XY model in 2D and 3D

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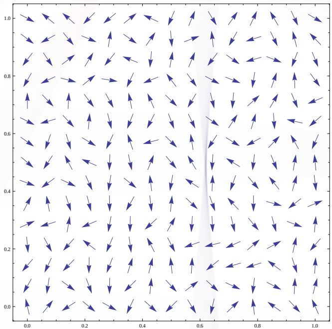
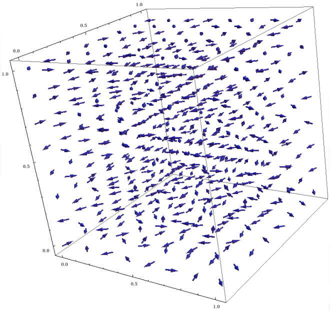
Part I

The XY model, duality and loop expansion

Why the XY model?

- An important model used, for example, in superconductivity.
- It presents a particular phase transition involving topological excitations (vortices).
- It is dual to other interesting models (Coulomb gas model, sos model…).





The XY model in d dimensions

Towards a dual model

The Hamiltonian of the XY model in d dimensions is given by

$$\beta\mathcal{H} = -\beta \sum_{\mu,i} \cos(\nabla_\mu \theta_i) \Rightarrow Z = \int \delta\theta \exp\left[\beta \sum_{\mu;i} \cos(\nabla_\mu \theta_i)\right], \quad \theta_i \in (-\pi, \pi],$$

where $\nabla_\mu \theta_i = \theta_i - \theta_{i-\hat{\mu}}$ is the discrete derivative in the μ direction. It is **invariant** under the transformation $\nabla_\mu \theta_i \mapsto \nabla_\mu \theta_i + 2\pi n_{\mu,i}$, $n_{\mu,i} \in \mathbb{Z}$. We can expand the hamiltonian using the identity

$$\mathrm{e}^{a \cos b} = \sum_{n=-\infty}^{+\infty} I_n(a)\, \mathrm{e}^{inb}, \quad I_n(x) \text{ modified Bessel function.}$$

Then for $k_{\mu;i} \in \mathbb{Z}$, substituting and integrating in θ,

$$Z = \int \delta\theta \prod_{\mu,i} \mathrm{e}^{\beta \cos(\nabla_\mu \theta_i)} = \boxed{\sum_{\{k_{\mu;i}\}} \mathrm{e}^{\sum_{\mu;i} \ln I_{k_{\mu;i}}(\beta)} \underbrace{\delta_{\boldsymbol{\nabla}\cdot\mathbf{k}_i,0}}_{\text{null divergence}}} \quad \mathbf{k}_i = (k_{\mu;i})_{\mu=1,\dots d}$$

XY model in 2D and 3D

Using Einstein's notation $\nabla_\mu k_{\mu;i} = 0$ so

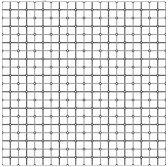
$d = 2$	$d = 3$
$k_{\mu;i} = \epsilon_{\mu\nu} \nabla_\nu \phi_i$	$k_{\mu;i} = \epsilon_{\mu\nu\lambda} \nabla_\nu \phi_{\lambda,i}$
$\phi_i \in \mathbb{Z}$	$\boldsymbol{\phi}_i \in \mathbb{Z}^3$

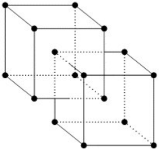
the new field is located on the dual lattice

$$Z = \sum_{\{\phi_i\}} \exp\left(\sum_{\mu;i} \ln I_{\epsilon_{\mu\nu} \nabla_\nu \phi_i}(\beta) \right) \quad \Big| \quad Z = \sum_{\{\boldsymbol{\phi}_i\}} \exp\left(\sum_{\mu;i} \ln I_{\epsilon_{\mu\nu\lambda} \nabla_\nu \phi_{\lambda,i}}(\beta) \right)$$

$$\boxed{\equiv \sum_{\{\phi_i\}} \exp\left(\sum V(\{\epsilon \nabla \phi\}) \right)}$$

From now on we will work on the dual lattice.





