Topics in T-duality

Essay for Philip Candelas' course on String Theory

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Abstract

I review several basic aspects of a relation between geometrically distinct but nevertheless physically equivalent compactified string theories. This T-duality, which is a consequence of the extended nature of strings, challenges the traditional notion of length. First, I consider the theory of a closed bosonic string on a circle and point at the invariance of its spectrum under inversion of the radius. The one-loop partition function and higher-order contributions are also shown to be invariant, so that T-duality can be considered as a symmetry of the interacting theory. I explain a surprising phenomenon of symmetry enhancement for a special value of the compactification radius. I finally turn on to toroidal (Narain) compactification and apply it to the case of heterotic string theories.

Contents

1 String on a circle

Compared to the case of point particles, the analysis of the theory of a closed bosonic string on a manifold with one circular dimension presents a novelty, namely the possibility for the string to wind a number w of times around the compact dimension of radius R (say the twenty-fifth) :

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi R w. \tag{1}$$

This new feature will lead to the appearance of an original symmetry of the theory, called T-duality.[1]

The most straightforward way to illustrate T-duality is to look at the mass spectrum of this string on a circle [1]. In terms of complex coordinate $z \equiv \sigma^1 + i\sigma^2$ and $\bar{z} \equiv \sigma^1 - i\sigma^2$, the Euclidean action reads

$$S = \frac{1}{2\pi\alpha'} \int d^2 z \partial X^\mu \bar{\partial} X_\mu, \tag{2}$$

where $\partial \equiv \partial_z = (\partial_1 - i\partial_2)/2$ and $\bar{\partial} \equiv \partial_{\bar{z}} = (\partial_1 + i\partial_2)/2$. The equation of motion is

$$\bar{\partial}\partial X^\mu(z, \bar{z}) = 0, \tag{3}$$

[1] This name comes from the custom of denoting by T the massless complex scalar whose VEV is associated with the compactification size R and the antisymmetric tensor B as $T = R^2/\alpha' + iB$ (see below).

which implies that ∂X^μ is holomorphic and $\bar\partial X^\mu$ antiholomorphic. They thus admit Laurent expansions, whose coefficients read

$$\alpha_m^\mu = \left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z), \tag{4}$$

$$\tilde\alpha_m^\mu = -\left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{d\bar z}{2\pi} \bar z^m \bar\partial X^\mu(\bar z). \tag{5}$$

The total Noether momentum along the periodic dimension is quantized in units of the inverse radius

$$p^{25} = \frac{n}{R} = \frac{1}{2\pi\alpha'} \oint (dz \partial X^{25} - d\bar z \bar\partial X^{25}) = (2\alpha')^{-1/2}(\alpha_0^{25} + \tilde\alpha_0^{25}), \tag{6}$$

with n an arbitrary integer. The total change in the X^{25} coordinate while going once around the winded string is

$$2\pi R w = \oint (dz\partial X^{25} + d\bar z \bar\partial X^{25}) = 2\pi(\alpha'/2)^{1/2}(\alpha_0^{25} - \tilde\alpha_0^{25}) \tag{7}$$

Equations (6) and (7) are solved as

$$\left(\frac{2}{\alpha'}\right)^{1/2} \alpha_0^{25} = \frac{n}{R} + \frac{Rw}{\alpha'} \equiv p_L^{25}, \tag{8}$$

$$\left(\frac{2}{\alpha'}\right)^{1/2} \tilde\alpha_0^{25} = \frac{n}{R} - \frac{Rw}{\alpha'} \equiv p_R^{25}. \tag{9}$$

The Virasoro generators are given by

$$L_0 = \frac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \equiv \frac{\alpha'}{4}\left(p^\mu p_\mu + (p_L^{25})^2\right) + N \tag{10}$$

$$\tilde L_0 = \frac{1}{2}\tilde\alpha_0 \cdot \tilde\alpha_0 + \sum_{n=1}^{\infty} \tilde\alpha_{-n} \cdot \tilde\alpha_n \equiv \frac{\alpha'}{4}\left(p^\mu p_\mu + (p_R^{25})^2\right) + \tilde N \tag{11}$$

where the dot product involves all of the twenty-six dimensions, whereas the indices μ only ranges over non-compact dimensions. In particular, $p^\mu p_\mu$ gives (minus) the mass-squared M^2 observed after compactification. From

