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ZOLI

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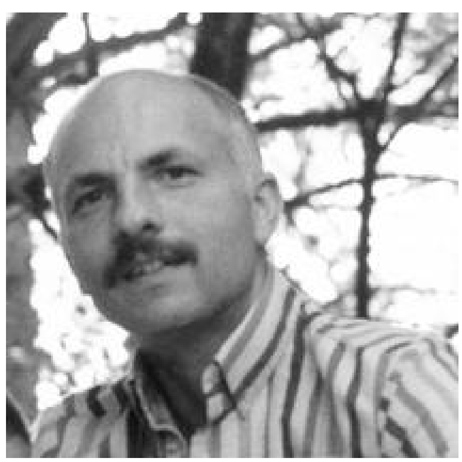
1. Introduction

Zoltan "Zoli" Tibor Balogh died at his home in Oxford, Ohio, in the early morning hours of Wednesday, June 19, 2002. He was 48 years old. In this article, we give a brief sketch of his life and then discuss his mathematical contributions. He will be sorely missed, both as a leader in the field of set-theoretic topology and as our friend.



2. Biographical snapshot

Zoli was born on December 7, 1953, in Debrecen, Hungary, the son of Tibor Balogh and Ilona Kelemen. His father, a mathematician working in the area of "matrix-valued stochastic processes," was a professor at Kossuth University in Debrecen. His mother had a graduate degree in chemistry and also was a professor at Kossuth University. A younger sister Agnes later acquired an MD in the field of internal medicine and is currently a practicing physician in Debrecen. Zoli grew up in Debrecen, attending the local elementary schools and high-school.

In 1972, Zoli began his university education by entering Lajos Kossuth University as a mathematics student in the Faculty of Sciences and received the B.Sc+ degree in 1977, completing a five year program. This degree would be comparable to a very strong Master of Science degree in the US, with a research specialization in topology. Indeed, his research ability began to show up early– Zoli presented a paper, *Relative compactness and recent common generalizations of metric and locally compact spaces*, at the Fourth Prague Topological Symposium in 1976, in which he introduces the concept of "relative compactness." (The paper, later published in the conference proceedings, was the precursor of the paper by the same name published in Fundamenta Mathematicae in 1978.) In 1977, he received the **Renyi Kato Memorial Prize**, awarded by the Bolyai Janos Mathematical Society to outstanding young researchers in mathematics.

Zoli continued his graduate studies at Kossuth University from 1977 to 1980, during which time he was a Teaching Assistant and then a Research Fellow. A very active research period for Zoli, he gave one or more presentations at international conferences each year and produced results which account for approximately five publications. Certainly, his results were beginning to be noticed by others in the field. In 1979, the Bolyai Janos Mathematical Society presented Zoli with the **Grunwald Geza Memorial Prize**, an annual award to outstanding researchers under the age of 30. Zoli received his Doctorate from Kossuth University in 1980 and was awarded the Ph.D. in Topology and Set-theory by the Hungarian Academy of Sciences.

Zoli's time at Kossuth University was not entirely devoted to academics and research. In 1976, he married Eva Balicza, a classmate. In 1978, their first daughter Agnes was born, followed by Judit in 1979. Eva and Zoli were divorced in 1981.

He remained at Kossuth University as a Research Fellow and Senior Research Fellow from 1980 until May 1984. During this period, his results began to show more of the use of notation and techniques from set-theory and more use of set-theoretic axioms.

The summer of 1984 took Zoli to the University of Toronto, Canada, as a Visiting Professor for his first significant visiting position outside of Debrecen. Although only a three month visit, this period accounted for a significant boost in his interest in the use of set-theory applied to topological problems. He returned to Kossuth University from September through December 1984.

