Geometrical and Graphical Solutions of Quadratic Equations

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One of the unfortunate consequences of the rapid technological advances in recent years is that some time-honored methods of calculation and approximation now exist only in textbooks of the past. Many of the topics studied by our parents and grandparents will soon be only memories (if they are not now), and with them goes our appreciation of the difficulties experienced before the days of calculators and computers. A classic example of a topic no longer studied is that of geometrical and graphical solutions of higher degree equations. For example, how many of today's students know that imaginary roots of real quadratic equations can be found from real graphs? This article will present several geometrical and graphical methods of solving quadratic equations. While definitely old-fashioned, these methods are nonetheless quite interesting, and illustrate the power and beauty of coordinate geometry.

Greek Origins

The geometric solution of quadratic equations goes back to the ancient Greeks, long before the rectangular coordinate system of René Descartes. Methods for the equivalent of finding the positive real roots of quadratic equations can be found in Euclid's *Elements* as early as Proposition 11, Book II [2]. Propositions VI 28 and VI 29 considered such methods for equations of the form

$$x^2 - ax + b^2 = 0 \quad \text{and} \quad x^2 - ax - b^2 = 0,$$

where a and b represent lengths of given line segments. After negative roots had been recognized, later developments included additional methods that yielded all real roots.

Figure 1 illustrates simple Euclidean constructions for the following cases:

1) $x^2 - bx + c = 0$ 2) $x^2 + bx + c = 0$

3) $x^2 - bx - c = 0$ 4) $x^2 + bx - c = 0.$

Each can be verified by simple geometry and algebra. For example, to verify that $x_1 = AC$ and $x_2 = CB$ are the roots of $x^2 - bx + c = 0$, note that $AC/\sqrt{c} = \sqrt{c}/CB$ and thus $c = AC \cdot CB$. It follows that $c = x_1(b - x_1)$ or $x_1^2 - bx_1 + c = 0$.
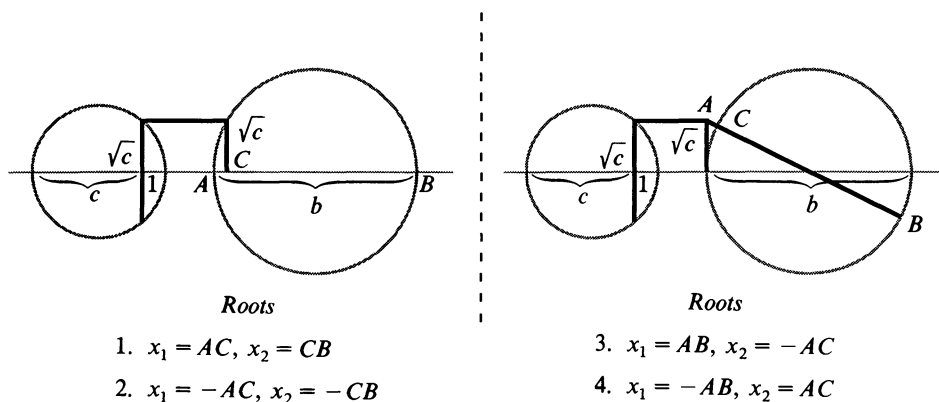
Figure 1

Below the figure:

Roots

1. $x_1 = AC$, $x_2 = CB$
2. $x_1 = -AC$, $x_2 = -CB$

Roots

3. $x_1 = AB$, $x_2 = -AC$
4. $x_1 = -AB$, $x_2 = AC$

The verification for $x_2 = CB$ is done in the same way. (This proof is not from Euclid; it is suggested by a problem in Howard Eves' *An Introduction to the History of Mathematics* [**3**, p. 70].)

