

Introduction to derived algebraic geometry

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Our main goal throughout these lectures will be to explicate the notion of a *derived Artin stack*. We will avoid homotopy theory wherever possible.

Throughout, we will keep the following conventions: Everything will be over a base field k of characteristic 0, and schemes will always be (at least locally) finite type.

1 Motivations

1.1 Moduli theory

Definition 1. A *moduli problem* is a functor $F : k\text{-comm} \simeq k\text{-Aff}^{op} \rightarrow \mathbf{Sets}$ off the category of commutative k -algebras, or equivalently the opposite category of affine k -schemes.

The idea here is to think of $F(A)$ as the set of “families of objects in F parametrized by $\text{Spec } A$ ”.

The main question is: Is this moduli problem *representable*? Ideally, such a representing object would be a scheme X , and then we’d have bijections $F(A) \simeq \text{Hom}_{\text{Sch}}(\text{Spec } A, X)$ that are functorial in A . When such an X exists, it’s called the (*fine*) *moduli space* associated to the moduli problem F .

In practice, X almost never exists. Moreover, even when it does exist, it can be rather complicated to understand the local structure of the scheme X in terms of F . For example, if X isn’t smooth, we’d like to know about the (completed) local rings $\widehat{O}_{X,x}$ at the singularities $x \in X$, and this is difficult to extract directly from F .

Example 1. Fix an associative and finitely presented k -algebra B . Let us consider the moduli problem M_B of finite-dimensional B -modules. That is, $M_B : k\text{-comm} \rightarrow \mathbf{Sets}$ is given by

$$A \mapsto \{M \in \text{Mod}_{B \otimes_k A} : M \text{ projective of finite type over } A\}/\text{iso}.$$

(A map $A \rightarrow A'$ induces a natural base change function $M_B(A) \rightarrow M_B(A')$ given by $M \mapsto M \otimes_A A'$.)

This moduli problem is *not* representable by a scheme. There’s an obvious reason, and then there’s a deep reason. The obvious reason is that M_B is not a sheaf (which we can see even by taking $B = k$). But moreover, even if we sheafify M_B , the result still isn’t represented by a scheme.

To see this, let us first rigidify our moduli problem to remove the isomorphisms; we do this by fixing a particular M to which our modules are required to be isomorphic, and defining the functor $X_B^n : k\text{-comm} \rightarrow \mathbf{Sets}$ given by $A \mapsto \text{Hom}_{k\text{-alg}}(B, M_n(A))$. (This is the “functor of B -module structures on k^n ”.) Now, X_B^n is representable by an affine scheme, given by a quotient $B = k\langle x_1, \dots, x_p \rangle / (\sigma_1, \dots, \sigma_q)$. This sits in the fiber product diagram

$$\begin{array}{ccc} X_B^n & \hookrightarrow & M_n^p \\ \downarrow & & \downarrow \sigma \\ \{*\} & \longrightarrow & M_n^q \end{array}$$

Now, to return to our original functor M_B , we need to quotient by GL_n . This acts on X_B^n by conjugation on $M_n(A)$. Now, we might hope that M_B can be given by X_B^n/GL_m , but this is a “bad quotient” (comparable to $\mathbb{A}^1/\mathbb{G}_m$), and so we’re stuck.

This first issue is solved by using *algebraic stacks*. We’ll try to avoid these, but we’ll have to use them at least a bit.

The second issue is the local structure of $X = X_B^n$. The first thing to ask is: Is X smooth? If not, how bad is it? To answer this question, we will consider *infinitesimal lifting problems*. Let $A \in k\text{-comm}$ and suppose $I \subset A$ is a square-0 ideal. Write $A_0 = A/I$. Then we have the following diagram:

$$\begin{array}{ccc} \text{Spec } A_0 & \xrightarrow{x} & X \\ \downarrow & \nearrow \tilde{x} & \\ \text{Spec } A & & \end{array}$$

Essentially by definition, X is smooth iff we have a lift for all x and all A . But more generally, we can attempt to understand the obstructions via our fiber product diagram. Let's write this now as

$$\begin{array}{ccc} \text{Spec } A_0 & \hookrightarrow & \text{Spec } A \\ \downarrow x & & \\ X & \xrightarrow{f} & Y \\ \downarrow g & \searrow z & \downarrow h \\ \{*\} & \xrightarrow{j} & S \end{array}$$