Poisson summation formula

We can use now the **Poisson summation formula**

$d = 2$

$$\sum_{\{\phi\}} e^{\sum V(\{\epsilon\nabla\phi\})} = \overbrace{\int \delta\phi \sum_{\{m_i\}} e^{\sum V(\{\epsilon\nabla\phi\}) + 2\pi i \sum_i m_i \phi_i}}^{\text{must be invariant under } \phi_i \mapsto \phi_i + \phi_0,\ \phi_0 \in \mathbb{R}}, \qquad \underbrace{\int \delta\phi \stackrel{\text{def}}{=} \prod_i \int_{-\infty}^{\infty} d\,\phi_i}_{\text{now } \phi \text{ is a continuous field!}}$$

In the previous formula $m_i \in \mathbb{Z}$. Imposing the invariance $\sum_i m_i = 0$.

$d = 3$

$$\sum_{\{\phi\}} e^{\sum V(\{\epsilon\nabla\phi\})} = \overbrace{\int \delta\boldsymbol{\phi} \sum_{\{\mathbf{m}_i\}} e^{\sum V(\{\epsilon\nabla\phi\}) + 2\pi i \sum_i \mathbf{m}_i \cdot \boldsymbol{\phi}_i}}^{\text{must be invariant under } \phi_{\mu;i} \mapsto \phi_{\mu;i} + \nabla_\mu \rho_i}, \qquad \int \delta\boldsymbol{\phi} \stackrel{\text{def}}{=} \prod_{\mu;i} \int_{-\infty}^{\infty} d\,\phi_{\mu;i}$$

In the previous formula $\mathbf{m}_i \in \mathbb{Z}^3$. Imposing the invariance $\boldsymbol{\nabla} \cdot \mathbf{m}_i = 0$.

Low temperature expansion

Now expanding in n, considering $\beta \gg 1$ (low temperature behaviour) and omitting multiplicative constants, we can substitute

$$\ln I_n(\beta) \longrightarrow -\frac{1}{2\beta} n^2$$

$d = 2$

$$Z \approx \int \delta\phi \sum_{\{m_i\}} e^{-\frac{1}{2\beta}\sum(\nabla_\mu \phi_j)^2 + 2\pi i \sum m_j \phi_j}$$

$d = 3$

$$Z \approx \int \delta\boldsymbol{\phi} \sum_{\{\mathbf{m}\}} e^{-\frac{1}{2\beta}\sum(\epsilon_{\mu\nu\lambda}\nabla_\nu \phi_{\lambda,j})^2 + 2\pi i \mathbf{m}_j \cdot \boldsymbol{\phi}_j}$$

By a gaussian integration over ϕ_i we obtain

$$Z \approx \sum_{\{m_i\}} e^{\pi\beta \sum_{\mathbf{r}\neq\mathbf{r}'} m(\mathbf{r}) V_2(\mathbf{r}-\mathbf{r}') m(\mathbf{r}')}$$

$$Z \approx \sum_{\{\mathbf{m}\}} e^{\pi\beta \sum_{\mathbf{r}\neq\mathbf{r}'} \mathbf{m}(\mathbf{r}) \cdot \mathbf{m}(\mathbf{r}') V_3(\mathbf{r}-\mathbf{r}')}$$

where $V_d(\mathbf{r})$ is the Green function on the lattice.

2D XY model for low temperature

The Coulomb gas

For $d = 2$, after some calculations we obtain, removing the divergence

$$V_2(\mathbf{r}) \mapsto \tilde{V}_2(\mathbf{r}) \stackrel{\text{def}}{=} V_2(\mathbf{r}) - V_2(\mathbf{0}) = \ln r - c,\, c \in \mathbb{R}^+ \Rightarrow$$

$$Z = \sum_{\{m_i\}} \mathrm{e}^{\pi\beta \sum_{\mathbf{r}\neq\mathbf{r}'} m(\mathbf{r})m(\mathbf{r}') \ln r - \pi c \beta \sum_{\mathbf{r}} m^2(\mathbf{r})}$$

i.e., the system is equivalent to a **neutral ($\sum_{\mathbf{r}} m(\mathbf{r}) = 0$) Coulomb gas in 2D!** The variables $m(\mathbf{r})$ are called **vortex variables** (topological excitations), because it can be shown ▸ GO! that

$$m(\mathbf{r}) \leftrightarrow \frac{1}{2\pi} \oint_\gamma \delta\theta,$$

γ walk surrounding $\mathbf{r}$ on the lattice.