Carlyle's Method

The Scottish writer Thomas Carlyle (1795–1881) developed a geometrical solution of quadratic equations based upon coordinate geometry. In his early years Carlyle was a mathematics teacher, and among his accomplishments was the translation of Legendre's 1794 revision of *Elements* into English. This translation, as later revised by Charles Davies in 1851 and J. H. van Amringe in 1885, went through 33 American editions [**3**, p. 338]. Thus the Legendre revision rather than the original Euclid became the pedagogical basis for the study of *Elements* in the United States.

Carlyle's method, according to Eves [**3**, p. 61], appeared in the popular *Elements of Geometry* of Sir John Leslie (1766–1832). Leslie remarked: "The solution of this important problem...was suggested to me by Mr. Thomas Carlyle, an ingenious young mathematician, and formerly my pupil."

Carlyle's method, as described by Eves, provides the solutions to the equation $x^2 + bx + c = 0$ by considering the points of intersection of a particular circle with the x-axis. Graph the circle that has a diameter with endpoints $(0, 1)$ and $(-b, c)$. If there are two real solutions, the circle will intersect the x-axis at two points. The abscissas of these two points are the solutions. Figure 2 illustrates this method for the general case. If only one real solution (a double solution) exists, the circle will be tangent to the x-axis, and the double solution is the abscissa of the point of tangency. If there are no real solutions, the circle will not intersect the x-axis.

The verification of Carlyle's method is based upon the fact that the equation of the circle is $x^2 + y^2 + bx - (1 + c)y + c = 0$. Setting $y = 0$, we find that the abscissas of the intersections of the circle with the x-axis are given by $x^2 + bx + c = 0$, and so are the roots of the given equation.

Figure 3 illustrates Carlyle's method applied to the equation $x^2 + 2x - 8 = 0$. The abscissas of the points of intersection of the circle and the x-axis are -4 and 2, the solutions of the equation.
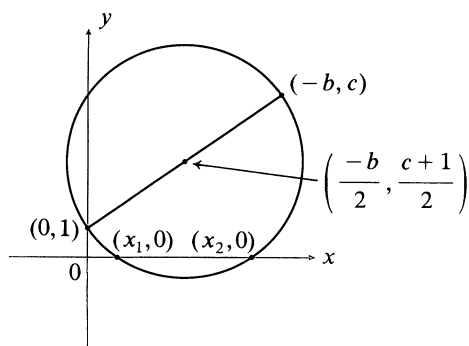
Figure 2

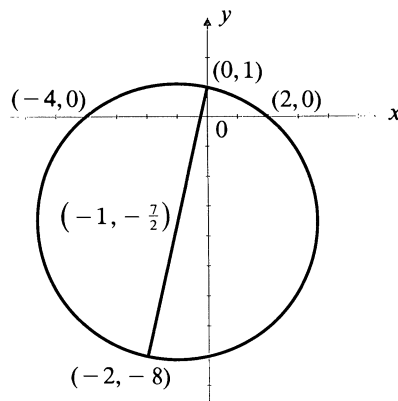Carlyle's Method: x_1 and x_2 are solutions of $x^2 + bx + c = 0$.

Figure 3

The solutions of $x^2 + 2x - 8 = 0$ are -4 and 2.

von Staudt's Method

The German geometer Karl Georg Christian von Staudt (1798–1867) was one of the many eminent mathematicians who made noteworthy contributions to elementary fields of mathematics. He held the chair of mathematics at Erlangen for a time, and is known for his *Geometrie der lage* (1847) in which he built up projective geometry without any reference to magnitude or number. His method of solving quadratic equations geometrically is described in Eves' *History* [**3**, pp. 69–70] as follows:

> The quadratic equation $x^2 - gx + h = 0$ is given. On a rectangular Cartesian frame of reference, plot the points $(h/g, 0)$ and $(4/g, 2)$, and let the join of these two points cut the unit circle of center $(0, 1)$ in points R and S. Project R and S from the point $(0, 2)$ onto the points $(r, 0)$ and $(s, 0)$ on the x-axis... r and s are the roots of the given equation.