By smoothness, there exists a lift $\text{Spec } A \rightarrow Y$, and the set of these form a torsor for $T_{S,fx} \otimes_{A_0} I$, where we write $T_{S,fx} = (fx)^*(T_S)$ (for T_S the tangent sheaf). A similar story applies to lifts $\text{Spec } A \rightarrow \{*\}$, but of course here the tangent sheaf is trivial. Thus, we obtain

$$o(x) \in \text{coker}(T_{Y,fx} \otimes_{A_0} I \rightarrow T_{S,ix} \otimes_{A_0} I),$$

and $o(x) = 0$ iff \tilde{x} exists (although to obtain the correct generalization, we should give the source of this morphism the direct summand $T_{\{*\},gx} \otimes_{A_0} I$.) We call this cokernel the *obstruction space*.

However, this construction depends on the choice of “generators and relations” for X . We'd like something which doesn't depend on such a presentation. This is what *derived schemes* are all about: X should carry some extra structure in a way that the obstruction space and the obstructions themselves are natural with respect to the moduli problem.

Remark 1. Instead of the generators and relations framework, suppose that $B = k[G]$ and $G = \pi_1(T)$ for some space T . Then B is supposed to give us “local systems over T ”. From this, we could cook up an obstruction space from T that might seem to have nothing to do with B . This, however, is highly nonunique and depends on the way we think of the moduli functor.

The conclusion of this talk is that we can mix these two stories: M_B should ultimately be a *derived Artin stack*, which will be given concretely by $[\tilde{X}_B^n/GL_n]$, where the tilde denotes “with a derived structure” and the square-brackets denote “stack quotient”. This is a typical example of a derived Artin stack.

Definition 2. An *Artin stack* is a functor $F : k\text{-comm} \rightarrow \text{Groupoids}$ such that:

- there is a groupoid-object in schemes $s, b : X_1 \rightrightarrows X_0$ with s and b (the source and target morphisms) smooth;
- there are functorial equivalences of groupoids $X(A) = \text{Hom}(\text{Spec } A, X) \xrightarrow{\sim} F(A)$.

Then, if X is a scheme admitting an action of a group scheme G , then $[X/G]$ just means the quotient stack associated to $X \times G \rightrightarrows X$.

2 Local derived algebraic geometry

Local DAG is the same thing as the homotopy theory of commutative dg algebras.

Let us recall that we had a fiber product diagram of affine varieties

$$\begin{array}{ccc} X_B^n & \longrightarrow & M_n^p \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & M_n^q. \end{array}$$

This corresponds to a pushout diagram

$$\begin{array}{ccc} A & \longleftarrow & C \\ \uparrow & & \uparrow \\ k & \longleftarrow & B. \end{array}$$

Thus $A \cong C \otimes_B k$. Of course, in general C is not flat over B , and indeed neither is k . To deal with this, we will consider the *derived* tensor product $C \otimes_B^{\mathbb{L}} k$; this will be $\mathcal{O}(\tilde{X}_B^n)$. To do this requires us to move from commutative algebras to commutative dg algebras.

Definition 3. A *commutative dg algebra* (or *cdga*) consists of:

- a complex A of k -modules;
- a morphism of complexes $\mu : A \otimes_k A \rightarrow A$;
- a morphism of complexes $e : k \rightarrow A$;

such that μ is associative, unital (for e), and commutative. That is, A is a commutative unital associative monoid-object in the tensor-category of complexes over k .

We have the following alternate descriptions. We begin with a graded k -vector space $A = \bigoplus A^i$. This is to have a graded multiplication $\mu : A^i \otimes A^j \rightarrow A^{i+j}$, and commutativity in this graded setting implies that $\mu(a \otimes b) = (-1)^{|a|\cdot|b|} \mu(b \otimes a)$. (Of course, this is all wrapped up in the signed definitions of the tensor product of complexes.) We also have a unit $e \in A^0$ for the multiplication μ . Lastly, we have a k -linear differential $d : A^i \rightarrow A^{i+1}$ such that $d^2 = 0$, which is a *graded derivation* over μ , i.e. $d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$. (This sign, too, is wrapped up in the categorical notation.)

Example 2. The prototypical example is the cdga of differential forms $\Omega^*(M)$ on a manifold M , equipped with the usual de Rham differential.