3D XY model for low temperature

QED analogy and smoke rings

In 3D the partition function looks as the generating function for free photons with $\phi(\mathbf{r}) \leftrightarrow \mathbf{A}(\mathbf{r})$:

$$Z \approx \int \delta\phi \, \mathrm{e}^{-\frac{1}{2\beta} \sum_j (\epsilon_{\mu\nu\lambda} \nabla_\nu \phi_{\lambda,j})^2}$$

Ignoring gauge issues

$$V_3(\mathbf{r}) \sim \frac{1}{r} \Rightarrow \text{“Biot-Savart law”}$$

Moreover $\boldsymbol{\nabla} \cdot \mathbf{m}(\mathbf{r}) = 0$ so “currents” $\mathbf{m}(r)$ generate **closed loops** (smoke rings). Introduce a new “current loop” variable L:

$$Z = \sum_{\{\mathbf{m}^L\}} \mathrm{e}^{\pi\beta \sum_{L,L'} \sum_{\mathbf{r}\neq\mathbf{r}'} \mathbf{m}^L(\mathbf{r})\cdot\mathbf{m}^{L'}(\mathbf{r}') V_3(\mathbf{r}-\mathbf{r}')}$$

Part II

Scaling and Renormalization group

Phase transition in the 2D model?

Mermin–Wagner–Hohenberg theorem and Kosterlitz–Thouless argument

Mermin–Wagner–Hoenberg theorem

There cannot be any spontaneous breaking of a continuous symmetry in a system in $d \leq 2$ dimensions.

but

1. suppose that vortices are separated by a distance L_0 in a lattice with step a;
2. by analogy with electromagnetic theory, they have energy $\beta E = \frac{\beta}{2} \int_a^{L_0} \left(\frac{m}{r}\right)^2 \mathrm{d}^2 r = \pi m^2 \beta \ln \frac{L_0}{a}$;
3. there are $\sim \left(\frac{L_0}{a}\right)^2$ different position for them

$$\beta f \sim (\pi m^2 \beta - 2) \ln \frac{L_0}{a} \Rightarrow \begin{cases} \beta > \frac{2}{\pi} & f > 0 \text{ free vortices suppressed} \\ \beta < \frac{2}{\pi} & f < 0 \text{ free vortices with } m = \pm 1 \text{ proliferate} \end{cases}$$

There is a “transition” regarding vortices for $\beta \sim \frac{2}{\pi}$! **No spontaneous magnetization appears for $\beta \to +\infty$.**

The fugacity: a control parameter for vorticity

$$Z = \sum_{\{m_i\}} \exp\left(\pi\beta \sum_{\mathbf{r}\neq\mathbf{r}'} m(\mathbf{r}) \ln \frac{r}{a} m(\mathbf{r}') - \pi c\beta \sum_{\mathbf{r}} m^2(\mathbf{r})\right)$$

$$\equiv \sum_{\{m_i\}} \exp\left(\pi\beta \sum_{\mathbf{r}\neq\mathbf{r}'} m(\mathbf{r}) \ln \frac{r}{a} m(\mathbf{r}')\right) y^{\sum_{\mathbf{r}} m^2(\mathbf{r})},$$

with $y = \mathrm{e}^{-\pi c\beta}$ fugacity, $y \to 0$ for $\beta \to +\infty$.

Renormalization Group procedure

First step: integrate

- We consider near the transition only **vortex pairs** $m(\mathbf{r}_1) + m(\mathbf{r}_2) = 0$, $m(\mathbf{r}) = \pm 1$ (lowest energy excitations).
- Integration on pairs at distances $a < |\mathbf{r}_1 - \mathbf{r}_2| < a(1+\delta\ell)$ so:

$$Z = Z^+ + \delta Z$$

1. Z^+ = sum of configurations with vortex separations greater than $a(1+\delta\ell)$;
2. δZ interaction between one pair of separation between a and $a(1+\delta\ell)$ and the others pairs

$$\delta Z = \sum_{\{m\}^+} e^{-\beta\mathcal{H}^+} \sum_{\substack{m(\mathbf{r}_1),m(\mathbf{r}_2)=\pm 1 \\ a<|\mathbf{r}_1-\mathbf{r}_2|<a(1+\delta\ell)}} y^2\, e^{\pi\beta \sum_{\mathbf{r}} m(\mathbf{r}) \left[m(\mathbf{r}_1) \ln \frac{|\mathbf{r}_1-\mathbf{r}|}{a} + m(\mathbf{r}_2) \ln \frac{|\mathbf{r}_2-\mathbf{r}|}{a} \right]}$$