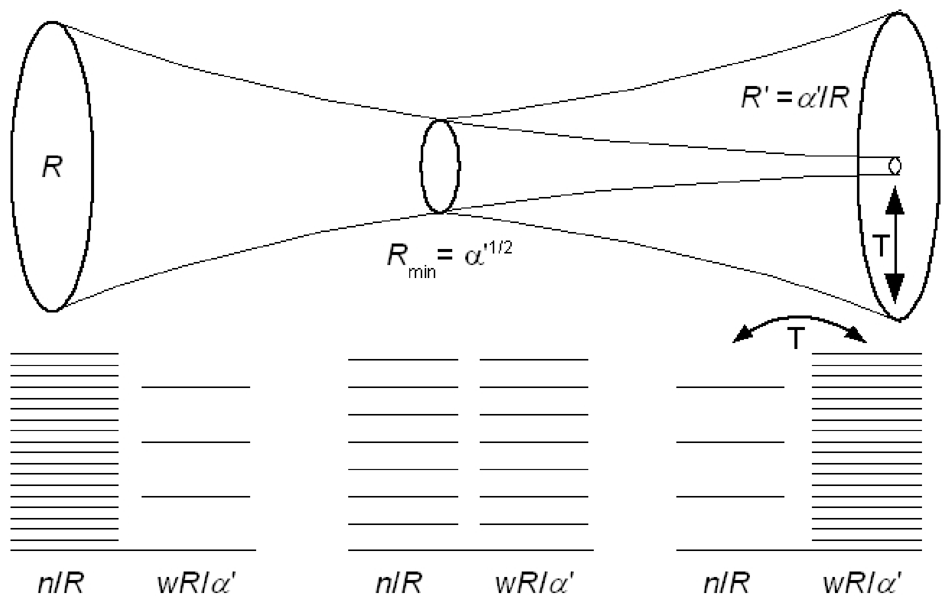

Figure 1: Spectra of a string on circles of various radii. Circles smaller than $R_{\rm min} = \alpha'^{1/2}$ are T-dual to larger circles.

the physical states conditions $L_0 - 1|\phi\rangle = 0 = \tilde{L}_0 - 1|\phi\rangle$, we can then extract the mass formula

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{12}$$

and the level-matching condition

$$\tilde{N} - N = nw. \tag{13}$$

Notice that in the decompactification limit $R \to \infty$ the winding states become infinitely heavy while the momentum excitations become infinitesimal and generate a continuum (see figure 1). Conversely, as $R \to 0$ the momentum excitations decouple while the winding states form a continuum. The two limits thus present identical spectra !

In fact, the mass formula (12) is invariant under the T-duality transformation

$$R \to R' = \frac{\alpha'}{R}, \qquad n \leftrightarrow w. \tag{14}$$

A string winding around a circle of large radius therefore produces the same mass spectrum as a string winding around a circle of small radius. In the following sections, we will show that T-duality actually is a symmetry of the full interacting theory, so that large and small radii are equivalent in every respect.

An important implication of T-duality is the existence of a minimum radius of order of the Planck scale, given by the self-dual radius

$$R_{\text{min}} = \sqrt{\alpha'}. \tag{15}$$

The argument presented in this section for a periodic dimension also holds for any circle in space-time. One is thus driven to the conclusion that "small circles don't exist" [2]. The impossibility to compress a circle below the Planck length is maybe not so surprising given that the fundamental—structureless—objects of the theory are one-dimensional. Length scales much smaller than the string scale simply cannot be probed [3].

This duality strongly suggests that the traditional manifold conception of space-time is not applicable at the Planck length, but should rather be considered as an 'emergent' notion, only approximately valid at much larger length-scales [4].