We will take the following conventions:

- All our cdgas will be concentrated in nonpositive degree.
- We will write $\pi_i(A) = H^{-i}(A)$.
- If A is a cdga then $\pi_*(A) = \bigoplus \pi_i(A)$ is naturally a commutative graded algebra; the multiplication is induced by that of A .

Example 3. Let A be a commutative k -algebra, let M be an A -module, and let $s : M \rightarrow A$ be a map of A -modules. Then one important source of cdgas is the construction $K(A, M, s) = \text{Sym}_A(M[1])$. Note that $M[1]$ means that we're considering M in degree -1 . Thus this object is equivalent to $\Lambda_A^*(M)$, since now M is in odd degree. Now, the map $s : M \rightarrow A$ extends in a unique way, and we obtain the *Koszul dg algebra* of (A, M, s) .

Geometrically, we should think of $X = \text{Spec } A$, M is a vector bundle $V \rightarrow X$ (i.e. $M = \Gamma(X, V)$ and $V = \text{Spec}(\text{Sym}_A(M))$), s is a section $V \leftarrow X$, and then $K(A, M, s)$ computes the “derived zero-locus of s ”, i.e. the derived intersection of X and $s(X)$ in V . Thus, this corresponds to the pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{0} & V \\ \uparrow & & \uparrow s \\ \text{“Spec } K(A, M, s)\text{”} & \longrightarrow & X \end{array}$$

Then we have that

$$\pi_i(K(A, M, s)) \simeq \text{Tor}_i^{\text{Sym}_A(M)}(A, A).$$

We should warn that we must consider cdga’s up to *quasi-isomorphism*, since this is the only sense in which the expression $B \otimes_C^{\mathbb{L}} k$ is well defined.

Example 4. Take $A = k[T]$, $M = k[T]$, and $s : k[T] \rightarrow k[T]$ given by $s(T) = 0$. Then

$$K(A, M, s) = (0 \rightarrow k[T] \xrightarrow{\cdot T} k[T] \rightarrow 0)$$

(in degrees -2 through 1). The unique map $K(A, M, s) \rightarrow k$ is a quasi-isomorphism.

Now, the “right” way to work “up to weak equivalence” is to introduce a model structure on cdgas; we don’t want to go that far afield, so we’ll just squeak by by only defining homotopies (and hence homotopy classes of maps).

2.1 Mapping spaces

We define the *standard algebraic n -simplex* to be

$$\Delta^n = \text{Spec}(k[T_0, \dots, T_n] / (\sum T_i - 1)).$$

This is an algebraic version on an n -simplex. Then, we define the cdga $C^n = \Omega_{DR}^*(\Delta^n)$, where product is the wedge product of forms. Note that we have the two points $\{0\} \hookrightarrow \Delta^1 \hookleftarrow \{1\}$. These give two maps $C^1 \rightrightarrows k$. Analogously, we get all the usual simplicial identities.

Definition 4. Let $A, B \in \text{cdga}$. Then $\text{Map}(A, B)$ is the topological space obtained as follows:

- the 0-simplices are maps $A \rightarrow B$ of cdgas;
- the 1-simplices are maps $A \rightarrow B \otimes C^1$ of cdgas;
- we glue 1-simplices to 0-simplices by the two compositions $A \rightarrow B \otimes C^1 \rightrightarrows B \otimes k \cong B$;
- the 2-simplices are maps $A \rightarrow B \otimes C^2$ of cdgas;
- these are glued to 1-simplices as before;
- rinse and repeat.

Note that we can compose: we have $\text{Map}(A, B) \times \text{Map}(B, C) \rightarrow \text{Map}(A, C)$.

Definition 5. Call a cdga a *cell cdga* if there exists a filtration

$$k = A_0 \hookrightarrow A_1 \hookrightarrow \dots \hookrightarrow A$$

such that $A_{m+1} \cong A_m \otimes k[x_\alpha]$ (and the inclusion $A_m \hookrightarrow A_{m+1}$ is the obvious one), and $d(x_\alpha) \in A_m$. (These are the *cofibrant* objects in the model structure; all objects are fibrant, since fibrations are just morphisms that are levelwise surjections except possibly in degree 0.)

Example 5. If A is a polynomial ring, then $K(A, M, s)$ is a cell cdga.

