Dear Ngo Bao Chau,
Dear Professor Remmert,
Ladies and Gentlemen,

It is a great pleasure to give the laudatory speech for Ngô Bao Châu who is the recipient of the 2007 Oberwolfach prize. This prize is awarded approximately every three years to a young European mathematician below the age of 36 by the Oberwolfach Foundation in cooperation with the Mathematical Research Institute Oberwolfach and its Scientific Committee. The field of mathematics within which the recipient of this year's prize was selected is algebra and number theory. Ngô Bao Châu was chosen for his work on the Fundamental Lemma conjecture of Langlands and Shelstad. With his proof of this long standing conjecture, Ngo has established himself as a leader in a central area of mathematics at the crossroads between algebraic geometry and automorphic forms.

I have structured my talk as follows. First, I will give a short curriculum vitae of Ngo in the form of a table. Then I will place the result of Ngo in its historical context. Finally, I will state a special case of his result and give some comments on his proof.

1. **Short curriculum vitae of Ngô Bao Châu**

- 1972 born in Hanoi, Vietnam
- 1990 moves to France
- 1992-1995 student at the ENS, rue d'Ulm
- 1993-1997 doctoral studies at U. de Paris Sud, with G. Laumon
- 1997 dissertation 'Le lemme fondamental de Jacquet et Ye'
- 1998-2004 chargé de recherches au CNRS, at Univ. de Paris Nord
- 2004 Habilitation
- 2004– Professor U. de Paris-Sud
- 2006– IAS, Princeton
- *distinctions*: Clay Research Award 2004, Speaker at ICM 2006.

2. **Background**

The conjecture of Langlands and Shelstad lies in the field of automorphic forms. In the beginning of the 20^{th} century this theory was the theory of modular forms, i.e., of holomorphic functions on the upper half plane transforming in a prescribed way under the action of discrete groups of conformal motions. It was only in the 1950's, under the influence of I. Gelfand and Harish-Chandra, that the theory of automorphic forms on arbitrary semi-simple Lie groups, or semi-simple algebraic groups, was developed. In the 1960's the theory was dramatically refocused through the introduction by R. Langlands of his functoriality principle. This principle is a conjecture that stipulates correspondences between automorphic forms on semi-simple groups which are related by a homomorphism between their Langlands dual groups. This principle is surely among the most ingenious ideas of the last century and constitutes the deepest statement about automorphic forms known to us today (as a conjecture!). Langlands himself also showed how his functoriality principle bears upon one of the central problems of arithmetic algebraic geometry, that of calculating the zeta function of Shimura varieties and of determining the ℓ-adic Galois representations defined by their cohomology.

At the same time, Langlands emphasized the importance of the Selberg trace formula as a tool for a proof of the functoriality principle in many cases, for instance for establishing correspondences between automorphic forms on classical groups. He also pointed to the relevance of the Selberg trace formula for the zeta function problem.

One of the first tests of these radically new ideas is contained in the paper by J.-P. Labesse and Langlands on SL_2. At a certain point in their paper they prove an innocuous-looking statement that later turned out to be an instance of a general phenomenon. This result allowed them to construct a transport of certain functions between groups, dual to the desired transport of automorphic forms.

Langlands soon recognized the importance of this statement in the general context of the functoriality principle, and named this conjecture 'fundamental lemma'; a more appropriate name would have been the *fundamental matching conjecture*. In joint work with D. Shelstad, he formulated a precise conjecture in the general case. Already the formulation of this conjecture is very complicated, and, in fact, the conjecture comes in several variants (e.g., endoscopic version, or base change version, etc.), depending on which homomorphism on the Langlands dual group one uses to transport automorphic forms.

In the ensuing 25 years the matching conjecture has turned out to be absolutely essential in achieving progress on the functoriality principle. Furthermore, R. Kottwitz showed that the matching conjecture is also crucial in the zeta function problem. In spite of its importance and its proof in special cases, the fundamental lemma resisted intense efforts and its proof seemed out of reach. Indeed, quite a number of papers were written during this period which were conditional on the fundamental matching conjecture.

Ngo has now finally proved this conjecture and has thereby removed this major stumbling block to further progress. More precisely, he proved the endoscopic fundamental lemma for unitary groups in joint work with G. Laumon. Very recently, he posted a 188 page manuscript with a solution in the general endoscopic case.

What is the fundamental lemma about? As indicated above, it arises in the comparisons of trace formulas. The trace formula is an identity, where on one side, the 'geometric side', there appear sums of orbital integrals. The fundamental lemma is an identity between orbital integrals of simple functions, like characteristic functions of open compact subgroups.

The field of automorphic forms, and in particular the fundamental lemma, has the reputation of being impenetrable, with results only appreciable by an insider. In the rest of my talk I want to show that this is not necessarily so. I will state a special case of the FL theorem of Laumon/Ngo which is highly non-trivial, yet can be understood by many. And my hope is that the beauty of the statement, if not of its proof, can be appreciated by all.