2 One-loop partition function

The one-loop partition function of a closed string can be expressed as the path integral on a torus with no vertex operators [1]. The coordinates on the worldsheet are subject to the following identifications :

$$(\sigma_1, \sigma_2) \cong (\sigma_1 + 2\pi, \sigma_2) \cong (\sigma_1 + 2\pi\tau_1, \sigma_2 + 2\pi\tau_2), \tag{16}$$

with modular parameter $\tau \equiv \tau_1 + i\tau_2$. Tori related by transformations of the modular group $SL(2, \mathbb{Z})$ are equivalent. Any modular transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{17}$$

where a, b, c, d are integers satisfying $ad - bc = 1$, can be generated by the transformations $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$. The space of conformally inequivalent tori can thus be restricted to the fundamental region

$$-1/2 \leq \tau_1 \leq 1/2, \quad \tau_2 \geq 0, \quad |\tau| \geq 1. \tag{18}$$

One can think of the path integral on a torus as formed by a field theory on a circle that has been evolved for Euclidean time $2\pi\tau_2$, translated in σ_1 by $2\pi\tau_1$, and identified with the initial circle. The generator of translations along σ_2 is the Hamiltonian $H = L_0+\tilde{L}_0-(c+\tilde{c})/24$ (with the central charges c and $\tilde{c}$ given by the number d of transverse oscillators), while the momentum operator $P = L_0 - \tilde{L}_0$ generates translations along σ_1. The identifications of the ends of the cylinder thus formed is realized by taking the trace. The one-loop partition function then reads

$$\begin{aligned} Z_1(\tau) &= \mathrm{Tr}\left[\exp(-2\pi\tau_2 H + 2\pi i\tau_1 P)\right] && (19)\\ &= (q\bar{q})^{-d/24}\mathrm{Tr}\left(q^{L_0}\bar{q}^{\tilde{L}_0}\right), && (20) \end{aligned}$$

with $q \equiv \exp(2\pi i\tau)$. The trace splits into a sum (or integral) over momenta p and a sum over the occupation numbers N and $\tilde{N}$, following the decompositions (10) and (11) of the Virasoro generators into zero modes and oscillator sums :

$$\begin{aligned} Z_1(\tau) &= \sum_p \exp\left[-\pi\tau_2(\alpha_0^2+\tilde{\alpha}_0^2)+i\pi\tau_1(\alpha_0^2-\tilde{\alpha}_0^2)\right] \\ &\quad (q\bar{q})^{-d/24}\mathrm{tr}_N q^{\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n}\mathrm{tr}_{\tilde{N}}\bar{q}^{\sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n}. \qquad (21) \end{aligned}$$

The traces over the Fock space are easily computed if one writes them in a basis over all possible multiparticle states [5]. For example, in one dimension with the basis $|0\rangle$, $\alpha_{-n}|0\rangle$, $(\alpha_{-n})^2|0\rangle$ and so on, the first trace becomes (for one dimension)

$$\mathrm{tr}_N q^{\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n} = \prod_{n=1}^{\infty}(1+q^n+q^{2n}+\cdots) = \prod_{n=1}^{\infty}(1-q^n)^{-1}, \qquad (22)$$

and similarly for the right-moving sector. The second line of (21) then gives for each set of oscillators a contribution $|\eta(\tau)|^{-2}$, where the Dedekind eta function is defined as

$$\eta(\tau) \equiv q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \qquad (23)$$

The first line of (21) has to be treated separately for compact and non-compact dimensions. For a non-compact dimension, $\alpha_0^2+\tilde{\alpha}_0^2 = \alpha' p^2$ whereas

$\alpha_0^2 - \tilde{\alpha}_0^2 = 0$. The momenta form a continuum so that the sum turns into an integral, bringing a factor of space-time volume V :

$$\sum_k \exp\left[-\pi\tau_2\alpha' p^2\right] \to V \int \frac{dp}{2\pi} \exp\left[-\pi\tau_2\alpha' p^2\right] = V(4\pi^2\alpha'\tau_2)^{-1/2}. \quad (24)$$

For the periodic dimension X^{25}, the left- and right-momenta are quantized as in (8) and (9), so the momentum sum is replaced by a sum over n and w. The total contribution from the compact dimension to the partition function is then given by

$$Z_1^{25}(\tau) = |\eta(\tau)|^{-2} \sum_{n,w=-\infty}^{\infty} \exp\left[-\pi\tau_2\left(\frac{\alpha' n^2}{R^2} + \frac{w^2 R^2}{\alpha'}\right) + 2\pi i\tau_1 nw\right]. \quad (25)$$

This expression is manifestly invariant under the T-duality transformation (14). This implies that T-duality is a symmetry of the free string theory.