Definition 6. The opposite category of the *topological category of derived affine schemes* is the category whose objects are cell cdgas and whose mapping spaces are as defined above. We will write this as \mathbf{dAff}_k .

Now, we know how to define a scheme by gluing affine schemes. We can play the same game here, by introducing Zariski (or other) open subsets, etc. We won't do this today, but instead we'll explore some interesting classes of maps.

First, we'll give one more definition.

Definition 7. If A is a cdga over k , then an *A-dg module* consists of a complex M of k -modules along with an associative and unital action morphism $A \otimes_k M \rightarrow M$. We will write $D(A)$ for the *derived category* of A -dg modules, i.e. the category of A -dg modules with quasi-isomorphisms formally inverted.

Exercise 1. Define a notion of *cell A-dg modules* and mapping spaces thereof. (Hint: Replace C^n with the complex of cochains $C^*(\Delta^n; k)$.)

This gives us the *topological category of cell A-dg modules*, denoted \mathbf{Mod}_A . Then $D(A) \simeq \pi_0(\mathbf{Mod}_A)$, where by π_0 we mean to take π_0 of the mapping spaces.

2.2 Cotangent complexes

Let A be a cell cdga and M be a cell A -module. Then we have a cdga $A \oplus M$ defined by the multiplication in A , the action of A on M , and by setting $M^2 = 0$. (Note that this is not necessarily a cell A -dg module; if not, we should take the cellular approximation.) This comes with a natural augmentation $A \oplus M \rightarrow A$.

Definition 8. The *space of derivations of A into M* is the fiber at the basepoint of the map $\text{Map}(A, A \oplus M) \rightarrow \text{Map}(A, A)$; we write

$$\text{Der}(A, M) \rightarrow \text{Map}(A, A \oplus M) \rightarrow \text{Map}(A, A).$$

Proposition 1. *There is an A -dg module $\mathbb{L}_A \in \mathbf{Mod}_A$ such that*

$$\text{Der}(A, M) \xrightarrow{\sim} \text{Map}(\mathbb{L}_A, M).$$

(Note that we're only using homotopy equivalence of topological spaces here, not homeomorphism.)

Definition 9. \mathbb{L}_A is called the *cotangent complex* of A .

Proposition 2. *Let A be a smooth k -algebra, considered as a cell cdga. Then $\mathbb{L}_A \simeq \Omega_{A/k}^1 \in D(A)$.*

More generally, we always have that $\pi_0(\mathbb{L}_A) \simeq \Omega_{\pi_0(A)}^1$. However, the higher homotopy of \mathbb{L}_A is nonzero in general. Thus, \mathbb{L}_A detects the obstruction to smoothness.

We define the *derived tensor product* as a derived pushout:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \otimes_A^{\mathbb{L}} C, \end{array}$$

with $B \otimes_A^{\mathbb{L}} C = B' \otimes_A C$ where

$$\begin{array}{ccc} A & \xrightarrow{\text{cell inclusion}} & B' \\ & \searrow & \downarrow \wr \\ & & B. \end{array}$$

This factorization can be chosen *functorially*, by the “small-object argument” in model category theory. (One should think of B' as a projective resolution of B as an A -algebra.)

Cotangent complexes of derived tensor products are accessible in the following way. There is a triangle in $D(B \otimes_A^{\mathbb{L}} C)$,

$$\mathbb{L}_A \otimes_A (B \otimes_A^{\mathbb{L}} C) \rightarrow \mathbb{L}_B \otimes_B (B \otimes_A^{\mathbb{L}} C) \oplus \mathbb{L}_C \otimes_C (B \otimes_A^{\mathbb{L}} C) \rightarrow \mathbb{L}_D \xrightarrow{+1} \dots$$

This fact is the main ingredient used in computing cotangent complexes.

There is the following relation between cotangent complexes and mapping spaces.

Theorem 1. *Let $A, B \in \text{cdga}$ and $f : A \rightarrow B$. Then:*

1.

$$\pi_i(\text{Map}(A, B), f) \simeq \text{Ext}_{D(A)}^{-i}(\mathbb{L}_A, B)$$

for $i > 0$ (which is only an isomorphism of sets when $i = 1$; note that on the right side, the f shows up because we need to consider $B \in D(A)$).