$$1 + \frac{\delta Z}{Z^+} = 1 + y^2 \sum_{m(\mathbf{r}_1)=\pm 1} \iint_{\substack{|\mathbf{r}_1-\mathbf{r}_2| \\ \in [a, a(1+\delta\ell)]}} \frac{d^2 r_1}{a^2} \frac{d^2 r_2}{a^2} \prod_{\mathbf{r}\neq\mathbf{r}_1,\mathbf{r}_2} e^{\pi\beta m(\mathbf{r}_1) \ln \frac{|\mathbf{r}_1-\mathbf{r}|}{|\mathbf{r}_2-\mathbf{r}|} m(\mathbf{r})}$$

$\rightarrow$  $\rightarrow$

$$\underbrace{e^{y^2 \delta\ell \frac{2\pi L^2}{a^2}}}_{\text{scaling law for } f} e^{-\pi^4 y^2 \beta^2 \delta\ell \sum_{\mathbf{r}'\neq\mathbf{r}''} m(\mathbf{r}') \ln \frac{|\mathbf{r}'-\mathbf{r}''|}{a} m(\mathbf{r}'')}$$

Renormalization Group procedure

Second step: rescale

Rescaling to the new "minimum scale" $a \mapsto a(1+\delta\ell)$ we obtain two contributions to fugacity

$$\ln\frac{r}{a} = \ln\frac{r}{a(1+\delta\ell)} + \delta\ell \to \mathrm{e}^{-\pi\beta\delta\ell\sum_{\mathbf{r}} m^2(\mathbf{r})}$$

$$\int\frac{\mathrm{d}^2 r}{a^2} = (1+2\delta\ell)\int\frac{\mathrm{d}^2 r}{a^2(1+\delta\ell)^2} \to (1+2\delta\ell)^{\sum_{\mathbf{r}} m^2(\mathbf{r})} \approx \mathrm{e}^{2\delta\ell\sum_{\mathbf{r}} m^2(\mathbf{r})}$$

So finally

$$Z = \mathrm{e}^{y^2\delta\ell\frac{2\pi L^2}{a^2}}\sum_{\{m_i\}}\mathrm{e}^{\pi(\beta-\pi^3 y^2\beta^2\delta\ell)\sum_{\mathbf{r}\neq\mathbf{r}'} m(\mathbf{r})\ln\frac{r}{a'} m(\mathbf{r}')}\left(y\,\mathrm{e}^{(2-\pi\beta\delta)\ell}\right)^{\sum_{\mathbf{r}} m^2(\mathbf{r})}$$

$$\xrightarrow{\text{up to an infinite constant}} \sum_{\{m_i\}}\mathrm{e}^{\pi\beta'\sum_{\mathbf{r}\neq\mathbf{r}'} m(\mathbf{r})\ln\frac{r}{a'} m(\mathbf{r}')}\, y'^{\sum_{\mathbf{r}} m^2(\mathbf{r})}$$

Kosterlitz–Thouless scaling equations

$$\frac{d\beta}{d\ell} = -\pi^3\beta^2 y^2, \qquad \frac{dy}{d\ell} = (2-\pi\beta)y \Rightarrow T_{\mathrm{KT}} = \frac{\pi}{2}$$

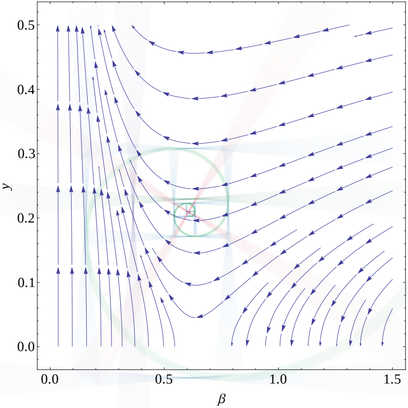
- For $T < T_{\mathrm{TK}}$ the model can have thermally generated topological excitations in pairs of vortices with $m = \pm 1$ Show me!:

$$G(r) \sim \frac{1}{r^{\frac{1}{\pi\beta}}}.$$

- For $T > T_{\mathrm{TK}}$ pairs are separated and Show me!

$$G(r) \sim e^{-\frac{r}{\xi}} \quad \text{with } \xi \sim e^{(T-T_{\mathrm{KT}})^{-\frac{1}{2}}}.$$

Note that $\xi \not\sim (T - T_{\mathrm{KT}})^{-\nu}$.