For the consistency of the theory, the partition function must be invariant under modular transformations [6]. The contribution from the non-compact dimensions goes as $\tau_2^{-1/2}|\eta(\tau)|^{-2}$. Using the following properties of the Dedekind function [1] :

$$\eta(\tau+1) = \exp(i\pi 12)\eta(\tau), \quad (26)$$
$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau), \quad (27)$$

we see that it is invariant under $\tau \to \tau+1$ as well as under $\tau \to -1/\tau$ (which sends τ_2 to $\tau_2/|\tau|^2$), and thus invariant under the full modular group.

Modular invariance of the contribution (25) from the compact dimension can be made explicit by using the Poisson resummation formula [1]

$$\sum_{n=-\infty}^{\infty} \exp\left(-\pi a n^2 + 2\pi i b n\right) = a^{-1/2} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{\pi(m-b)^2}{a}\right] \quad (28)$$

to rewrite

$$Z^{25}(\tau) = R(\alpha'\tau_2)^{-1/2}|\eta(\tau)|^{-2} \sum_{m,w=-\infty}^{\infty} \exp\left(-\frac{\pi R^2|m-w\tau|^2}{\alpha'\tau_2}\right). \quad (29)$$

The coefficient in front of the sum has just been shown to be modular invariant. The sum is also invariant under $\tau \to \tau + 1$ and $\tau \to -1/\tau$ if we redefine the dummy summation variables as $m \to m + w$, and $m \to -w$, $w \to m$, respectively (note that such a shift is allowed because sum is finite, given that $\tau_2 \geq 0$ in the fundamental region).

The partition function (29) can be interpreted as a sum over topologically distinct sectors labeled by m and w, which correspond to non-trivial closed curves on the toric world-sheet winding around the compact direction :

$$\begin{aligned} X^{25}(\sigma^1 + 2\pi, \sigma^2) &= X^{25}(\sigma^1, \sigma^2) + 2\pi R w, && (30) \\ X^{25}(\sigma^1 + 2\pi\tau_1, \sigma^2 + 2\pi\tau_2) &= X^{25}(\sigma^1, \sigma^2) + 2\pi R m. && (31) \end{aligned}$$

In the next section we generalize this construction to the case of higher-genus world-sheets.

3 Higher genus contributions

Interacting string theory involves a summation over worldsheets of different topologies. In this section we show, following [7] and [8], that the T-duality symmetry extends to the higher-genus contributions to the partition function as well. This will confirm the claim that a small compactification radius is completely equivalent to a very large one (at least perturbatively).

Each non-trivial closed curve on a multiloop torus can wrap around the compact direction. The possibles windings (30) and (31) of X^{25} thus generalizes on a compact Riemann surface Σ of genus g to

$$\oint_{a^i} dX^{25} = 2\pi R w^i, \qquad \oint_{b_i} dX^{25} = 2\pi R m_i, \tag{32}$$

where a^i and b_i (with $i = 1, \ldots, g$) are cycles defining a canonical homology basis. We can write a general solution $dX^{25} = \lambda^i \omega_i + \bar{\lambda}^i \bar{\omega}_i$ in terms of a normalized basis of holomorphic $(1,0)$-forms $\omega_i \equiv \omega_i dz$ on the surface Σ, with

$$\oint_{a^i} \omega_j = \delta^i_j, \qquad \oint_{b_i} \omega_j = \tau_{ij}. \tag{33}$$