2. If we write $B_{\leq n} = (0 \rightarrow B^n/\text{Im}(d) \rightarrow B^{n-1} \rightarrow \dots \rightarrow B^0 \rightarrow 0)$ for the truncation of B , then the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{f_n} & B_{\leq n} \\ & \searrow \exists f_{n+1} ? & \uparrow \\ & & B_{\leq n+1} \end{array}$$

is controlled by an obstruction

$$o(f_n) \in \text{Ext}_{D(A)}^{n+2}(\mathbb{L}_A, \pi_{n+1}(B));$$

that is, $o(f_n) = 0$ iff f_{n+1} exists, and in that case, the set of such lifts (up to homotopy) is a torsor for $\text{Ext}_{D(A)}^{n+1}(\mathbb{L}_A, \pi_{n+1}(B))$.

3 Derived schemes and stacks

3.1 Derived schemes

One defines derived schemes in total analogy with ordinary schemes.

Definition 10. A *derived scheme* is a pair (X, \mathcal{A}_X) of a topological space and a sheaf of cdgas, such that:

- $(X, \pi_0(\mathcal{A}_X))$ is a scheme;
- $\pi_i(\mathcal{A}_X)$ is quasicoherent for the above scheme;

Example 6. We should first illustrate what an affine derived scheme looks like. Let $A \in \text{cdga}$. Then $(\text{Spec } A, \mathcal{A})$ has for its topological space $\text{Spec}(\pi_0(A))$, and for any $f \in \pi_0(A)$, $\mathcal{A}(D(f)) = A[\bar{f}^{-1}]$ for D denoting the usual “distinguished open” and \bar{f} being any lift of f to A^0 . Of course, this lift is not unique. This ends up meaning that on intersections we may not have equalities, but rather quasi-isomorphisms. Of course, we have a projection $A^0 \twoheadrightarrow \pi_0(A)$, which gives an inclusion $\text{Spec } A \hookrightarrow \text{Spec } A^0$, and we can work entirely on the target.

Proposition 3. *Spec defines a full embedding $\text{dAff}_k \hookrightarrow \text{dSch}$.*

We haven’t actually defined morphisms of derived schemes, because it takes a little bit of heavy machinery. Note also that for topological categories, a *full embedding* is only one whose induced maps on mapping-spaces are *weak equivalences*.

Of course, there is a full embedding $i : \text{Sch} \hookrightarrow \text{dSch}$, given by considering k -algebras as cdgas concentrated in degree 0. This embedding is actually faithful, too, essentially because the homotopy theory of cdgas restricted

to objects concentrated in degree 0 reduces to the usual theory of k -algebras. The embedding has an adjoint $h^0 : \mathbf{dSch} \rightarrow \mathbf{Sch}$, given by $(X, \mathcal{A}_X) \mapsto (X, \pi_0(\mathcal{A}_X))$.

Note that i does not preserve fiber products! But that's sort of the point: fiber product locally corresponds to tensor product, so derived fiber product should locally correspond to *derived* tensor product. By gluing, we defined the *derived fiber product* $X \times_S^h Y$ of derived schemes. This satisfies the following universal property: the diagram

$$\begin{array}{ccc} X \times_S^h Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

commutes up to a chosen homotopy h , i.e. a path in $\text{Map}(X \times_S^h Y, S)$ from the upper composition to the lower composition, and $X \times_S^h Y$ is *universal* with respect to this property. That is, if Z is any derived scheme, then the map of spaces

$$\text{Map}(Z, X \times_S^h Y) \rightarrow \text{Map}(Z, X) \times_{\text{Map}(Z, S)}^h \text{Map}(Z, Y)$$

is a weak equivalence, where \times^h denotes homotopy fiber product in spaces.

Remark 2. This is just a *homotopy limit*. However, \mathbf{dSch} does not have a model structure; rather, the category of spaces with sheaves of cdgas has a model structure, on which \mathbf{dSch} happens to be closed under taking holims.

On the other hand, $h^0(X \times_S^h Y) = h^0(X) \times_{h^0(S)} h^0(Y)$; thus, homotopy fiber product is a strengthening of the usual notion of fiber product.

Remark 3. The π_0 map always induces a map $h^0(X) \rightarrow X$.

Remark 4. The adjunction $i : \mathbf{Sch} \rightleftharpoons \mathbf{dSch} : h^0$ is very similar to the adjunction $\mathbf{Sch}^{\text{reduced}} \rightleftharpoons \mathbf{Sch}$. (In both cases, we're adding some sort of infinitesimal structure.)

