-linear algebra the symbol δ conventionally denotes a pseudo-derivation and σ is an injective endomorphism, which is the case of the shift operator, defined by (3). To avoid confusion, we decided to keep the notation typical of papers dealing with discrete-time nonlinear systems; that is, the symbol δ represents the forward-shift operator.

2.1. Construction of the division ring of fractions

One common way of constructing fields is to take the field of fractions of an integral domain, a process exactly like that of constructing the field of rational numbers from the ring of integers. However, unlike in a commutative case, this construction does not work for every non-commutative integral domain. The non-commutative ring can be embedded into its quotient field (or field of fractions) if the so-called Ore condition is satisfied.

Lemma 1 (Left Ore condition). *For all nonzero $a, b \in \mathcal{K}[\delta]$ there exist nonzero $a_1, b_1 \in \mathcal{K}[\delta]$ such that $a_1 b = b_1 a$, that is, a and b have a common left multiple.*

The ring $\mathcal{K}[\delta]$ can, therefore, be embedded into a non-commutative quotient field [17,18] by defining quotients as

$$\frac{a}{b} = b^{-1} \cdot a, \tag{5}$$

where $a, b \in \mathcal{K}[\delta]$ and $b \neq 0$. Addition is defined by reducing two quotients to the same denominator

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{\beta_2 a_1 + \beta_1 a_2}{\beta_2 b_1}, \tag{6}$$

where $\beta_2 b_1 = \beta_1 b_2$ by the Ore condition. Multiplication is defined by

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{\alpha_1 a_2}{\beta_2 b_1}, \tag{7}$$

where $\beta_2 a_1 = \alpha_1 b_2$ again by the Ore condition. The resulting quotient field of twisted polynomials is denoted by $\mathcal{K}\langle\delta\rangle$.

3. TRANSFER FUNCTION OF THE NONLINEAR DISCRETE-TIME SYSTEM

In Section 2 we associated with system of equations (1) its linearized system of equations (2) in terms of one-forms. The same can be done with a higher-order input-output difference equation,

$$\varphi(y(t+n), \ldots, y(t), u(t+s), \ldots, u(t)) = 0, \tag{8}$$

obtained by eliminating the states from Eq. (1). By taking the total differential of (8) we get

$$a_n \mathrm{d}y(t+n) + \cdots + a_0 \mathrm{d}y(t) = b_s \mathrm{d}u(t+s) + \cdots + b_0 \mathrm{d}u(t)$$

or, alternatively,

$$(a_n \delta^n + \cdots + a_0)\mathrm{d}y(t) = (b_s \delta^s + \cdots + b_0)\mathrm{d}u(t), \tag{9}$$

where $\varphi \in \mathcal{K}$, $a_i = \partial\varphi/\partial y(t+i)$, $i = 0, \ldots, n$ and $b_j = -\partial\varphi/\partial u(t+j)$, $j = 0, \ldots, s$.

Once we have defined the fraction of two polynomials, the transfer function can be introduced.

Definition 1. *An element* $F(\delta) \in \mathcal{K}\langle\delta\rangle$ *such that* $\mathrm{d}y(t) = F(\delta)\mathrm{d}u(t)$ *is said to be a transfer function of discrete-time nonlinear system* (**??**) *or* (8), *respectively.*

Now it follows from (9) that the transfer function of (8) (or (**??**)) is as follows:

$$F(\delta) = \frac{b_s\delta^s + \cdots + b_0}{a_n\delta^n + \cdots + a_0}. \tag{10}$$

Remark 2. In the case of the multi-input multi-output system, we think of $F(\delta)$ as a matrix with the entries in $\mathcal{K}\langle\delta\rangle$, and $F(\delta)$ is then referred to as a transfer matrix.