The complex period matrix τ_{ij} is symmetric and has positive imaginary part [9]. The periods are then expressed as

$$\lambda^i + \bar{\lambda}^i = 2\pi R w^i, \qquad \tau_{ij}\lambda^j + \bar{\tau}_{ij}\bar{\lambda}^j = 2\pi R m_i, \tag{34}$$

which solves to

$$\lambda^i = -i\pi R(m_j - w^k\bar{\tau}_{kj})(\mathrm{Im}\tau_{ij})^{-1}. \tag{35}$$

The term concerning the periodic coordinate in the action (2) can be expressed as

$$S^{25} = \frac{1}{4\pi\alpha'}\int_\Sigma dX^{25}\wedge\star dX^{25}, \tag{36}$$

where the star indicates the Hodge dual, acting on the complex coordinates as $\star dz = -idz$ and $\star d\bar{z} = id\bar{z}$. Using that [9]

$$\int_\Sigma \omega_i\wedge\bar{\omega}_j = \sum_{k=1}^{g}\left[\oint_{a^k}\omega_i\oint_{b_k}\bar{\omega}_j - \oint_{b_k}\omega_i\oint_{a^k}\bar{\omega}_j\right] = -2i\mathrm{Im}\tau_{ij}, \tag{37}$$

we derive the action concerning the (m, w) sector

$$S^{25}_{m,w} = \frac{\pi R^2}{\alpha'}(m - w\bar{\tau})(\mathrm{Im}\tau)^{-1}(m - w\tau). \tag{38}$$

The contribution from the circular direction to the genus g partition function can be written in a form similar to (25) (with an additional coefficient of R due to the X^{25} zero mode [7])

$$\begin{aligned} Z_g^{25} &= R\,|\eta(\tau)|^{-2}\sum_{m,w=-\infty}^{\infty}\exp(-S^{25}_{m,w}) = R^{1-g}|\eta(\tau)|^{-2}(\det\alpha'\mathrm{Im}\tau)^{1/2} \\ &\quad \sum_{n,w=-\infty}^{\infty}\exp\left[-\pi\mathrm{Im}\tau\left(\frac{\alpha' n^2}{R^2}+\frac{w^2R^2}{\alpha'}\right)+2\pi i\mathrm{Re}\tau\, nw\right], \end{aligned} \tag{39}$$

where we performed a Poisson resummation in matrix form [10]. From the previous section we know that such an expression is modular invariant. Under T-duality transformation (14) we find

$$Z_g^{25}\left(\frac{\alpha'}{R}\right) = \left(\frac{R^2}{\alpha'}\right)^{g-1} Z_g^{25}(R). \tag{40}$$

The complete partition function is given by

$$Z(\Phi, R) = \sum_{g=0}^{\infty} e^{\chi\Phi} Z_g(R), \tag{41}$$

where Φ is the (constant) dilaton and $\chi = 2 - 2g$ is the Euler number of a Riemann surface of genus g. To verify its invariance, we need to know the action of a T-duality transformation on the dilaton. The vacuum expectation value of Φ sets the value of the gravitational coupling $\kappa \propto \exp(-\Phi)$. After dimensional reduction, the 25-dimensional coupling constant $\kappa_{25} = (2\pi R)^{-1/2}\kappa$ must be invariant under T-duality, and so κ varies as

$$\kappa \to \kappa' = \frac{\alpha'^{1/2}}{R}\kappa, \tag{42}$$

which implies for the dilaton

$$\Phi \to \Phi' = \Phi + \frac{1}{2}\ln\left(\frac{R^2}{\alpha'}\right). \tag{43}$$

This is just what is needed to make the complete partition function invariant. In conclusion, T-duality can be considered as a genuine symmetry of the (perturbative) interacting string theory.

4 Enhanced gauge symmetry

In standard Kaluza-Klein theory with point particles, the gauge symmetries that are present in the compactified theory correspond to the isometries of the compact dimension—for instance a $U(1)$ gauge symmetry for a circular dimension. A striking feature of the theory of a string on a circle is the appearance of non-Abelian symmetries at the fixed point of the duality transformation (14) [11].

