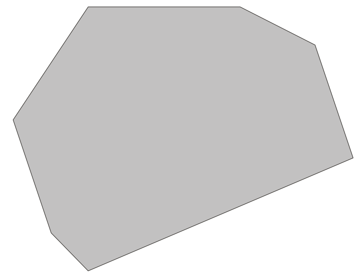
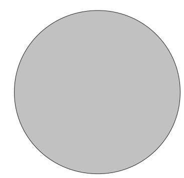
Maximum area of polygon



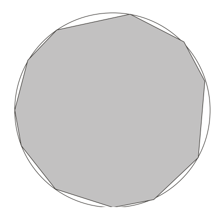
Suppose I give you *n* sticks. They might be of different lengths, or the same length, or some the same as others, etc. Now there are lots of polygons you can form with those sticks. Your job is to find the one with maximum (planar) area.

I was given this problem by a friend at a mathematics meeting. [For a mathematician a new problem is a gift—the most wonderful gift a friend can bestow.] Good timing, because at math meetings there are always talks that lose me after a while, and it's good to have something to click over in my mind. Actually this problem was enticing enough that even the good talks had serious competition.

The problem seems at first to be hard. You are not given much information and there are lots and lots of things you can do. Actually there are two different kinds of decisions you have to make. The first is to put the sticks in the "right" order (sequentially) and the second is to determine the angles between them.



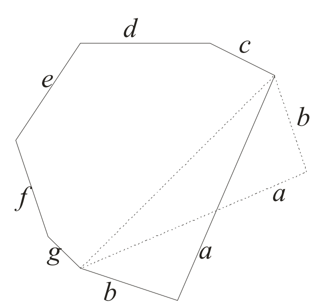
It's hard to know how to begin. One idea is to start with a small value of *n*, and the smallest non-trivial value is *n*=4. Another is to take a very large value of *n* and make the edges quite short. That's what I first thought of, because it made it feel a lot like the classic fence problem. You are given a fixed length of fencing and you want to make an enclosure of maximum area, and most people know that the answer is a circle.



So I figured that if there were a lot of short edges, the answer might be that the vertices should lie in a circle. And then I wondered if that just might be the answer for *all* polygons. Well, maybe that was a bit much to expect. On the other hand, the problem had been given to me by a mathematician and mathematicians like to known by the quality of the problems they talk about, and so this was likely a pretty nice problem, and circles are pretty nice too. So I kept that circle idea in the back of my mind.

I began by thinking about the order of the edges. Because that seemed to make things quite complicated, and I was hoping something simple would appear.

Could I imagine a situation in which I could increase the area by rearranging the order of the sides? As a particular special case—could I increase the area by permuting two adjacent sides? [Such a permutation is called a *transposition.*]



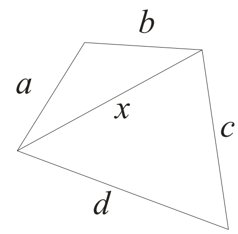
Well the answer to that is clearly no. If I take any polygon, and reverse the order of two adjacent sides, leaving the other sides fixed in place, the area remains the same. That insight is a real breakthrough. For example, if I have a 5-sided polygon with sides in order *abcdefg*, and I know the maximum area of that, I don't have to worry about *bacdefg*—it must have the same maximum area.

What about other permutations? What about *dbgeafc*? Well it's not hard to argue that any permutation can be obtained by a sequence of transpositions of adjacent terms. For example, starting with *abcdefg*, use a sequence of adjacent transpositions to put *d* first, then a sequence to put *b* next, then a sequence to put *g* next, etc. It follows from this that the area cannot be increased by any permutation of the edges.

That's a big step. It means we don't have to worry about the order of the edges. We can take any particular ordering of the edges and work with that. So we are left with the question of what the angles should be.

The case n=4

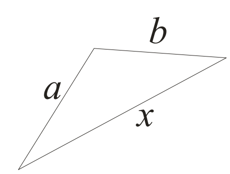
Perhaps this is the moment to focus on the case n=4. Take a 4-sided polygon *abcd*. What are the possible shapes we can have for fixed side lengths a, b, c and d? Well we have one degree of freedom. For example, if we let x be the length of the diagonal between the a-b sides and the c-d sides, then the area $A = A(x)$ of the quadrilateral is determined by x.



So the problem becomes: given *a*, *b*, *c* and *d*, choose *x* to maximize *A*(*x*).

How might we do that? Well A is the sum of the areas of the two triangles and we know the side lengths of each triangle. Is there a formula for the area of a triangle in terms of the lengths of the three sides?