Scaling and renormalization in 3D

Preparation to the analysis

Let us rewrite the partition function for the 3D XY model as

$$Z = \sum_{\{\mathbf{J}^{(L)}\}} \prod_L y^{(L)} \exp\left(-\pi\beta \sum_{L'\neq L} \sum_{\mathbf{r}\neq\mathbf{r}} \mathbf{J}^{(L)}(\mathbf{r}') \cdot \mathbf{J}^{(L')}(\mathbf{r}') U(\mathbf{r}-\mathbf{r}')\right)$$

where $y^{(L)}$ is the fugacity for each loop L of diameter a_L and core dimension a_c:

$$y^{(L)} = \exp\left(-\pi\beta \sum_{\mathbf{r}\neq\mathbf{r}} \mathbf{J}^{(L)}(\mathbf{r}) \cdot \mathbf{J}^{(L)}(\mathbf{r}') U(\mathbf{r}-\mathbf{r}')\right) \xrightarrow[\mathcal{J}_\mu(\mathbf{r})=0,\pm 1,\text{circular loop of diameter } a_L]{\textbf{Warning!}\text{ Approximation in progress}}$$

$$\rightarrow \exp\left[-2\pi^2\beta \frac{a_L}{a} \ln \frac{a_L}{a_c}\right] = \Big(\underbrace{e^{-2\pi^2\beta \ln \frac{a}{a_c}}}_{y}\Big)^{\frac{a_L}{a}} e^{-2\pi^2\beta \frac{a_L}{a} \ln \frac{a_L}{a}}$$

$$Z = \sum_{\{\mathbf{J}^{(L)}\}} \prod_L y^{\frac{a_L}{a}} e^{-2\pi^2\beta \frac{a_L}{a} \ln \frac{a_L}{a}} \exp\left(-\pi\beta \sum_{L'\neq L} \sum_{\mathbf{r}\neq\mathbf{r}'} \mathbf{J}^{(L)}(\mathbf{r}) \cdot \mathbf{J}^{(L')}(\mathbf{r}') U(\mathbf{r}-\mathbf{r}')\right)$$

RG procedure: a sketch

1. For N loops we can explicit the sum over loop configurations assuming $\mathcal{J}_\mu(r) = 0, \pm 1$ **(low energy configurations)**, **circular** loops of center $\mathbf{R}$ so that $\mathbf{r}^{(L)} = \mathbf{R}^{(L)} + \boldsymbol{\rho}^{(L)}$:

$$\sum_{\{J\}} \leftrightarrow \sum_{N=0}^{\infty} \sum_{\mathcal{J}_\mu^{(L)}=0,\pm 1} \underbrace{\frac{1}{N!}}_{\text{shuffling centers}} \prod_{L=1}^{N} \int \frac{\mathrm{d}^3 R^{(L)}}{a^3} \int \frac{\mathrm{d}^3 \rho^{(L)}}{a^3}$$

2. We can repeat the Kosterlitz procedure for $d = 2$, decomposing $Z = Z^{+} + \delta Z$ integrating over loops of radius $\rho \in \left(\frac{a}{2}, \frac{a}{2}(1 + \delta\ell)\right)$.
3. Rescaling all explicit scale dependences:

$$\frac{a_L}{a} = \frac{a_L}{a(1+\delta\ell)}(1+\delta\ell), \quad U(r) \sim \frac{1}{r} = \frac{1}{a(1+\delta\ell)r}(1+\delta\ell).$$

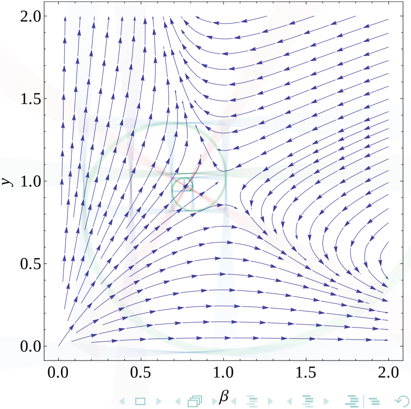
Finally we can extract the renormalization equation that leave invariant the partition function.

