Chapter 2

Particle Motions in a Penning Trap

A single charged particle in a Penning trap is a bound system that has many analogies to the hydrogen atom except that the atomic nucleus has been replaced by an external electrostatic quadrupole potential superimposed on a spatially uniform, stable magnetic field.

In this chapter, the essential elements of particle motions in the Penning trap are summarized. Detailed accounts are reviewed elsewhere [13].

2.1 Perfect Trap

If a particle of charge e and mass m is placed in a uniform magnetic field $\mathbf{B} = B_0\hat{z}$, the particle travels around the field line in a cyclotron orbit, with the free space cyclotron frequency

$$\boldsymbol{\omega}_{\mathbf{c}} = \frac{|eB|}{mc}\hat{z} = \omega_c\hat{z}. \tag{2.1}$$

The motion of the particle is bound in a Penning trap by superimposing an electric quadrupole potential

$$V(\mathbf{r}) = \frac{V_0}{2}\frac{z^2 - \rho^2/2}{d^2}. \tag{2.2}$$

The variables z and ρ are cylindrical coordinates and d is a characteristic trap dimension.

The quadrupole potential has traditionally been produced by placing electrodes along equipotentials of V(**r**). Two 'endcaps' follow the hyperbola of revolution

$$z = \pm\sqrt{z_0^2 + \rho^2/2}, \tag{2.3}$$

and one 'ring' electrode is along the hyperbola of revolution

$$z^2 = \frac{1}{2}(\rho^2 - \rho_0^2). \tag{2.4}$$

The characteristic trap dimension is defined by

$$d^2 \equiv \frac{1}{2}(z_0^2 + \frac{\rho_0^2}{2}) \tag{2.5}$$

in terms of the minimum axial and radial distances to the trap electrodes, z_0 and ρ_0.

The equations of motion result from the Lorentz force on the charged particle

$$\mathbf{F} = -e\boldsymbol{\nabla}V + \frac{e}{c}\mathbf{v} \times \mathbf{B}. \tag{2.6}$$

The axial motion along $\hat{z}$ decouples since $v_z\hat{\mathbf{z}} \times \mathbf{B} = 0$. The resulting equation of motion is that of a simple harmonic oscillator

$$\ddot{z} + \omega_z^2 z = 0, \tag{2.7}$$

with angular [1] axial frequency

$$\omega_z^2 = \frac{eV_0}{md^2}. \tag{2.8}$$

The radial equation of motion is

$$m\ddot{\boldsymbol{\rho}} = e[\mathbf{E}_\rho + (\frac{\dot{\boldsymbol{\rho}}}{c}) \times \mathbf{B}] \tag{2.9}$$

where E_ρ is the radial component of the quadrupole electric field which we can express in terms of the axial frequency

$$E_\rho = -\frac{\partial V}{\partial \rho} = \frac{V_0}{2d^2}\rho = \frac{1}{2}\frac{m}{e}\omega_z^2\rho. \tag{2.10}$$

[1]Throughout this thesis, either the frequency ν or the equivalent angular frequency $\omega = 2\pi\nu$ is used depending upon which is more convenient in the immediate context.

Thus the radial equation of motion in terms of the free space cyclotron and axial frequencies ω_c and ω_z, in a Penning trap is

$$\ddot{\boldsymbol{\rho}} - \boldsymbol{\omega_c} \times \dot{\boldsymbol{\rho}} - \frac{1}{2}\omega_z^2 \boldsymbol{\rho} = 0. \tag{2.11}$$

When $\omega_z \to 0$ (i.e. the voltage $V_0 \to 0$), we recover the equation of motion describing uniform circular motion at the free space cyclotron frequency ω_c.

Solving Eq. 2.11 yields two eigenfrequencies given by

$$\omega_c' = \omega_c - \frac{\omega_z^2}{2\omega_c'} = \omega_c - \omega_m \tag{2.12}$$

and

$$\omega_m = \frac{\omega_z^2}{2\omega_c'}. \tag{2.13}$$

The frequency ω_c' is the modified cyclotron frequency, and it reflects the deviation from the free space cyclotron frequency ω_c resulting from the presence of the electric quadrupole field. The magnetron frequency ω_m describes the slow circular motion that results from a balance between the radially inward motional electric field and the radially outward electric field. The magnetron motion is unstable in that removing energy from it increases the orbit size. Fortunately, the radiation damping is usually so small that the motion is stable for many years.