The verification of the method is also given by Eves. If we let A be the point $(0, 2)$, L be the point $(h/g, 0)$ where $\overleftrightarrow{RS}$ crosses the x-axis, and T be the point $(4/g, 2)$ where $\overleftrightarrow{RS}$ crosses the tangent to the circle at A, we obtain the following equations:

$$\text{circle: } x^2 + y(y - 2) = 0$$

$$\text{line } AR\text{: } 2x + r(y - 2) = 0$$

$$\text{line } AS\text{: } 2x + s(y - 2) = 0.$$

It follows that the graph of

$$[2x + r(y - 2)][2x + s(y - 2)] - 4[x^2 + y(y - 2)] = 0$$

passes through the points A, R, and S. But this equation simplifies to

$$(y - 2)[2x(r + s) + rs(y - 2) - 4y] = 0,$$

which represents the pair of straight lines

$$y - 2 = 0 \quad \text{and} \quad 2x(r + s) + rs(y - 2) - 4y = 0.$$

Since neither R nor S lie on the first line, it follows that the second line must be

the line RS. Setting $y = 0$ and $y = 2$, in turn, in the equation of line RS then yields

$$OL = \frac{rs}{r+s} = \frac{h}{g}$$

and

$$AT = \frac{4}{r+s} = \frac{4}{g}.$$

It follows that $r + s = g$ and $rs = h$. Thus r and s are the roots of $x^2 - gx + h = x^2 - (r+s)x + rs = (x-r)(x-s) = 0$.

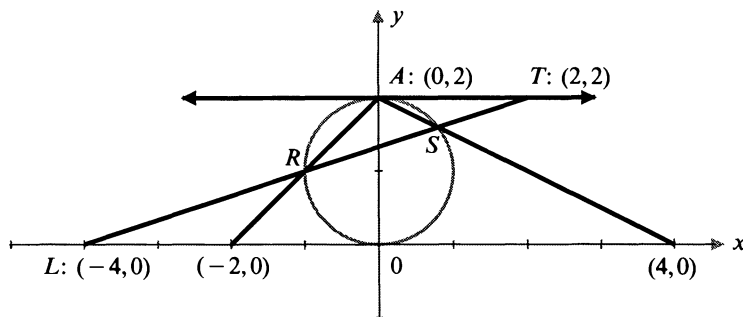The method of von Staudt is applied to the equation $x^2 - 2x - 8 = 0$ in Figure 4.

Figure 4
$r = -2$ and $s = 4$ are the two solutions of $x^2 - 2x - 8 = 0$, by the method of von Staudt.

Solving $x^2 + bx + c = 0$ Using the Fixed Graph $y = x^2$

The usual method of solving $x^2 + bx + c = 0$ by coordinate geometry is to graph the parabola $y = x^2 + bx + c$ and locate the points where the parabola intersects the x-axis. However, another procedure [9, p. 32], which simplifies the process considerably, involves the use of the standard parabola $y = x^2$. If we have a supply of graph paper, with each sheet containing the graph of $y = x^2$, we can solve $x^2 + bx + c = 0$ by simply graphing the line $y = -bx - c$ on one such sheet, and then finding the abscissas of the points of intersection of the parabola and the line. Figure 5 illustrates this method for the equation $x^2 - x - 6 = 0$. Of course, if the
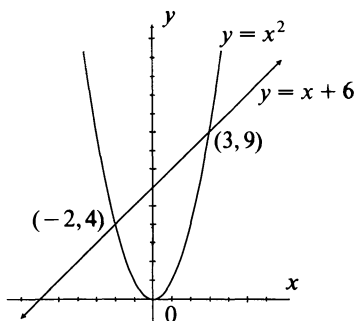

Figure 5
The solutions of $x^2 - x - 6 = 0$ are 3 and -2, the abscissas of the points of intersection of the line and the fixed parabola.

parabola and the line do not intersect, the roots are imaginary. A technique for finding imaginary roots will be described later.