Let us examine the massless states in the spectrum (12) that satisfy the level-matching condition (13). The states constructed by acting on the vacuum with one holomorphic and one anti-holomorphic oscillator and without any momentum or winding are always massless :

$$N = \tilde{N} = 1,\, n = w = 0 \quad \Rightarrow \quad M = 0. \tag{44}$$

When the excitations are along the non-compact dimensions, the states $\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0\rangle$ correspond to the 25-dimensional graviton, the antisymmetric tensor and the dilaton. When either the left- or the right-moving excitation lies along the compact dimension, we obtain the vector states $\alpha^{25}_{-1}\tilde{\alpha}^{\mu}_{-1}|0\rangle$ and $\alpha^{\mu}_{-1}\tilde{\alpha}^{25}_{-1}|0\rangle$, arising from the 26-dimensional graviton and antisymmetric tensor; these two massless vectors generate a $U(1)_L \times U(1)_R$ gauge symmetry.

Finally, $\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}|0\rangle$ is a massless scalar. All those massless states are also present in Kaluza-Klein compactification.

But there are further massless states for special value of the radius. The states with non-vanishing winding number

$$N=0,\ \tilde{N}=1,\ n=w=\pm 1, \quad \text{and} \quad N=1,\ \tilde{N}=0,\ n=-w=\pm 1 \tag{45}$$

correspond to four vector states and four scalars, all with mass

$$M=\frac{\alpha'-R^2}{R\alpha'}. \tag{46}$$

They are massless for the self-dual radius $R_{\min}=\alpha'^{1/2}$. At this special radius, the four new massless vectors combine to the two that are present for any radius to enhance the gauge symmetry to $SU(2)_L\times SU(2)_R$. Moreover, the states with

$$n=\pm 2,\ w=N=\tilde{N}=0, \quad \text{and} \quad w=\pm 2,\ n=N=\tilde{N}=0 \tag{47}$$

correspond to four scalars that are also massless at the minimum radius. Remarkably, as the radius moves away from the minimum value, a stringy Higgs effect takes place : the additional gauge bosons 'eat' the four scalars (47) and acquire a mass, which breaks the symmetry back to $U(1)_L\times U(1)_R$.

The enhanced $SU(2)_L\times SU(2)_R$ symmetry can be exhibited by studying the current algebra [1]. We focus on the holomorphic currents, since the antiholomorphic ones behave in a similar way. The oscillator state $\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{25}|0\rangle$ is created by the operator

$$J^3(z)=i\alpha'^{-1/2}\partial X_L^{25}, \tag{48}$$

while the additional vector states from (45) are created by the operators

$$J^{\pm}(z)=\exp(\pm 2i\alpha'^{-1/2}X_L^{25}). \tag{49}$$

These operators obey the algebra

$$J^+(z)J^-(w) \sim \frac{1}{(z-w)^2}+\frac{2}{z-w}J^3(w), \tag{50}$$

$$J^3(z)J^{\pm}(w) \sim \frac{\alpha'^{-1}}{2(z-w)}J^{\pm}(w). \tag{51}$$

If we define $J^{\pm}(z) \equiv (J^1(z) \pm iJ^2(z))$, the algebra is rewritten as (setting α' to 1/2)

$$J^i(z)J^j(w) \sim \frac{\delta^{ij}}{2(z-w)^2} + i\frac{\epsilon^{ijk}}{z-w}J^k(w). \tag{52}$$

The holomorphic currents can be expanded as

$$J^i(z) = \sum_{m=-\infty}^{\infty} \frac{J^i_m}{z^{m+1}}, \tag{53}$$

where the Laurent coefficients form an affine Kac-Moody $SU(2)$ algebra with level one :

$$\left[J^i_m, J^j_n\right] = \frac{m}{2}\delta_{m+n}\delta^{ij} + \epsilon^{ijk}J^k_{m+n}. \tag{54}$$

Notice that the $J^{\pm}$ operators (49) that create the two additional vector states are single-valued under $X^{25}_L \to X^{25}_L + 2\pi R$ only for the self-dual radius. On the other hand, the J^3 operator (48) is well-defined for any values of the radius.