The condition for which a charged particle will be bound in the Penning trap is

$$\omega_z \leq \frac{\omega_c}{\sqrt{2}}. \tag{2.14}$$

which requires that the inward motional field be larger than the outward motional field. For typical trap sizes and field strengths

$$\omega_m \ll \omega_z \ll \omega_c'. \tag{2.15}$$

describes the hierarchy in the trap eigenfrequencies.

In a perfect Penning trap the free space cyclotron frequency is given exactly by Eq. 2.12 and 2.13 so that a measurement of ν_c' and ν_z is sufficient to measure ν_c . A comparison of charge to inertial mass ratios can be made in a Penning trap by measuring the free space cyclotron frequency of two different particles.

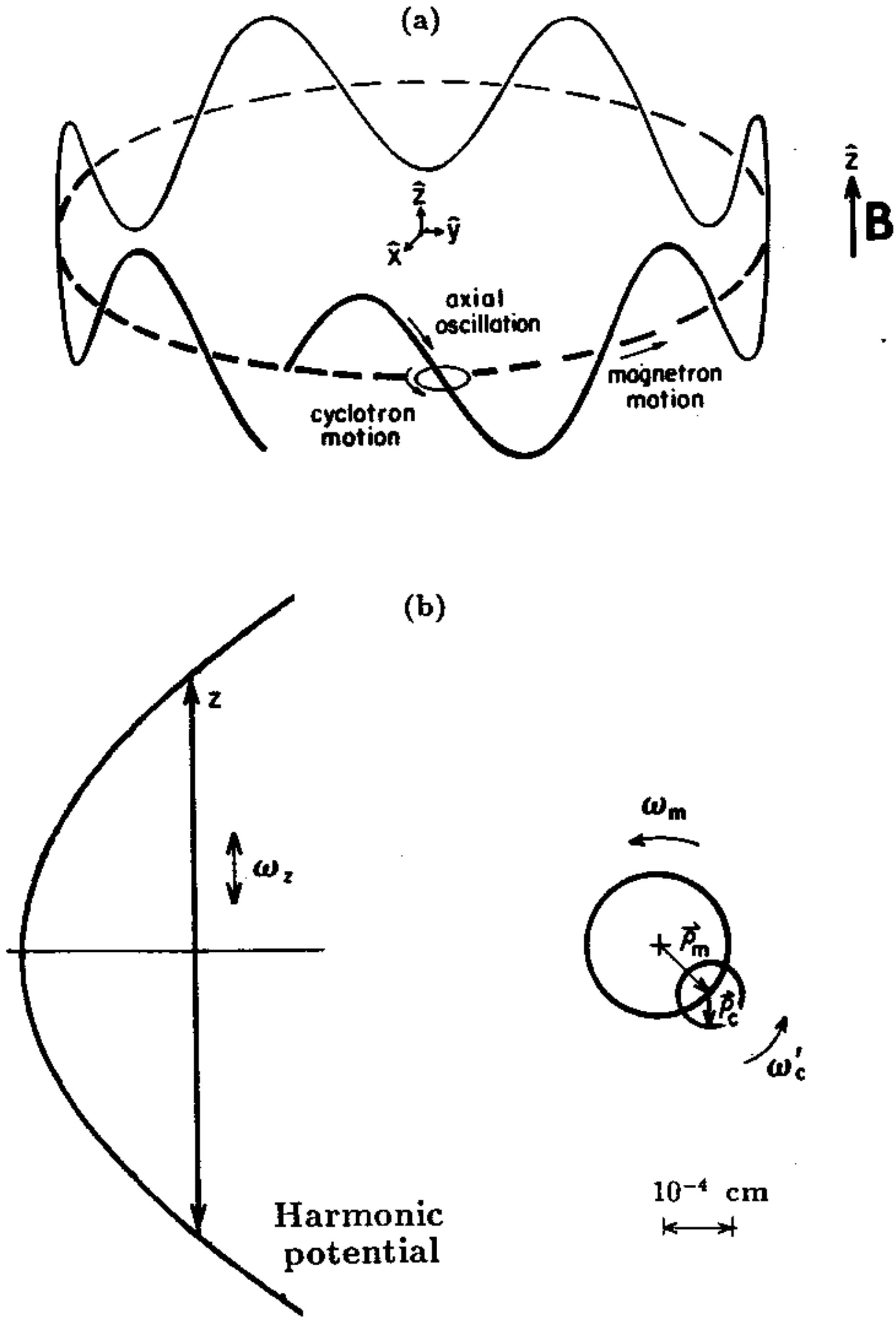

Figure 2.1: (a) Schematic representation of the orbit of a charged particle confined in a Penning trap. (b) A scaled representation of the three oscillatory motions for a confined antiproton ($T_z = T_c = 4$K, $T_m = (\omega_m/\omega_c)4$ K).

For example, the ratio of the cyclotron frequencies for a proton with mass m_p and charge q_p, to that of an antiproton with mass $m_{\bar{p}}$ and $q_{\bar{p}}$ is

$$\frac{\nu_c(p)}{\nu_c(\bar{p})} = \frac{q_p B/m_p c}{q_{\bar{p}} B/m_{\bar{p}} c} = \frac{q_p}{m_p}\frac{m_{\bar{p}}}{q_{\bar{p}}} \tag{2.16}$$

This last equality requires the magnetic field to be the same for both species over the duration of the comparison, a topic we return to in Chapter 10.

A comparison of cyclotron frequencies is a often referred to as a comparison of the inertial masses. For matter ions, reference to a cyclotron frequency being a mass comparison is justified by charge quantization and the stringent experimental limits put on charge neutrality between the proton and electron of $(q_{p^+} + q_{e^-}) < 10^{-19}e$ [62]. Strictly speaking, the measurements reported in this thesis are a comparison of the charge to mass ratio of the antiproton and the proton.