Example 7. Let $x : \text{Spec } k \rightarrow X = \text{Spec } A$ be a point (for A a k -algebra). Then the derived self-intersection $x \times_X^h x$ is opposite to

$$B = k \otimes_A^{\mathbb{L}} k \simeq \text{Sym}_k(\mathbb{L}_A \otimes_A k[1]).$$

If A is smooth, then this simplifies to

$$k \otimes_A^{\mathbb{L}} k \simeq \text{Sym}(\Omega_{X,x}^1[1]) = \text{Sym}(\bigwedge^* \Omega_{X,x}^1).$$

Example 8. Let X be a scheme. Then the derived self-intersection of the diagonal

$$\begin{array}{ccc} \mathcal{L}X & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \longrightarrow & X \times X \\ & & \Delta \end{array}$$

is given by $\mathcal{L}X = \text{Spec}(\text{Sym}_{\mathcal{O}_X}(\mathbb{L}_X[1]))$. The \mathcal{L} really does stand for “loopspace”: $\mathcal{L}X \simeq \text{Map}(S^1, X)$, where we consider $S^1 = B\mathbb{Z}$.

In general, it is hard to compute self-intersections $Y \times_X Y$. There is something very specific about the diagonal that makes the above example at all tractable.

3.2 Derived stacks

Here is a first approximation to derived Artin stacks. Morally, we'd like to think about these as the *smooth groupoids* inside \mathbf{dSch} . First, we must define the topological category of topological functors $\mathbf{dAff}^{\text{op}} \rightarrow \mathbf{Top}$ (although we should define the higher descent conditions). Denote this by \mathbf{dSt} , the category of *derived stacks*. There is a full Yoneda embedding $\mathbf{dSch} \rightarrow \mathbf{dSt}$ given by $X \mapsto \text{Map}(-, X)$. Finally, we obtain a *derived Artin stack* as the image (up to equivalence) of a smooth groupoid object in \mathbf{dSch} .

Let us at least attempt to define these last terms. A map $A \rightarrow B$ in \mathbf{cdga} is called *smooth* if:

- B is finitely presented over A (which roughly means that it is equivalent to a *finite* cell object over A);
- $\text{cofib}(\mathbb{L}_A \otimes_A B \rightarrow \mathbb{L}_B)$ is a projective B -dg module.

Then, a *smooth groupoid* is simply one whose source and target morphisms are smooth morphisms.

4 Representability of derived moduli problems

The relation between schemes and stacks is encoded in the following diagram:

$$\begin{array}{ccc}
\text{Sch} & \xrightarrow{i} & \text{dSch} \\
\downarrow & & \downarrow \\
\text{Artin stacks} & \hookrightarrow & \text{der. Artin stacks} \\
\downarrow & & \downarrow \\
\text{Fun}(\text{Aff}_k^{\text{op}}, \text{Gpds}) & \hookrightarrow & \text{dSt} = \text{Fun}(\text{dAff}_k^{\text{op}}, \text{Top}).
\end{array}$$

(The bottom-left we often just call the category of *stacks*.) The truncation h^0 for the top arrow actually extends for all horizontal arrows. We will mainly focus on the middle arrow.

Example 9. Derived Artin stacks have derived fiber products; thus, we can obtain derived Artin stacks by starting with ordinary Artin stacks and taking their derived intersection.

Let us return to our original motivating example of “finite-dimensional B -modules” for B an associative k -algebra. Let us assume that B is finitely presented (as well as an additional condition on its *homological dimension*, i.e. on Ext-groups of B -modules; this is analogous to having a group G such that BG can be taken to have only finitely many cells in each dimension).

Let us define a derived Artin stack that classifies this functor. Let $\mathcal{M}_B : \text{dAff}_k^{\text{op}} \rightarrow \text{Top}$ be defined as follows. Recall that $\text{dAff}_k^{\text{op}}$ is just the category of cell cdgas. Let A be one. Then $\mathcal{M}_B(A)$ should be the *space* of $(B \otimes_k A)$ -dg modules M such that M is projective of finite type over A , i.e. M is a direct summand of some A^p in $D(A)$. We make $\mathcal{M}_B(A)$ into a functor by, for a map $A \rightarrow A'$ of cell cdgas, defining $\mathcal{M}_B(A) \rightarrow \mathcal{M}_B(A')$ by $- \otimes_A^{\mathbb{L}} A'$.