5 Narain compactification

We now generalize our analysis to the case of q dimensions compactified on a torus T^q :

$$X^m \cong X^m + 2\pi R, \qquad \text{with} \qquad 26 - q \leq m \leq 25. \tag{55}$$

The spectrum of momenta (p^m_L, p^n_R) in the compact directions forms a lattice Γ in a $2q$-dimensional space, spanned by the momentum and winding excitations (cf. (8) and (9)). We will work with dimensionless lattice momenta $l \equiv (l^m_L, l^n_R) = (\alpha'/2)^{1/2}(p^m_L, p^n_R)$, and define the product $\circ$ with Lorentzian signature (q, q) as $l \circ l' \equiv l_L \cdot l'_L - l_R \cdot l'_R$.

The requirement of modular invariance of the partition function (cf. (25))

$$Z_\Gamma(\tau) = |\eta(\tau)|^{-2q} \sum_{l \in \Gamma} \exp(\pi i \tau l_L^2 - \pi i \bar{\tau} l_R^2) \tag{56}$$

severely constrains the lattice Γ [12]. Under $\tau \to \tau + 1$, $Z_\Gamma(\tau)$ changes by a phase $\exp(\pi i(l \circ l))$, so this implies that the lattice Γ must be even :

$$l \circ l \, \in \, 2\mathbb{Z}. \tag{57}$$

Invariance under $\tau \to -1/\tau$ can be analysed by making use of Poisson resummation

$$\sum_{l' \in \Gamma} \delta(l - l') = V_\Gamma^{-1} \sum_{l'' \in \Gamma^*} \exp(2\pi i l'' \circ l), \tag{58}$$

which relates the lattice Γ to its dual Γ^* (i.e. the set of all vectors whose product with a vector of Γ gives an integer). Here V_Γ is the volume of the unit cell of the lattice Γ. Using this representation, the partition function can be easily shown to obey the relation [1] :

$$Z_\Gamma(\tau) = V_\Gamma^{-1} Z_{\Gamma^*}(-1/\tau). \tag{59}$$

Modular invariance is thus respected if the lattice is self-dual :

$$\Gamma = \Gamma^*, \tag{60}$$

in which case of course $V_\Gamma = V_{\Gamma^*}^{-1} = 1$.

All even self-dual lattices can be obtain from one such lattice, say Γ_0, by $O(q,q)$ transformation. However, physical quantities like the mass always involve the products p_L^2 and p_R^2 and are thus invariant under the group $O(q) \times O(q)$ that acts on the left- and right-moving momenta separately. The space of inequivalent theories is then

$$O(q,q)/(O(q) \times O(q)). \tag{61}$$

The number of parameters in this coset space is $\mathrm{dim}O(q,q) - \mathrm{dim}O(q) - \mathrm{dim}O(q) = q^2$. These parameters can be understood as the compact components of the background fields [13]. Their number indeed matches the number of freely adjustable parameters represented by the background metric g_{mn} and antisymmetric tensor B_{mn} : $q(q+1)/2 + q(q-1)/2 = q^2$.

In addition, there exits also some discrete subgroup of $O(q,q)$ that takes the lattice Γ_0 into itself. This is the T-duality group $O(q,q,\mathbb{Z})$. It includes $R \to \alpha'/R$ dualities on the q individual circles, linear redefinitions of the axes that respect the periodicity, and discrete shifts of the antisymmetric tensor background B_{mn}. The space of inequivalent lattices and inequivalent backgrounds is finally given by

$$O(q,q,\mathbb{Z}) \backslash O(q,q)/(O(q) \times O(q)). \tag{62}$$

6 Heterotic T-duality

In this section we consider the bosonic construction of heterotic string theory, with 26 left-movers and 10 right-movers [12]. After toroidal compactification of q dimensions, the dimensionless momenta in the compact directions (l_L^m, l_R^n), with $26-q \leq m \leq 25$ and $10-q \leq n \leq 9$, take value in some lattice Γ. In analogy to the case of the bosonic string, this lattice is constrained by the requirement of modular invariance of the partition function to be an even self-dual Lorentzian lattice of signature $(q+16, q)$.