Scaling equation in the 3D XY model

$$\frac{d\beta}{d\ell} = \beta - \frac{2\pi^3}{3} y\beta^2, \qquad \frac{dy}{d\ell} = y\left[6 - \frac{\pi^2}{2}\beta \underbrace{\left(\ln\frac{a}{a_c} + 1\right)}_{h\text{“self energy”}}\right]$$

- β is **not marginal**.
- Trivial fixed point: $\beta^* = y^* = 0$.
- High temperature: $\beta = 0 \Rightarrow y = y_0\, e^{6\ell} \to +\infty$ i.e., vorticity proliferates.
- Low temperature: $y = 0$, $\beta = \beta_0\, e^{\ell} \to \infty$.
- Nontrivial fixed point

$$\beta^* = \frac{12}{\pi^2 h}, \quad y^* = \frac{h}{8\pi}$$



Linearized solutions

Writing $y = y^*(1+\delta y)$ and $\beta = \beta^*(1+\delta\beta)$

$$\begin{cases} \frac{d\beta}{d\ell} = \beta - Ay\beta^2, \\ \frac{dy}{d\ell} = y(6 - B\beta) \end{cases} \Rightarrow \frac{d}{d\ell}\begin{pmatrix}\delta y \\ \delta\beta\end{pmatrix} = \begin{pmatrix} 6 - B\beta^* & -B\beta^* \\ 1 - 2Ay^*\beta^* & -A\beta^* y^* \end{pmatrix}\begin{pmatrix}\delta y \\ \delta\beta\end{pmatrix}$$

$$\text{being } \begin{cases} \beta^* = \frac{6}{B} \\ y^* = \frac{B}{6A} \end{cases} \Rightarrow \frac{d}{d\ell}\begin{pmatrix}\delta y \\ \delta\beta\end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -6 \\ -1 & -1 \end{pmatrix}}_{A,\,B \text{ indipendent}}\begin{pmatrix}\delta y \\ \delta\beta\end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\underbrace{\lambda_+ = 2,\ \mathbf{v}_+ = \frac{1}{\sqrt{10}}\begin{pmatrix}-3 \\ 1\end{pmatrix}}_{\text{relevant}}, \quad \underbrace{\lambda_- = -3,\ \mathbf{v}_- = \frac{1}{\sqrt{5}}\begin{pmatrix}2 \\ 1\end{pmatrix}}_{\text{irrelevant}}$$

We assume that the relevant axis is the **temperature axis**, i.e., λ_+ is the temperature eigenvalue, so, from general theory

$$\text{loop size } \xi \sim (T - T_c)^{-\nu} \quad \text{with } \nu = \frac{1}{\lambda_+} = \frac{1}{2}$$

Conclusions

We have studied the XY model in 2D and 3D in absence of external fields. We have obtained the following results.

2D model In 2D **there is no spontaneous magnetization** at $T \to 0$ but at $T = T_{\mathrm{KT}}$ the system undergoes to a Kosterlitz–Thouless transition:

- for $T < T_{\mathrm{KT}}$ free vortices are suppressed but there are pairs vortex–antivortex; $\xi = +\infty$
- for $T > T_{\mathrm{KT}}$ free vortices proliferate; ξ finite.

3D model A duality transformation shows that the model is equivalent to an abelian gauge theory on a lattice (e.g., QED).

- Vortex lines are found as topological excitations.
- A phase transition occurs for a certain T_c between a phase where linear vortices are suppressed ($T < T_c$) and a phase with linear vortices are favoured ($T > T_c$).
- It can be shown that **spontaneous magnetization** for $T \to 0$ is present.

Thank you for your attention!