In the linear time-invariant case, one can associate to each proper rational function an input-output difference equation of a control system. However, things are different in the nonlinear case. Though we can always associate with a proper rational function $F(\delta) = a^{-1}(\delta) \cdot b(\delta)$ a corresponding input-output differential form, $\omega = a(\delta)\mathrm{d}y(t) - b(\delta)\mathrm{d}u(t)$, this one-form is not necessarily integrable. If the input-output differential form is integrable, or can be made integrable by multiplying an integrating factor, then there exists an input-output difference equation of the form (8) such that the transfer function of this input-output equation is $F(\delta)$. In other words, not every quotient of skew polynomials necessarily represents a control system.

However, since the transfer function can be found from the system equations, it directly implies that in such a case the condition of integrability is satisfied.

The transfer function can be alternatively calculated from the linearized state equations (2). Rewrite first equation in (2) as

$$(\delta I - A)\mathrm{d}x(t) \quad = \quad B\mathrm{d}u(t). \tag{11}$$

Now, it follows from (2) that

$$F(\delta) = C(\delta I - A)^{-1}B + D. \tag{12}$$

In spite of the formal similarity to transfer functions of linear discrete-time systems, inverting matrix $(\delta I - A)$ over the non-commutative field is now far from trivial, since entries of $(\delta I - A)$ are twisted polynomials. Inversion requires finding the solutions of a set of linear equations over the non-commutative field (see [[17]]). To compute the transfer function (12), one has to find the left-hand inverse of $(\delta I - A)$. One possibility is to use the classical Gauss–Jordan elimination algorithm, using the definitions of addition and non-commutative multiplication, given, respectively, by (6) and (7).

We now study some illustrative examples.

Example 1. Consider the system described by state equations

$$\begin{aligned}
x_1(t+1) &= x_2(t) + u^2(t),\\
x_2(t+1) &= u(t),\\
y(t) &= x_1(t).
\end{aligned}$$

The matrices A, B, and C in Eqs (2) are as follows:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 2u(t) \\ 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

and the transfer function, computed according to (12), is

$$\begin{aligned} F(\delta) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\delta} & \frac{1}{\delta^2} \\ 0 & \frac{1}{\delta} \end{pmatrix} \begin{pmatrix} 2u(t) \\ 1 \end{pmatrix} \\ &= \frac{2u(t)}{\delta} + \frac{1}{\delta^2} = \frac{\delta 2u(t) + 1}{\delta^2} = \frac{2u(t+1)\delta + 1}{\delta^2}. \end{aligned}$$

Example 2. Consider the system described by state equations

$$\begin{aligned} x_1(t+1) &= u(t), \\ x_2(t+1) &= x_3(t), \\ x_3(t+1) &= x_1(t) + u(t)x_2(t), \\ y_1(t) &= x_1(t), \\ y_2(t) &= x_3(t). \end{aligned}$$

Then

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & u(t) & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 0 \\ x_2(t) \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the left-hand inverse of $(\delta I - A)$ is

$$(\delta I - A)^{-1} = \begin{pmatrix} \frac{1}{\delta} & 0 & 0 \\ \frac{1}{\delta(\delta^2 - u(t))} & \frac{\delta}{\delta^2 - u(t)} & \frac{1}{\delta^2 - u(t)} \\ \frac{1}{\delta^2 - u(t+1)} & \frac{u(t+1)}{\delta^2 - u(t+1)} & \frac{\delta}{\delta^2 - u(t+1)} \end{pmatrix}.$$

Since now we have the system with two outputs, we obtain the transfer matrix

$$F(\delta) = C(\delta I - A)^{-1}B = \begin{pmatrix} \frac{1}{\delta} \\ \frac{y_2(t)\delta + 1}{\delta^2 - u(t+1)} \end{pmatrix}. \tag{13}$$