3. **The theorem**

As mentioned above, the result of Laumon/Ngo concerns orbital integrals for unitary groups. As a warm-up, let us first consider orbital integrals for $\mathrm{GL}(n)$:

$$\mathrm{O}^G_\gamma(1_K) = \int_{G_\gamma\backslash G} 1_K(g^{-1}\gamma g)\frac{\mathrm{d}g}{\mathrm{d}g_\gamma}\ ,$$

where we used the following notation.

- F non archimedean local field, O_F the ring of integers of F
- $G = \mathrm{GL}(n, F)$, $K = \mathrm{GL}(n, O_F)$ maximal compact open subgroup.
- $1_K =$ the characteristic function of K
- $\gamma \in G$ regular semi-simple, hence its centralizer G_γ is a maximal torus in G
- $\mathrm{d}g$ and $\mathrm{d}g_\gamma$ Haar measures on G and G_γ.

This orbital integral has a combinatorial description as the cardinality of a set of lattices, as follwos.

$$\mathrm{O}^G_\gamma = |X_\gamma/\Lambda_\gamma|.$$

Here:

- $X_\gamma = \{O_F\text{-lattices } M \subset F^n \mid \gamma(M) = M\}$,
- γ is regular semi-simple, i.e., the F-subalgebra $F[\gamma]$ of $\mathrm{M}_n(F)$ generated by γ is commutative semi-simple of dimension n, hence $F[\gamma] = \prod_{i\in I} E_i$, where $(E_i)_{i\in I}$ is a finite family of finite separable extensions of F,
- after choosing uniformizers $\pi_i = \pi_{E_i}$ in the E_i we have $F[\gamma]^\times \cong \Lambda_\gamma \times K_\gamma$, where $\Lambda_\gamma = \mathbb{Z}^I$ and $K_\gamma = \prod_{i\in I} O^\times_{E_i}$ is a maximal compact open subgroup of $G_\gamma = F[\gamma]^\times$,
- $\Lambda_\gamma \subset G_\gamma$ acts freely on X_γ,
- we normalized the Haar measures by $\mathrm{vol}(K, \mathrm{d}g) = \mathrm{vol}(K_\gamma, \mathrm{d}g_\gamma) = 1$.

Thus we see that this simple orbital integral unwinds as a cardinality, namely the number of lattices fixed under translation by γ, taken up to the obvious homotheties commuting with the action of γ.

Next, we want to describe the orbital integrals for unitary groups. We will use the following general notation to describe the relevant unitary groups.

- F is a local field of equal characteristic p
- F' is an unramified quadratic field extension of F, with Galois group $\mathrm{Gal}(F'/F) = \{1, \tau\}$.
- E is a totally ramified separable extension of F.
- $\Phi_{(\alpha)}$ is a non degenerate hermitian form on the F'-vector space $E' = E \otimes_F F'$

  $$\Phi_{(\alpha)}(x, y) = \mathrm{tr}_{E'/F'}(\alpha x^\tau y),$$

  ($\alpha \in E^\times$).
- The discriminant of $\Phi_{(\alpha)}$ only depends on the valuation of α. Fix α^+, resp. α^- such that $\Phi^+_{E'} = \Phi_{(\alpha^+)}$ has *even* parity of the order of the discriminant, and $\Phi^-_{E'} = \Phi_{(\alpha^-)}$ has *odd* parity of the order of the discriminant.

Now we can exhibit a typical *endoscopic subgroup* of a unitary group. Fix totally ramified separable finite extensions E_1 and E_2 of F of degrees n_1 and n_2. Let E'_1 and E'_2 denote the unramified quadratic field extensions E_1F' and E_2F' of E_1 and E_2.

Let $E' = E'_1 \oplus E'_2$ (a F'-vector space of dimension n_1+n_2). Endow E' with the non degenerate hermitian forms

$$\Phi^+ = \Phi^+_{E'_1} \oplus \Phi^+_{E'_2}$$

and

$$\Phi^- = \Phi^-_{E'_1} \oplus \Phi^-_{E'_2}.$$

These two forms are equivalent. Therefore we can find $g \in GL_{F'}(E')$ such that

$$\Phi^-(x,y) = \Phi^+(gx,gy) \qquad (\forall x,y \in E').$$

Let us now fix $\gamma_1 \in E_1'^{\times}$ and $\gamma_2 \in E_2'^{\times}$ such that $\gamma_1\gamma_1^{\sigma} = \gamma_2\gamma_2^{\sigma} = 1$. We assume that $E'_i = F'[\gamma_i]$, i.e. the minimal polynomial $P_i(T) \in F'[T]$ of γ_i has degree n_i. We assume moreover that the polynomials $P_1(T)$ and $P_2(T)$ are separable and prime to each other.