This is a space simply because dg modules form a topological category, and these have classifying spaces; in particular, we will have $\pi_0(\mathcal{M}_B(A))$ equal to the set of isomorphism classes of such modules. Then, $\pi_1(\mathcal{M}_B(A), E) = \text{Aut}_{D(B \otimes_k A)}(E)$, and more generally $\pi_i(\mathcal{M}_B(A), E) \simeq \text{Ext}_{D(B \otimes_k A)}^{1-i}(E, E)$ for $i > 1$.

Theorem 2. \mathcal{M}_B is representable by a derived Artin stack.

Proof. We essentially already saw this in the first lecture. The idea of the proof is to define the forgetful functor $\pi : \mathcal{M}_B \rightarrow \mathcal{M}_k$ by forgetting the B -module structure. But $\mathcal{M}_k = \coprod_n B\text{Gl}_n$, which we know is an Artin stack. So it remains to check the fibers of this map. So, let E be a finite-dimensional k -vector space, and consider $E \in \mathcal{M}_k$. Then $\pi^{-1}(E) = \text{Map}(B, \underline{\text{End}}(E))$, which takes A to $\text{Map}_{k\text{-dg alg}}(B, \text{End}(E) \otimes_k A)$. Thus, using a cell decomposition, we can see that $\pi^{-1}(E)$ is an iterated derived fiber product of affine schemes (namely Mat_n^p). Thus, $\pi^{-1}(E) = \text{Spec } A$ for some $A \in \text{cdga}$. This implies that

$$\mathcal{M}_B \simeq \coprod_n [\text{Spec } A^{(n)} / \text{Gl}_n].$$

□

Let us now move on to derived mapping stacks.

Theorem 3. Let X be a smooth proper variety and Y be a derived Artin stack (which is of finite type over k). Then there exists a derived Artin stack which takes A to $\text{Map}_{\text{ast}}(X \times \text{Spec } A, Y)$.

This is a difficult theorem. It's a consequence of a derived version of Artin's representability theorem, due to Lurie. (For certain Y or for finite X , this can be done by hand; cf. the exercise sheet.)

Definition 11. We call this object the *derived mapping stack*, and denote it by $\mathbb{R}\underline{\mathrm{Map}}(X, Y)$.

If we truncate this, we just get $h^0(\mathbb{R}\underline{\mathrm{Map}}(X, Y)) \simeq \mathrm{Map}(X, Y)$, the usual Artin stack classifying maps. But in general, $\mathbb{R}\underline{\mathrm{Map}}(X, Y) \neq \mathrm{Map}(X, Y)$; the former behaves much better, infinitesimally. For instance, we can completely compute the cotangent complex, whereas it is computationally impossible to access that of the latter. To reach this result, we need to say a bit more about cotangent complexes.

Let X be a derived scheme. Then the cotangent complex will be a sheaf of dg-modules over X . First, we need the topological category $D_{\mathrm{qcoh}}(X)$, the derived category of quasicoherent \mathcal{A}_X -modules. These are just sheaves of \mathcal{A}_X -dg modules \mathcal{E} such that locally on $U = \mathrm{Spec} A$, $\mathcal{E}|_U$ is given by an A -dg module M (via the usual “tilde construction”) up to quasi-isomorphism.

Now, we simply obtain $\mathbb{L}_X \in D_{\mathrm{qcoh}}(X)$ by applying the functor \mathbb{L}_- locally and then gluing; one might write $\mathbb{L}_X = \mathbb{L}_{\mathcal{A}_X}$, i.e. on $U = \mathrm{Spec} A \subset X$, $\mathbb{L}_X|_U \simeq \widehat{\mathbb{L}}_A$.

This extends to derived Artin stacks as follows. Let X be one. We need to figure out what $D_{\mathrm{qcoh}}(X)$ is. Suppose X is given by a groupoid in \mathbf{dSch} , $s, b : X_1 \rightrightarrows X_0$. Then a quasicoherent sheaf on X is the data:

- a qcoh sheaf \mathcal{E}_0 on X_0 ;
- a quasi-isomorphism $\alpha : s^*\mathcal{E}_0 \xrightarrow{\sim} b^*\mathcal{E}_0$ of qcoh sheaves on X_1 ;
- higher coherence data.