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Part III

Appendix

Additional notes: meaning of ϕ and m in 2D XY model

The Villain approximation

At low temperature in a lattice of N sites $\nabla_\mu\theta_i \approx 2\pi n_{\mu;i}$, $n_{\mu;i} \in \mathbb{Z}$.
To take into account the periodicity, we consider $\theta_i \in \mathbb{R}$. In this case $n_{\mu;i} \mapsto n_{\mu;i} + \nabla_\mu\rho_i$, $\rho_i \in \mathbb{Z}$, is a redefinition of θ_i and we have to fix a gauge to do not overcount the configurations.

$$Z = (2\beta\,\mathrm{e}^{2\beta})^N \sum_{\{n\}}{}' \int_{-\infty}^{+\infty} \delta\theta \int_{-\infty}^{+\infty} \delta k_\mu \exp \sum \left[-\frac{k_{\mu;j}^2}{2\beta} + ik_{\mu;j}(\nabla_\mu\theta_j - 2\pi n_{\mu;j}) \right]$$

$$= (4\pi\beta\,\mathrm{e}^{2\beta})^N \sum_{\{n\}}{}' \int_{-\infty}^{\infty} \delta k_\mu \exp \sum \left[-\frac{k_{\mu;j}^2}{2\beta} - 2\pi i k_{\mu;j} n_{\mu;j} \right] \prod_i \delta_{\boldsymbol{\nabla}\cdot\mathbf{k}_i,0} \xrightarrow[k_{\mu;i}=\epsilon_{\mu\nu}\nabla_\nu\phi_i]{\text{null divergence condition}}$$

$$= (4\pi\beta\,\mathrm{e}^{2\beta})^N \sum_{\{n\}}{}' \int_{-\infty}^{\infty} \delta\phi \exp \sum \left[-\frac{1}{2\beta}(\nabla_\mu\phi_j)^2 + 2\pi i\phi_j \epsilon_{\mu\nu}\nabla_\mu n_{\nu;j} \right]$$

i.e, $\boxed{m_i = \epsilon_{\mu\nu}\nabla_\mu n_{\nu;i}}$. If $\nabla_\mu\Theta_i = \nabla_\mu\theta_i - 2\pi n_{\mu;i}$ then for a loop γ

$$\oint_\gamma \nabla_\mu\Theta = 2\pi \sum_s m_s, \quad s \text{ dual sites in } \gamma$$

The correlation lenght in the 2D XY model

Below the critical temperature

Below the critical temperature **we suppose $\Delta\theta$ small**; in the continuum limit

$$\beta\mathcal{H} = -\beta \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \to \text{constant} - \frac{\beta}{2}\int (\nabla\theta)^2 \, \mathrm{d}^2 r \Rightarrow$$

$$G(r) - G(0) \sim -\frac{1}{2\pi\beta} \ln r + \text{constant}$$

So

$$\langle \mathbf{s}(\mathbf{r}) \cdot \mathbf{s}(\mathbf{0}) \rangle \sim \Re \langle \mathrm{e}^{i(\theta_{\mathbf{r}} - \theta_{\mathbf{0}})} \rangle = \mathrm{e}^{-\frac{\langle (\theta_{\mathbf{r}} - \theta_{\mathbf{0}})^2 \rangle}{2}} = \mathrm{e}^{G(r) - G(0)} = \frac{1}{r^{\frac{1}{2\pi\beta}}} \Rightarrow \boxed{\eta = \frac{1}{2\pi\beta}}$$

Back!

The correlation lenght in the 2D XY model

Above the critical temperature

Above the critical temperature, being $\xi \sim \mathrm{e}^{\ell}$, we denote $\begin{cases} x = 2 - \pi\beta \\ t = \frac{T - T_{\mathrm{KT}}}{T_{\mathrm{KT}}} \end{cases}$ so

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\ell} \approx Ay^2 \\ \frac{\mathrm{d}y}{\mathrm{d}\ell} = xy \end{cases} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{Ay} \Rightarrow Ay^2 - x^2 = \text{constant} \approx 2Ay(0)^2 t \text{ near the critical line}$$

$$\ell = \int_0^{\ell} \mathrm{d}\,\ell' = \int_{x(0)}^{x(\ell)} \frac{\mathrm{d}\,x}{Ay^2} = \int_{x(0)}^{x(\ell)} \frac{\mathrm{d}\,x}{x^2 + 2Ay(0)^2 t} \approx \int_{-\infty}^{+\infty} \frac{\mathrm{d}\,x}{x^2 + 2Ay(0)^2 t} = \frac{\pi}{\sqrt{2Ay(0)^2 t}}$$

$$\boxed{\xi \sim \mathrm{e}^{\frac{\text{constant}}{\sqrt{t}}}}.$$

Back!