The method above generalizes. For example, the solution of the quadratic equation $x^2 + bx + c = 0$ can be found by graphing a line and the standard cubic $y = x^3$ on the same set of axes [9]. The solutions of

$$0 = (x - b)(x^2 + bx + c)$$
$$= x^3 + (c - b^2)x - bc$$

are b and the solutions of $x^2 + bx + c = 0$. Graphically, they are the abscissas of the points of intersection of the graph of $y = x^3$ and the line $y + (c - b^2)x - bc = 0$.

Solving Quadratics Using $xy = 1$

In 1908 there appeared a small text by Arthur Schultze titled *Graphic Algebra* [9]. This remarkable little book contains numerous methods of solving higher degree equations geometrically. One of the more interesting methods is a variation on the one just described. Rather than using the standard parabola, this method employs the rectangular hyperbola $xy = 1$.

In the equation $ax^2 + bx + c = 0$, we make a partial substitution, using $x = 1/y$: $ax/y + b/y + c = 0$ or $ax + cy = -b$.

Now consider the system

$$ax + cy = -b, \quad y = 1/x.$$

The solution of the above system yields the desired root(s) of $ax^2 + bx + c = 0$.

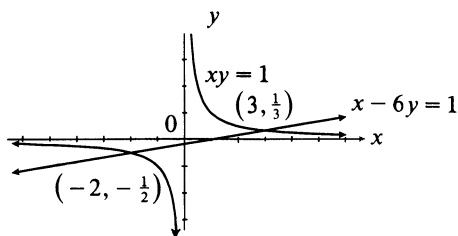Figure 6 provides an illustration of this method, applied to $x^2 - x - 6 = 0$.

Figure 6

The abscissas of the points of intersection of $xy = 1$ and $x - 6y = 1$ are the solutions of $x^2 - x - 6 = 0$. They are -2 and 3.

Imaginary Solutions of $x^2 + bx + c = 0$

Earlier we considered the method of finding the real solutions of $x^2 + bx + c = 0$ using the fixed graph $y = x^2$ and the line $y = -bx - c$. If there is no point of intersection, the equation has two imaginary solutions which are, of course, complex conjugates. A graphical method that yields the real part and the absolute value of the imaginary parts is also described in the Schultze text [9, pp. 37–38].

For $x^2 + bx + c = 0$ to have imaginary solutions, the discriminant $b^2 - 4c$ must be negative. The two solutions are

$$\frac{-b}{2} \pm \frac{i\sqrt{4c - b^2}}{2}.$$

Figure 7 illustrates the method in the case when $b < 0$. Below the parabola $y = x^2$, graph the line L given by $y = -bx - c$. Let $\overline{AB}$ be any chord of the parabola parallel to L. From the midpoint M of $\overline{AB}$ drop a perpendicular to the x-axis intersecting L at N and the parabola at P. Locate Q on $\overleftrightarrow{MN}$ above P such that $PN = PQ$. Construct a chord of the parabola through Q parallel to L, intersecting the parabola at S and T.

Now let V and W be the feet of the perpendiculars to the x-axis from Q and T, respectively. The abscissa of V is the real part of each solution, and the length of $\overline{VW}$ is the absolute value of each imaginary part.

To verify that the abscissa of V is $-b/2$, we need only show that such is the case for M. If A and B are denoted by (x_1, x_1^2) and (x_2, x_2^2) then

$$\frac{x_2^2 - x_1^2}{x_2 - x_1} = -b \quad \text{(the slope of } L\text{)},$$

$$x_2 + x_1 = -b,$$

$$\frac{x_2 + x_1}{2} = \frac{-b}{2}.$$

Since M is the midpoint of $\overline{AB}$, its abscissa is $(x_2 + x_1)/2 = -b/2$, the real part of each solution.