However, it is much more easier to find the transfer function from the input-output equations

$$\begin{aligned} y_1(t+1) &= u(t), \\ y_2(t+1) &= y_2(t)u(t+1) + u(t) \end{aligned} \tag{14}$$

like in (10). By taking the total differential of (14), we obtain

$$\begin{aligned} \mathrm{d}y_1(t+1) &= \mathrm{d}u(t), \\ \mathrm{d}y_2(t+1) &= u(t+1)\mathrm{d}y_2(t) + y_2(t)\mathrm{d}u(t+1) + \mathrm{d}u(t) \end{aligned}$$

that yield immediately (see (9))

$$\begin{aligned} \mathrm{d}y_1(t) &= \frac{1}{\delta}\mathrm{d}u(t), \\ \mathrm{d}y_2(t) &= \frac{y_2(t)\delta + 1}{\delta^2 - u(t+1)}\mathrm{d}u(t). \end{aligned} \tag{15}$$

3.1. Properties of the transfer function

The transfer function of the nonlinear discrete-time system satisfies many of the properties we expect from the transfer function according to our linear intuition.

First, each nonlinear discrete-time system (1) has a unique transfer function, no matter what state-space realization is used. To show this, we have, in fact, to prove the following proposition.

Proposition 1. *Transfer function* (12) *of nonlinear discrete-time system* (1) *is invariant with respect to the state transformation* $\xi(t) = \phi(x(t))$.

Proof. For any state transformation $\xi(t) = \phi(x(t))$ one has $\mathrm{rank}_{\mathcal{K}} T = n$, where $T = (\partial\phi/\partial x(t))$. Since $\mathrm{d}\xi(t) = T\mathrm{d}x(t)$, in the new coordinates we have

$$\begin{aligned} \mathrm{d}\xi(t+1) &= \delta(T)AT^{-1}\mathrm{d}\xi(t) + \delta(T)B\mathrm{d}u(t), \\ \mathrm{d}y(t) &= CT^{-1}\mathrm{d}\xi(t) + D\mathrm{d}u(t), \end{aligned} \tag{16}$$

where $\delta(T)$ means δ applied pointwise to T. Thus, the transfer function reads as

$$\begin{aligned} F(\delta) &= CT^{-1}(\delta I - \delta(T)AT^{-1})^{-1}\delta(T)B + D \\ &= C(\delta(T^{-1})\delta \cdot T - A)^{-1}B + D. \end{aligned}$$

After applying the commutation rule $\delta \cdot T = \delta(T) \cdot \delta$ we get $F(\delta) = C(\delta I - A)^{-1}B + D$, which completes the proof. □

Example 3. Consider the system described by state equations

$$\begin{aligned} x_1(t+1) &= u(t), \\ x_2(t+1) &= x_1(t)u(t), \\ y(t) &= \frac{x_2(t)}{x_1(t)}. \end{aligned} \tag{17}$$

From (17) we get

$$A = \begin{pmatrix} 0 & 0 \\ u(t) & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ x_1(t) \end{pmatrix}, \; C = \begin{pmatrix} -\frac{x_2(t)}{x_1^2(t)} & \frac{1}{x_1(t)} \end{pmatrix}.$$

The left-hand inverse of $(\delta I - A)$ is

$$(\delta I - A)^{-1} = \begin{pmatrix} \frac{1}{\delta} & 0 \\ \frac{u(t+1)}{\delta^2} & \frac{1}{\delta} \end{pmatrix}$$

and the transfer function

$$\begin{aligned} F(\delta) &= C(\delta I - A)^{-1}B = -\frac{x_2(t)}{x_1^2(t)}\frac{1}{\delta} + \frac{1}{x_1(t)}\left(\frac{u(t+1)}{\delta^2} + \frac{x_1(t)}{\delta}\right) \\ &= -\frac{x_1(t)}{u(t)\delta} + \frac{1}{x_1(t)}\frac{u(t+1)+u(t)\delta}{\delta^2} = \frac{1}{\delta^2}. \end{aligned}$$

This result is not surprising if we notice that the input-output map of system (17) is linear, $y(t+2) = u(t)$, and the state equations can be linearized by the state transformation $\xi_1(t) = x_2(t)/x_1(t)$, $\xi_2(t) = x_1(t)$.