2.2 Imperfect Traps: The Invariance Theorem

Real traps have physical imperfections. For example, the trap electrodes can be slightly distorted and the quadrupole electrostatic field is not perfectly aligned along the uniform magnetic field.

Imperfections in the quadrupole potential field introduce anharmonic terms to the axial equation of motion expressed in Eq. 2.7. For this reason, the three electrode Penning trap is often modified by adding two 'compensation' electrodes [95,30]. The purpose of such electrodes is to tune out possible higher order anharmonic contributions to the potential. Nevertheless, the degree to which a perfect quadrupole can be produced is limited by the asymmetries and misalignment which exist.

The degree to which leading imperfections affect the achievable precision in a cyclotron frequency measurement has been discussed by Brown and Gabrielse [8,13]. Let $\overline{\omega}'_c$, $\overline{\omega}_z$, and $\overline{\omega}_m$ be the measured cyclotron, axial, and magnetron frequencies in a non-ideal trap with an asymmetry parameter ϵ representing deviations from the ideal quadrupole potential by

$$U = \frac{1}{2}m\omega_z^2[z^2 - \frac{1}{2}(x^2+y^2) - \frac{1}{2}\epsilon(x^2-y^2)], \tag{2.17}$$

and angles (θ, ϕ) representing a misaligned magnetic field given by

$$\mathbf{B} = B(\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \sin\theta\sin\phi\hat{k}). \tag{2.18}$$

Brown and Gabrielse [8,13] have derived an invariance theorem

$$\omega_c^2 = \overline{\omega}_c'^2 + \overline{\omega}_z^2 + \overline{\omega}_m^2. \tag{2.19}$$

This theorem provides a means of obtaining the free space cyclotron frequency (the one of interest for mass comparisons) from the measured trap eigenfrequencies.