We next determine the abscissa of W, which is the same as that of T. Since $PN = PQ$, the ordinate of Q is c. Thus an equation of $\overleftrightarrow{ST}$ is $y - c = -b(x + b/2)$. Solving this simultaneously with $y = x^2$ gives the quadratic equation $x^2 + bx + (-c + b^2/2) = 0$.
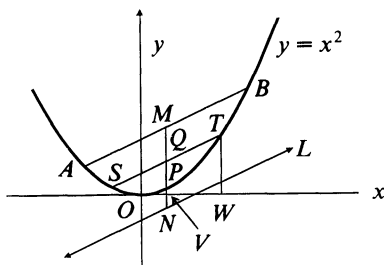
Figure 7
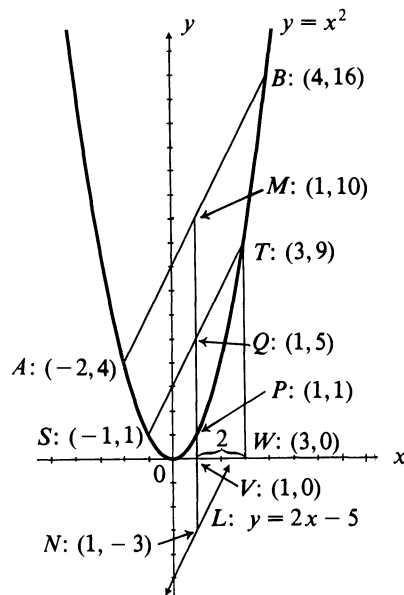
Figure 8

The abscissa of V: $(1, 0)$ is the real part of the complex conjugate solutions, and $VW = 3 - 1 = 2$ is the absolute value of the imaginary parts. The solutions of $x^2 - 2x + 5 = 0$ are $1 \pm 2i$.

An application of the quadratic formula yields

$$x = \frac{-b}{2} \pm \frac{\sqrt{4c - b^2}}{2}.$$

Choosing the $+$ sign to get the abscissa of point T, we obtain

$$VW = \left(\frac{-b}{2} + \frac{\sqrt{4c - b^2}}{2} \right) - \left(\frac{-b}{2} \right) = \frac{\sqrt{4c - b^2}}{2},$$

which is the absolute value of the imaginary parts of the solutions.

Figure 8 illustrates this method for the equation $x^2 - 2x + 5 = 0$, whose solutions are $1 \pm 2i$.

(Incidentally, the Schultze text [9, p. 41] also contains an analogous method for finding imaginary roots of a quadratic equation using the rectangular hyperbola $xy = 1$.)

In Conclusion

While this article has addressed methods of solving quadratic equations, methods are also known for graphically determining the real and imaginary roots of cubic and quartic polynomial equations with real coefficients. For example, the incomplete cubic equation $ax^3 + bx + c = 0$ can be solved geometrically by drawing the line $ay + bx + c = 0$ and the standard cubic $y = x^3$ on the same set of coordinate axes. (The well-known substitution for reducing a complete cubic polynomial equation to one that lacks a second degree term allows this method to be extended to *any* cubic equation.) The Schultze text [9] even contains methods to locate imaginary roots of cubics. It concludes with methods of solving fourth degree equations geometrically.

Several articles [4], [5], [8] dealing with graphical methods of solving polynomial equations appeared in the *American Mathematical Monthly* four decades ago. More recently, similar articles have appeared in *The Mathematics Teacher* [1], [6], and the *College Mathematics Journal* [7], [10].

In the preface of his *Graphic Algebra*, Arthur Schultze wrote:

> It is now generally conceded that graphic methods are not only of great importance for practical work and scientific investigation, but also that their educational value for secondary instruction is very considerable.

In a letter to this writer, Howard Eves points out that "though the material ... has little practical value ... it is beautiful and interesting." While after nearly a century one may argue that graphical methods of equation-solving are no longer "of great importance," we should be aware that these methods do indeed exist and were once studied in the classroom. And what was beautiful and interesting then is still so today.

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Acknowledgement