Like in the bosonic case, the moduli space is a coset space

$$O(q+16, q, \mathbb{Z}) \backslash O(q+16, q)/(O(q+16) \times O(q)). \tag{63}$$

The number of moduli after compactification on a torus T^q (regardless of the discrete symmetries) is $q(q+16)$. They can again be given an interpretation in terms of background fields [13]. To specify the vacuum states of the compactified heterotic theory, one need to fix the background metric g_{mn} and the antisymmetric two-form B_{mn}, which together account for q^2 parameters. The new feature is the presence of Yang-Mills fields A_m in the Lie algebra of $SO(32)$ or $E_8 \times E_8$ that may have non-trivial global holonomies. These correspond to non-trivial Wilson lines

$$U_m = P \exp \oint_{\gamma_m} A \cdot dx, \tag{64}$$

where P indicates path-ordering, and the closed curves γ_m belong to the fundamental group of the torus $\pi_1(T^q)$. This group being Abelian, the q Wilson lines must commute and thus can be simultaneously put into the maximal torus of the gauge group generated by exponentiating its Cartan subalgebra. The choice of q elements of the sixteen-dimensional Cartan algebra involves $16q$ parameters. Adding all the parameters, we obtain $q(q+16)$ as expected.

Note that such toroidal compactifications of the heterotic string do not lead to realistic models because none of the supersymmetries is broken. They result in $\mathcal{N}=4$ supersymmetry in four dimensions, which implies that the fermions are necessarily in the adjoint of the gauge group, in disagreement to what happens in the Standard Model.

As the coset space (63) is a general solution of the consistency conditions, it must to be the same whether we consider the $SO(32)$ or $E_8 \times E_8$ heterotic strings. This points to the fact that the two heterotic theories are just distinct limits of a single theory [13].

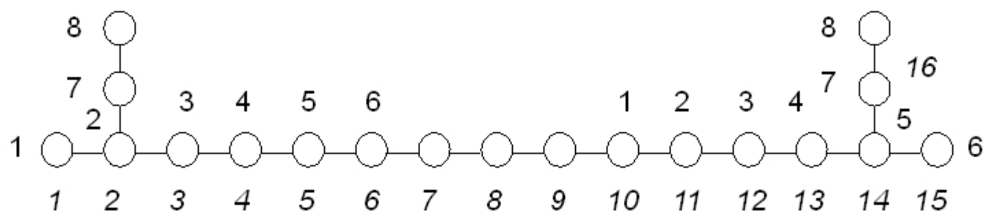

Figure 2: Dynkin diagram for the $\Pi_{17,1}$ lattice, with the nodes labeled to show the embeddings of $\Gamma_8 \oplus \Gamma_8$ and of Γ_{16} (italic).

It is actually possible to continuously interpolate between compactified versions of the $E_8 \times E_8$ and $SO(32)$ theories by turning on appropriate background fields and adjusting radii. A theorem by Hasse and Minkowski asserts that in any dimensions $8k+2d$ there exist an even integer self-dual Lorentzian lattice which is unique up to $SO(8k+d,d)$ transformations; for Euclidean signature, such lattices exist in dimensions $8k$ [10]. In 16 dimensions, there is only the $E_8 \times E_8$ lattice $\Gamma_8 \oplus \Gamma_8$ and the $SO(32)$ lattice Γ_{16}. If we append to each of them the even two-dimensional Lorentzian lattice U, we get an even self-dual Lorentzian lattice $\Pi_{17,1}$ of signature $(17,1)$. According to the uniqueness theorem, there must therefore be some $SO(17,1)$ transformation that relates the $\Gamma_8 \oplus \Gamma_8 \oplus U$ and $\Gamma_{16} \oplus U$ lattices. This transformation was exhibited in [14]. Compactified versions of the two heterotic string theories, insofar as they are continuously related, may be thus be regarded from a mathematical point of view as different ground states of the same theory. From a physical point of view, however, the intermediate $SO(17)$-transformed theories do not necessarily admit interpretations in terms of strings propagating in space-time. The natural notion of length eventually loses its validity somewhere between the two situations.

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