Yes there is. It's called Heron's formula. For the upper triangle with sides a, b and x, we let the semiperimeter be s:



$$s = \frac{a+b+x}{2}$$

and then the area is:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-x)}$$

Do the same for the lower triangle—if the semiperimeter is t:

$$t = \frac{c+d+x}{2}$$

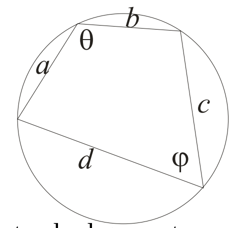
then the area of the entire quadrilateral is:

$$A(x) = \sqrt{s(s-a)(s-b)(s-x)} + \sqrt{t(t-c)(t-d)(t-x)}$$

and we want to choose x to make this a maximum. The way to do this is to set the x-derivative of this to be zero. But all this is *not* what I did.

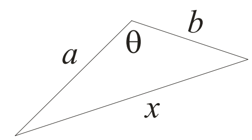
There are a couple of reasons for that. One is that I did not really want to differentiate that fairly complex expression with respect to x. [Note that s and t also depend on x, so we have a bit of a chain-rule to navigate.] And even if I did, could I solve the result for x? And even if I could do *that*, where would I then be?

But there's a much more significant reason I did not want to go the Heron's formula route. I still had the fence problem in mind and I was hoping that the maximum area would be found when the vertices of the quadrilateral lay in a circle. Now a quadrilateral with that property is called *cyclic*, and a standard theorem says that a quadrilateral is cyclic when opposite angles add to 180°. In the diagram at the right, that means that $\theta + \varphi = 180$. So that was the condition I was gunning for, and therefore I wanted angles to appear in my analysis. And Heron's formula doesn't use angles.



A standard geometry result is that a quadrilateral is cyclic if opposite angles add to 180°.

Okay. So what I want is a formula for the area of a triangle that involves one of the angles. Since a and b are the given side lengths, we'll look for a formula that involves a and b and the contained angle θ. [It's clear enough that the triangle—and therefore its area—are determined by these three quantities.]

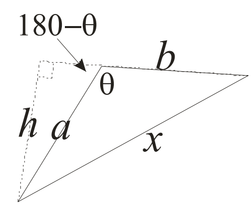


That's not so hard to do. Taking b to be the base, the altitude h of the triangle will satisfy:

$$h/a = \sin(180-\theta).$$

and since $\sin(180-\theta) = \sin\theta$:

$$h = a\sin\theta.$$



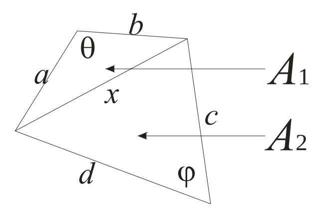
Finally, the area is one-half base times height:

$$\text{Area} = \frac{ab\sin\theta}{2}.$$

As an immediate check on this formula, if θ=0 we should get a zero area (and we do) and if θ=90, a will be an altitude and the area should be $ab/2$ (and it is).

Now the problem becomes: choose θ and φ to maximize the total area:

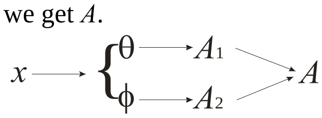
$$A(\theta, \varphi) = A_1 + A_2 = \frac{ab\sin\theta}{2} + \frac{cd\sin\phi}{2}$$



The trouble with this formulation is that it looks as if θ and φ are two independent variables, but they're not. If we choose one of them, the other is determined, so there's really only one variable here. We could try to find the relationship between them (hence express one of them in terms of the other) but in fact it's geometrically (and intuitively) nicer to work (as before) with the single variable x. That is (for fixed a, b, c and d), we will regard θ and φ as functions of x:

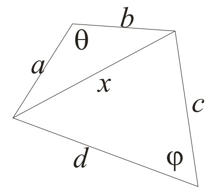
$$\theta = \theta(x)$$
$$\varphi = \varphi(x)$$

These in turn determine the areas of the two triangles and if we add those together we get A.