Proposition 2. *Suppose that the observable space $\mathcal{O}_\infty$ of nonlinear discrete-time system* (1) *is integrable. Then the transfer function describes only the accessible and observable subsystem of the state equations* (1).

Proof. Due to Proposition 1, the proof is quite straightforward. We begin by proving that the transfer function describes only the accessible subsystem.

For any nonlinear discrete-time system (1) there exists the state transformation $\xi(t) = \phi(x(t))$ with respect to which (12) is invariant and which yields a controllability canonical form given by

$$\begin{aligned} \xi_1(t+1) &= f_1(\xi_1(t)), \\ \xi_2(t+1) &= f_2(\xi_1(t), \xi_2(t), u(t)), \\ y(t) &= g(\xi_1(t), \xi_2(t), u(t)), \end{aligned} \tag{18}$$

where the components of the vector $\xi_1(t)$ describe the so-called autonomous elements. From above form $(\delta I - A)^{-1}B$ in (12) describes only the accessible subsystem. Really, in case of form (18),

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix},$$

where $A_{11} = (\partial f_1/\partial \xi_1)$, $A_{21} = (\partial f_2/\partial \xi_1)$, $A_{22} = (\partial f_2/\partial \xi_2)$, and $B_2 = (\partial f_2/\partial u)$. Then

$$(\delta I - A)^{-1} = \begin{pmatrix} (\delta I - A_{11})^{-1} & 0 \\ \dots & (\delta I - A_{22})^{-1} \end{pmatrix}$$

and thus $(\delta I - A)^{-1}B = (\delta I - A_{22})^{-1}B_2$ describes only the accessible subsystem.

The same applies also to the observable subsystem. The observability canonical form is given by

$$\begin{aligned} \xi_1(t+1) &= f_1(\xi_1(t), u(t)), \\ \xi_2(t+1) &= f_2(\xi_1(t), \xi_2(t), u(t)), \\ y(t) &= g(\xi_1(t), u(t)), \end{aligned}$$

where the vector $\xi_2(t)$ describes the unobservable states. This time, expression $C(\delta I - A)^{-1}$ in (12) describes only the observable subsystem. Really,

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \; C = \begin{pmatrix} C_1 & 0 \end{pmatrix},$$

where $A_{11} = (\partial f_1/\partial \xi_1)$, $A_{21} = (\partial f_2/\partial \xi_1)$, $A_{22} = (\partial f_2/\partial \xi_2)$, and $C_1 = (\partial g/\partial \xi_1)$. Then

$$(\delta I - A)^{-1} = \begin{pmatrix} (\delta I - A_{11})^{-1} & 0 \\ \dots & (\delta I - A_{22})^{-1} \end{pmatrix}$$

and thus $C(\delta I - A)^{-1} = C_1(\delta I - A_{11})^{-1}$ describes only the observable subsystem. □

Remark 3. Proposition 3 differs from its continuous-time counterpart [6], since in the discrete-time case the observable space cannot always be locally spanned by exact one-forms whose integrals would define the observable state coordinates (see [13]). In this case, of course, one cannot talk about the observable subsystem. In case of accessibility, we do not have this problem, since non-accessible subspace $\mathcal{H}_\infty$ is, like in the continuous-time case, always integrable.

Example 4. Consider again the system described in Example 2, but now only with its second output:

$$\begin{aligned} x_1(t+1) &= u(t), \\ x_2(t+1) &= x_3(t), \\ x_3(t+1) &= x_1(t) + u(t)x_2(t), \\ y(t) &= x_3(t). \end{aligned}$$

The transfer function of this system, computed from the input-output equation, is

$$F(\delta) = \frac{y(t)\delta + 1}{\delta^2 - u(t+1)}.$$

Since the denominator $\delta^2 - u(t+1)$ is a polynomial of degree 2 and the system is of order 3, the system obviously is either not accessible or not observable.