The diagonal element $(\gamma_1,\gamma_2) \in GL_{F'}(E')$ may be simultaneously viewed as

- an elliptic regular semi-simple element γ^+ in the unitary group

$$G \overset{\text{dfn}}{=\!=} U(E',\Phi^+) = gU(E',\Phi^-)g^{-1} \subset GL_{F'}(E'),$$

- as an elliptic regular semi-simple element γ^- in the unitary group

$$U(E',\Phi^-) \subset GL_{F'}(E')$$

- and as an elliptic (G,H)-regular semi-simple element δ in the endoscopic group

$$H = U(E'_1,\Phi^+_1) \times U(E'_2,\Phi^+_2) \subset GL_{F'}(E').$$

The elements γ^+ and $g\gamma^-g^{-1}$ of G are conjugate in $GL_{F'}(E')$ but are not conjugate in G. The conjugacy class of δ in H is equal to its stable conjugacy class. To see this, note that an element of $U(E'_i,\Phi^+_i) \subset GL_{F'}(E'_i)$ is stably conjugate to γ_i if and only if it has the same minimal polynomial as γ_i.

Define subgroups

$$K = \mathrm{Fix}_G(O_{E'_1} \oplus O_{E'_2}), \quad K^H = \mathrm{Fix}_H(O_{E'_1} \oplus O_{E'_2}) \ .$$

These are hyperspecial maximal open compact subgroups of G and H respectively.

Now we define *stable* and *unstable* orbital integrals. Let

- The *κ-orbital integral*,

$$O^{\kappa}_{\gamma}(1_K) = |\{L' \subset E' \mid L'^{\perp^+} = L' \text{ and } (\gamma_1,\gamma_2)L' = L'\}| \\ -|\{L' \subset E' \mid L'^{\perp^-} = L' \text{ and } (\gamma_1,\gamma_2)L' = L'\}|$$

  (L''s are $O_{F'}$-lattices, $(\cdot)^{\perp^{\pm}}$ denotes the duality for such lattices with respect to the hermitian form $\Phi^{\pm}$).

- The *stable orbital integral*,

$$SO_\delta^H(1_{K^H}) = |\{M_1' \subset E_1' \mid M_1'^{\perp_1^+} = M_1' \text{ and } \gamma_1 M_1' = M_1'\}| \times |\{M_2' \subset E_2' \mid M_2'^{\perp_2^+} = M_2' \text{ and } \gamma_2 M_2' = M_2'\}|.$$

  (M_i''s are $O_{F'}$-lattices and $(\cdot)^{\perp_i^+}$ denotes the duality for such lattices with respect to the hermitian form Φ_i^+).

Before we can state the main theorem, we need to define an additional numerical invariant of the situation. Let

$$r = r(\gamma_1, \gamma_2) = \mathrm{val}\big(\mathrm{Res}(P_1, P_2)\big),$$

where

$$\mathrm{Res}(P_1, P_2) = \prod_{k_1=0}^{n_1-1} \prod_{k_2=0}^{n_2-1} (\gamma_1^{(k_1)} - \gamma_2^{(k_2)}) \in O_{F'}$$

is the resultant of the minimal polynomials $P_1(T), P_2(T) \in F'[T]$ of γ_1, γ_2. Here $\gamma_i = \gamma_i^{(0)}, \ldots, \gamma_i^{(n_i-1)}$ are the roots of $P_i(T)$ in some algebraic closure of F' containing E_1' and E_2'.

A special case of the theorem of Laumon and Ngo (which confirms the matching conjecture of Langlands-Shelstad in this particular case) is now the following statement.

Theorem 0.1. *Under the above hypotheses, assume that the characteristic p of F is bigger than n. Then*

$$O_\gamma^\kappa(1_K) = (-q)^r SO_\delta^H(1_{K^H}),$$

where q is the number of elements in the residue field k.

As is obvious, the theorem is a purely combinatorial statement. However, the combinatorics are quite difficult. In earlier attempts, methods of combinatorial geometry based on Bruhat-Tits buildings were used; and these methods are successful in low-dimensional cases. In the proof of Laumon/Ngo, the whole arsenal of modern algebraic geometry is brought to bear on the problem. The starting point is the observation that $G/K = (LG/L^+G)(k)$ is the set of k-points of the *affine Grassmannian* of G, an ind-algebraic variety of infinite dimension. I cannot go here into this proof.

In the end, I stress that I have not given the history of the problem. Any such history would have to mention at least the following names, which are ordered here alphabetically : Chaudouard, Clozel, Goresky, Haines, Hales, Kazhdan, Kottwitz, Labesse, Langlands, MacPherson, Rogawski, Saito, Schröder, Shelstad, Shintani, Waldspurger, Weissauer, Whitehouse, . . .

And now I ask you all to join me in congratulating Ngô Bao Châu for his brilliant achievement.