The measured eigenfrequencies $\overline{\omega}_c'$, $\overline{\omega}_z$, and $\overline{\omega}_m$ can be expanded in terms of the distortion parameter ϵ and the tilting angles (θ, ϕ) by the following expressions:

$$\overline{\omega}_z^2 \approx \omega_z^2[1 - \frac{3}{2}\sin^2\theta(1 + \frac{1}{3}\epsilon\cos 2\phi)] \tag{2.20}$$

$$\overline{\omega}_m \approx \frac{\overline{\omega}_z^2}{2\overline{\omega}_c'}(1-\epsilon)^{1/2}[1 - \frac{3}{2}\sin^2\theta(1 + \frac{1}{3}\epsilon\cos 2\phi]^{-3/2} \tag{2.21}$$

Eliminating $\overline{\omega}_m$ from the invariance theorem, we obtain an expression for the free space cyclotron frequency in terms of the asymmetry parameter ϵ and tilt angle θ as

$$\frac{\omega_c}{\overline{\omega}_c'} = 1 + \frac{1}{2}\left(\frac{\overline{\omega}_z}{\overline{\omega}_c'}\right)^2 + \frac{9}{16}\left(\frac{\overline{\omega}_z}{\overline{\omega}_c'}\right)^4(\theta^2 - \frac{2}{9}\epsilon^2) + \dots \ . \tag{2.22}$$

From Eq. 2.15, terms proportional to $(\omega_z/\omega_c)^6$ and higher order terms are small and can be neglected. For vanishing θ and ϵ, we recover Eq. (2.12) for an ideal trap.

As an example, assuming that θ =1° or $|\epsilon| = 1\%$, the correction term for electrons in a typical trap in $(\overline{\nu}_z(e^-)/\overline{\nu}_c'(e^-))^4 \approx 10^{-14}$. Using protons, such imperfections can become significant. For the small trap (d=0.112 cm) used by VanDyck and Schwinberg [98] to measure the proton to electron mass, $(\overline{\nu}_z(p)/\overline{\nu}_c'(p))^4 \approx 3\times10^{-4}$ and the correction term in Eq. 2.22 is about 5 $\times10^{-8}$. Therefore trap imperfections and misalignments can play an important role when measuring protons or antiprotons. However, our use of a much larger trap (Chapter 3) makes the third term negligible at the precison reported in this thesis.

2.3 Extension to Many Particles

The equations in Sections 2.1 and 2.2 are derived only for the case of a single particle. For the comparisons reported here, we have always used more than one.

Wineland and Dehmelt generalized the equations of motion to the case of n harmonically bound particles of a single species [115]. They model the particles each of mass m and bound with a spring constant k, and driven by the potential between capacitor plates separated by a distance d. The equation of motion including mutual interactions (for simplicity assumed to be in the z direction only) is for the i^{th} particle

$$m\ddot{z}_i = -kz_i + \sum_{j\neq i}^{n} F_{ij} + F'_{ei}. \tag{2.23}$$

The term F_{ij} is the Coulomb force on the i^{th} particle due to the j^th particle. F'_{ei} is the external force on the i^{th} particle given by

$$F_{ei} = \frac{eV}{d} + \sum_{j}^{n} F_{ind}(i,j). \tag{2.24}$$

F'_{ind}(i,j) is the force on the i^{th} particle from the induced image charge in the capacitor plates (in actuality the trap electrodes) of the j^{th} particle. The spring constant k in Eq. 2.23 is a function of $F_{ind}(i,j)$, thus there is an n dependence.

For large particle numbers, such a force will yield a number dependent shift in the oscillation frequency (ref. [115] and Chapter 4). Assuming the numbers are kept small or the trap is sufficiently large, $F'_{ind} \approx 0$ we sum Eq. (2.23) over n particles giving

$$m\ddot{Z} = -kZ + \frac{eV}{d} + \sum_{i}^{n}\sum_{i\neq j}^{n} F_{ij} \tag{2.25}$$

where $Z = \sum_j^n Z_j/n$ is the center of mass coordinate for the n particle system. Newton's third law implies $F_{ij} = -F_{ji}$ so the sum vanishes. In the small number approximation, (2.25) becomes

$$m\ddot{Z} + -kZ = \frac{eV}{d}. \tag{2.26}$$

This equation for n particles also describes the motion of a harmonic oscillator with a center of mass oscillation at the frequency of a single particle of mass m

and charge q. Analogously, the cyclotron and magnetron center of mass motions are also similar to those of a single particle.