In fact, this system is not observable. One can compute observability filtration (see [13] for details) as $\mathcal{O}_0 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_3(t)\}$ and $\mathcal{O}_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_3(t), \mathrm{d}x_1(t) + u(t)\mathrm{d}x_2(t)\} = \mathcal{O}_\infty$. Since $\mathcal{O}_\infty$ is not integrable, we cannot find an observability canonical form, and consequently, the system cannot be decomposed into observable and unobservable subsystems.

Finally, we can also introduce for nonlinear systems an algebra of transfer functions. Each system structure can be divided into three basic types of connections: series, parallel, and feedback (see Fig. 1). For a series connection it follows that $\mathrm{d}y_B(t) = F_B(\delta)\mathrm{d}u_B(t) = F_B(\delta)F_A(\delta)\mathrm{d}u_A(t)$. Thus

$$F(\delta) = F_B(\delta)F_A(\delta).$$

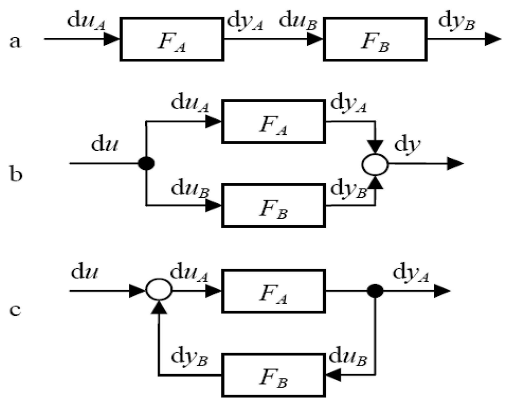

Fig. 1. Series (a), parallel (b), and feedback (c) connections of systems.

For parallel and feedback connection we get

$$F(\delta) = F_A(\delta) + F_B(\delta)$$

and

$$F(\delta) = (1 - F_A(\delta)F_B(\delta))^{-1} \cdot F_A(\delta),$$

respectively.

The following example demonstrates how to handle a series connection of two nonlinear systems.

Example 5. Consider two nonlinear discrete-time systems

$$y_A(t+1) + y_A^2(t) = u_A(t), \qquad y_B(t+2) = u_B(t+1) + u_B^2(t).$$

Transfer functions of these two systems are as follows:

$$F_A(\delta) = \frac{1}{\delta + 2y_A(t)}, \qquad F_B(\delta) = \frac{\delta + 2u_B(t)}{\delta^2}.$$

The systems are now combined together in a series connection. For the connection $A \to B$ the resulting transfer function is according to (7)

$$F(\delta) = F_B(\delta)F_A(\delta) = \frac{\delta + 2u_B(t)}{\delta^2} \cdot \frac{1}{\delta + 2y_A(t)} = \frac{1}{\delta^2}.$$

Therefore, the input-output map of the composite system is linear. However, when the systems are connected as $B \to A$, the resulting transfer function is

$$F(\delta) = F_A(\delta)F_B(\delta) = \frac{1}{\delta + 2y_A(t)} \cdot \frac{\delta + 2u_B(t)}{\delta^2} = \frac{\delta + 2u_B(t)}{\delta^2(\delta + 2y_A(t))},$$

which obviously does not result from the linear input-output map.

4. CONCLUSIONS

In this paper the notion of the transfer function of the discrete-time nonlinear control system was defined and some of its properties were proved. The resulting theory is, in principle, similar to that of the linear theory, except that the polynomials defining the transfer function belong to a non-commutative polynomial ring and the transfer function defines the relationship between the differentials of inputs and outputs. Transfer functions are thus more difficult to handle. We do hope, however, that the transfer function of the nonlinear control system introduces a new alternative algebraic framework for the modelling, analysis, and feedback design of nonlinear control systems.