Finding x to maximize A
The formula is:

$$A(x) = \frac{ab\sin\theta(x)}{2} + \frac{cd\sin\phi(x)}{2}.$$



As before, to find the value of x which maximizes A, we set the derivative dA/dx equal to zero. Using the chain rule:

$$\frac{dA}{dx} = \frac{ab\cos\theta}{2}\frac{d\theta}{dx} + \frac{cd\cos\phi}{2}\frac{d\phi}{dx} = 0$$

Now we need $d\theta/dx$ and $d\varphi/dx$. Let's do the first. We can write x as a function of θ using the cosine law:

$$x^2 = a^2 + b^2 - 2ab\cos\theta.$$

Now differentiate with respect to x (treating θ as a function of x):

$$2x = 2ab\sin\theta\,(d\theta/dx)$$

Solve for $d\theta/dx$:

$$\frac{d\theta}{dx} = \frac{x}{ab\sin\theta}$$

Similarly:

$$\frac{d\phi}{dx} = \frac{x}{cd\sin\phi}.$$

If we put these into the equation for dA/dx, we get:

$$\frac{dA}{dx} = \frac{ab\cos\theta}{2}\frac{x}{ab\sin\theta} + \frac{cd\cos\phi}{2}\frac{x}{cd\sin\phi} = 0$$

$$\frac{x}{2\tan\theta} + \frac{x}{2\tan\phi} = 0$$

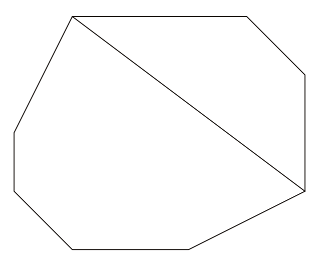
$$\tan\theta = -\tan\varphi.$$

Now when to two angles (in quadrants 1 and 2) have tangents which are negatives of one another?—when they add to 180°. Thus:

$$\theta + \varphi = 180.$$

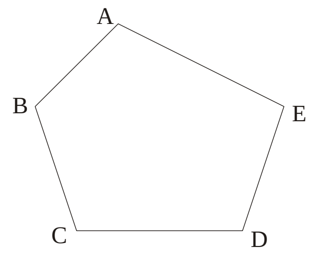
That's the result we were after. It tells us that the quadrilateral is cyclic.

Now it's time to move on to more sides. One idea is to use an inductive approach. If we cut the polygon with a chord, we have two polygons each with fewer sides than the original. If we know the result for these (that the vertices have to lie on a circle) maybe we can put these together to get the result for the original.

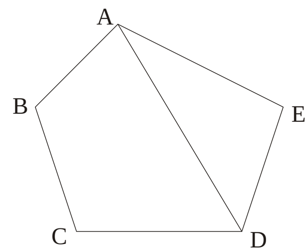


It's a good idea and it works. There are lots of ways to do it, though, and it turns out to be a bit of a challenge (at least it was for me) to find one that feels "right." Actually the nicest one I know of is not a traditional induction at all, but goes directly from n=4 to the general result. Take some time to play with this a bit.

It might be easier to take one step at a time. Try to prove the *n*=5 result. We've done four sides. Can you extend that result to do five?



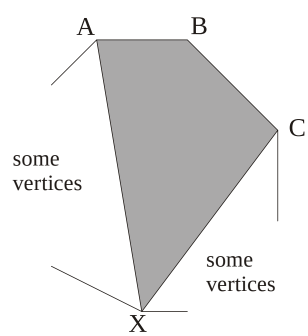
The case n=5.
Suppose we have a pentagon ABCDE and the vertices have been positioned to give maximal area. I show that the vertices must lie on a circle. Now the three vertices ABC determine a unique circle, so it's a question of showing the other two must lie on this circle.



Take D. Draw the chord AD. Now consider the polygon ABCD with fixed side lengths. I claim that it must be of maximum area. Indeed, if this were not so, I could increase the area by adjusting the position of B and C leaving A and D in position (since the distance AD won't change) and this would increase the area of the original polygon ABCDE. Thus, if ABCD is of maximal area, our *n*=4 result tells us that ABCD is cyclic C and therefore D lies on the circle determined by ABC.

A similar argument works for E, using the chord CE.

The general case.
As I said there are many ways to generalize the above argument to any value of *n*. Here is the most elegant I have encountered.



Suppose we have an *n*-gon with fixed side lengths whose vertices have been positioned to give maximal area. Take three adjacent vertices ABC. These determine a unique circle (it's easy to argue that they cannot lie in a straight line), and I show that all other vertices must lie on this circle. Take any other vertex X and consider the polygon ABCX with fixed side lengths. I claim that it must be of maximum area. Indeed, if this were not so, I could increase the area by adjusting the position of the vertices. Since AX and CX won't change in length in this adjustment, the other vertices can be "attached" to AX and CX giving us a new *n*-gon with the original side lengths but of greater area. That's a contradiction, so ABCX must be of maximal area. By the *n*=4 result, ABCX is cyclic and X must lie on the circle determined by ABC. And we are finished.