The next eighteen years would certainly be considered as "life-changing" in both his personal and his mathematical life. In November of 1984, Zoli and Agnes Polgar were married in Debrecen. Agnes was a chemistry student at Kossuth University, one semester away from finishing her program. In January 1985, less than two months after their marriage, they were on their way to Lubbock, Texas, where Zoli had a Visiting Associate Professor position waiting at Texas Tech University, where they remained for one and half years. In June of 1985 Zoli was faced with his first life-threatening medical emergency when he underwent open heart surgery for a bypass operation. Zoli had been having some chest pains and only a few days earlier, he and a friend had noticed an unusual shortness of breath. While in the hospital, being prepared for surgery, Zoli had a heart attack and only the proximity of immediate medical help saved his life. He was 31 years old. In every other way, Zoli was a healthy young man–certainly, his weight was good and he was a non-smoker. Because his father died at an early age, Zoli knew that he had a genetic predisposition for circulatory disease. He began to watch his diet more carefully and for many years became an avid practitioner of various types of aerobic exercises, including walking, running, and bicycling.

In July 1986, Zoli and Agnes moved back to Debrecen where Zoli served as an Associate Professor of Mathematics at Kossuth University until the summer of 1988. These were a busy two years: Agnes finished her degree in chemistry at Kossuth University ; their older son Adam was born in 1987; and Zoli continued to work with notable success on his research.

During this period (1987) Zoli also received a **Certification in English-Hungarian Translation of Mathematics** from Kossuth University. In 1988, he submitted his dissertation entitled (English translation) *Set-theoretic investigationson the classes of compact and locally compact spaces* for the Habilitation, which was conferred by the Hungarian Academy of Sciences in 1989. His original plans were to spend two years (academic years 1988-1990) in visiting positions in the US before returning to Debrecen; however, this was the last extended period of time that Zoli would spend in Hungary.

For the 1988-89 academic year the Department of Mathematics and Statistics at Miami University was fortunate to have Zoli as its **Distinquished Visiting Professor**. Next, he spent the fall semester of 1989 as a Visiting Associate Professor at the University of Wisconsin in Madison. This was a very exciting semester for him because several other topologists and set-theorists were there at the same time: Mary Ellen Rudin and Ken Kunen from Wisconsin, along with visitors Gary Gruenhage (Auburn University), Takao Hoshina (Tsukuba University), David Fremlin (University of Essex), and Adam Ostaszewski (London School of Economics). Also in 1989, one of his most important papers, solving the Moore-Mrowka problem (*On compact Hausdorff spaces of countable tightness*), appeared in the *Proceedings of the AMS*.

Zoli returned to Miami University as a Visiting Associate Professor in the spring semester of 1990, which would have completed his original plan of spending two years visiting in the US. However, Miami University and Zoltan Balogh were becoming mutually compatible and in the spring of 1990 the Department offered Zoli a tenure track position. He accepted the position and joined the Department as an Associate Professor in August 1990. Adding to this exciting turn of events, Zoli's younger son Daniel was born in January 1991. Zoli received tenure and promotion to Full Professor in 1994. Zoli's twelve years (1990–2002) at Miami University were a very productive research period. He continued to solve difficult and well-known problems at an amazing rate.

Unfortunately, Zoli's health remained precarious and his second life-threatening medical emergency occurred in the summer of 1999.

While riding his bicycle outside of Oxford, Ohio, on a late morning in July he suffered a massive stroke and collapsed near the road. Luckily, he was quickly found by a passerby who called an ambulance from a nearby farmhouse. The local emergency room physician quickly diagnosed the problem, started an intravenous dose of the experimental anticlotting drug t-Pa, and had Zoli transported by helicopter to the University Hospital in Cincinnati. After a further diagnosis revealed a clot still in his brain, the Cincinnati Stroke Team injected a dose of t-Pa directly into the clot. This remarkable procedure quickly alleviated the apparent symptoms of the stroke and ultimately provided for the full recovery of Zoli's cognitive abilities. (An accounting of this incident was described in a two-page article in *People Magazine*, March 6, 2000, titled "The Living Proof.") Just a few weeks after the stroke Zoli was on an airplane to New York, accompanying two Miami University graduate students to the 1999 Summer Conference on Topology and its Applications.

On the morning of his death, Zoli was scheduled to leave for Japan where he was invited to lecture in Tsukuba and then continue on to Matsue where he was an invited speaker for the International Conference on Topology and its Applications. It is unfortunate that he was not able to attend this last conference. Much of the joy of his mathematical life was meeting new friends and greeting old friends, and talking about mathematics with them.