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REFERENCES

1. Blomberg, H. and Ylinen, R. *Algebraic Theory for Multivariable Linear Systems*. Academic Press, London, 1983.
2. Zheng, Y., Willems, J. and Zhang, C. A polynomial approach to nonlinear system controllability. *IEEE Trans. Automat. Control*, 2001, **46**, 1782–1788.
3. Conte, G., Moog, C. H. and Perdon, A. M. *Nonlinear Control Systems: An Algebraic Setting*. Springer-Verlag, London, 1999.
4. Zheng, Y. and Cao, L. Transfer function description for nonlinear systems. *J. East China Normal Univ. (Nat. Sci.)*, 1995, **2**, 15–26.
5. Halás, M. An algebraic framework generalizing the concept of transfer functions to nonlinear systems. *Automatica* (accepted).
6. Halás, M. and Huba, M. Symbolic computation for nonlinear systems using quotients over skew polynomial ring. In *14th Mediterranean Conference on Control and Automation, Ancona, Italy*. 2006.
7. Zhang, C. and Zheng, Y. A polynomial approach to discrete-time nonlinear system controllability. *Int. J. Control*, 2004, **77**, 491–497.
8. Bartosiewicz, Z., Kotta, Ü., Nõmm, S. and Pawluszewicz, E. Input-output equivalence transformations for discrete-time nonlinear systems. In *Proceedings of the 2nd IFAC Symposium on Systems Structure and Control, Oaxaca, Mexico*. 2004, 706–711.
9. Kotta, Ü. Irreducibility conditions for nonlinear input-output difference equations. In *Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia*. 2000, 3404–3408.
10. Kotta, Ü. and Tõnso, M. Irreducibility conditions for discrete-time nonlinear multi-input multi-output systems. In *Proceedings of the 6th IFAC Symposium on Nonlinear Control (NOLCOS), Stuttgart, Germany*. 2004, 269–273.
11. Grizzle, J. W. A linear algebraic framework for the analysis of discrete-time nonlinear systems. *SIAM J. Control Optim.*, 1993, **31**, 1026–1044.
12. Aranda-Bricaire, E., Kotta, Ü. and Moog, C. H. Linearization of discrete-time systems. *SIAM J. Control Optim.*, 1996, **34**, 1999–2023.
13. Kotta, Ü. Decomposition of discrete-time nonlinear control systems. *Proc. Estonian Acad. Sci. Phys. Math.*, 2005, **54**, 154–161.
14. Kotta, Ü., Zinober, A. S. I. and Liu, P. Transfer equivalence and realization of nonlinear higher order input-output difference equations. *Automatica*, 2001, **37**, 1771–1778.
15. Farb, B. and Dennis, R. *Noncommutative Algebra*. Springer-Verlag, New York, 1993.
16. Bronstein, M. and Petkovšek, M. An introduction to pseudo-linear algebra. *Theor. Computer Sci.*, 1996, **157**, 3–33.
17. Ore, O. Linear equations in non-commutative fields. *Annals Math.*, 1931, **32**, 463–477.
18. Ore, O. Theory of non-commutative polynomials. *Annals Math.*, 1933, **34**, 480–508.

Diskreetsete mittelineaarsete juhtimissüsteemide ülekandefunktsioonid

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On defineeritud diskreetse mittelineaarse juhtimissüsteemi ülekandefunktsiooni mõiste. Definitsioon põhineb teatud mittekommutatiivsete polünoomide ringil, mida on võimalik Ore tingimuse täidetuse tõttu laiendada jagatiste ringiks. On uuritud ülekandefunktsiooni omadusi, mis on seotud süsteemi juhitavuse ja vaadeldavusega, ning näidatud, kuidas leida komposiitsüsteemide (järjestik- ja paralleelühendus ning tagasisidestatud süsteem) ülekandefunktsiooni. Esitatud tulemused on põhilises sarnased vastava lineaarse teooriaga, v.a see, et nüüd ülekandefunktsiooniga määratud polünoomid seovad sisendite ja väljundite diferentsiaale, mitte muutujaid endid, ning et vastav polünoomide ring on mittekommutatiivne.