Now, we define $\mathbb{L}_X \in D_{\mathrm{qcoh}}(X)$ to be given by

$$(\mathbb{L}_X)|_{X_0} = \mathrm{hofib} \left(\mathbb{L}_{X_0} \xrightarrow{\mathbb{L}_b} e^*(\mathbb{L}_{X_1/X_0}) \right).$$

(We must show that this is equivariant, of course.)

Theorem 4. *Let X be a smooth proper variety and Y be a derived Artin stack; this gives us a diagram*

$$\begin{array}{ccc} \mathbb{R}\underline{\mathrm{Map}}(X, Y) \times X & \xrightarrow{ev} & Y \\ \downarrow p & & \\ \mathbb{R}\underline{\mathrm{Map}}(X, Y) & & \end{array}$$

Then the tangent complex is given by

$$\mathbb{L}^\vee = \mathbb{T}_{\mathbb{R}\underline{\mathrm{Map}}(X, Y)} \simeq p_* ev^*(\mathbb{T}_Y).$$

This doesn't work in the non-derived setting; as far as is known, there isn't even a non-derived formula.

Theorem 5. *Recall our functor \mathcal{M}_B . Then*

$$\mathbb{T}_{\mathcal{M}_B} \simeq \mathbb{R}\underline{\mathrm{End}}_{B\text{-dg mod}}(\mathcal{E})[1],$$

where \mathcal{E} is the universal B -dg module on \mathcal{M}_B .

Again, this is wrong in the non-derived setting.

We care about tangent complexes because we care about *virtual classes*, but we won't have time to discuss these. Rather, we'll spend the last few minutes discussing symplectic forms on derived mapping spaces. These come from a nondegenerate pairing on tangent complexes. Let us begin by explaining how they come out of the formulas we've seen.

Example 10. Let X be a Calabi-Yau variety of dimension d , and let $Y = BG$ for G a reductive group equipped with a *trace* map $\mathrm{Tr} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow k$, i.e. a nondegenerate G -invariant bilinear form. (So e.g. we can take $G = \mathrm{GL}_n$, with the usual trace.) Then, $\mathbb{R}\underline{\mathrm{Map}}(X, BG)$ is the derived stack of G -bundles on X ; we often write this as $\mathbb{R}\mathrm{Bun}_G(X)$.

Take $V \in \mathbb{R}\mathrm{Bun}_G(X)$. Then

$$\mathbb{T} = \mathbb{T}_{\mathbb{R}\mathrm{Bun}_G(X), V} \simeq H^*(X, \mathrm{ad}(V))[1].$$

(Here, $\mathrm{ad}(V)$ is just a twisted version of the Lie algebra.) Since X is CY of dimension d , there is a nondegenerate pairing $\mathbb{T} \wedge \mathbb{T} \rightarrow k[2-d]$ which is induced by

$$H^*(X, \mathrm{ad}(V)) \otimes H^*(X, \mathrm{ad}(V)) \xrightarrow{[-, -]} H^*(X, \mathrm{ad}(V)) \xrightarrow{\mathrm{Tr}} H^*(X, \mathcal{O}_X) \rightarrow H^d(X, \mathcal{O}_x)[-d] \simeq k[-d].$$

(This is alternating by the graded sign rules.) Using this form, we obtain $\theta_\omega : \mathbb{T} \simeq \mathbb{L}[2-d]$. By Grothendieck duality, this can be globalized to yield a nondegenerate pairing

$$\omega : \mathbb{T}_{\mathbb{R}\mathrm{Bun}_G(X)} \wedge \mathbb{T}_{\mathbb{R}\mathrm{Bun}_G(X)} \rightarrow \mathcal{O}_X[2-d]$$

which induces

$$\mathbb{T}_{\mathbb{R}\mathrm{Bun}_G(X)} \xrightarrow{\sim} \mathbb{L}_{\mathbb{R}\mathrm{Bun}_G(X)}[2-d]$$

considered in $D_{\mathrm{qcoh}}(X)$.

Interpreted correctly, we're describing a 2-form (with some shift) on this Artin stack. In fact, ω is “closed” in the appropriate sense! Thus, it defines a symplectic structure on $\mathbb{R}\mathrm{Bun}_G(X)$ of degree $2-d$. For instance, in dimension 3, we obtain an important piece of the story for Donaldson-Thomas theory.