3. Zoli's research

Zoli's research spans over 25 years and includes many significant contributions in diverse areas of set-theoretic topology.[1] To help organize our discussion of his work, we have divided many of his papers into five "themes" which run through much of his work. It is not surprising that a researcher as strong and broad as Zoli would also have many papers that cannot be conveniently classified, so there is also a relatively large "miscellaneous" category.

Zoli's research especially stands out due to a series of solutions to several long-standing problems in the field, which he obtained at an amazing pace starting in the mid-1980s and continuing essentially

[1]This part of our article has also appeared, in somewhat condensed form, in the Hungarian journal *Publicationes Mathematicae Debrecen.*

until his death. We have singled out for special discussion six of his most remarkable results, which for easy identification we call his "greatest hits."

Our discussion of Zoli's research will be roughly chronological within themes. See Section 4 for a complete list of Zoli's papers.

3.1. Zoli's early work: relative compactness and hereditarily nice spaces

The 1960s were a kind of golden age for so-called "generalized" metrizable spaces. A. V. Arhangel'skiĭ defined p-spaces; K. Nagami defined Σ-spaces; K. Morita, M-spaces; and so on. Investigations of these classes, sometimes with an eye toward generalizing what were by then "classical" results in the area, were still going strong in the mid 1970s, when Zoli came on the scene. R. Hodel had generalized some metrization results to higher cardinals by defining "metrizability degree" and put them in the language of cardinal function theory[2], a hot topic at that time. Another topic of interest posed the question: What can be said of the whole space if one knows that every subspace is "nice" in the sense of belonging to a certain class of generalized metric spaces?

Zoli's first contributions of his career were in this area. Recall that a space X is a paracompact p-space if there is a perfect (i.e., closed with compact fibers) map f from X to a metrizable space Y. Let τ be the topology on X and τ' the weaker topology on X obtained by pulling back the metrizable topology on Y by the function f. Then if a filter on X has a cluster point in the topology τ', it is easy to see, using the perfectness of the map f, that the filter also has a cluster point in τ. Zoli's nice idea [Ba76][Ba78a][Ba79a] was to study exactly this relationship between topologies, calling τ *relatively compact* to τ' if they satisfy the above filter convergence condition. There was also a countably compact analogue [Ba79c], defined in terms of filters having a countable base. Zoli noticed that in many cases, especially when the space had a point-separating open cover of some sort, or when every subspace was "nice" in

[2]Typical cardinal functions that appear in our discussion are the weight $w(X)$, the character $\chi(X)$, and the Lindelöf degree $L(X)$, which are, respectively, the least cardinal of a base, the least cardinal such that each point has a local base not greater than that cardinality, and the least cardinal such that every open cover of X has a subcover of that cardinality.

the sense of being relatively compact to the topology τ, various cardinal functions on τ' were a bound for those of τ. General results of this form for relative compactness, and the similar notion of relative countable compactness, had many corollaries which superseded classical results and answered questions of Arhangel'skiĭ, Hodel, and others. The following example gives the flavor:

Theorem 3.1. [Ba76]. *Suppose (X,τ) is compact relative to a weaker topology τ' with metrizability degree $\leq \kappa$. If (X,τ) has a point $\leq \kappa$, T_1-separating open cover, then the metrizability degree of (X,τ) is $\leq \kappa$.*

Taking $\kappa = \omega$ and $\tau = \tau'$ gets J. Nagata's classical result that a paracompact p-space with a point-countable T_1-separating open cover is metrizable. A similar theorem with metrizability degree replaced by weight has cardinal function results of Hodel as corollaries, and answers a question of Arhangel'skiĭ on spaces whose every subspace is a paracompact p-space.

The proofs of these early results of Zoli already showed the style which he became well-known for later, involving heavy use of complicated combinatorics of sets and collections of sets, the arguments slowly but steadily making their way towards the final conclusion. The power of his mind was evident from the beginning!

Zoli also used relative compactness to investigate F_{pp}-spaces, i.e., spaces that are hereditarily paracompact p-spaces. A little later [Ba79b], he replaced relative compactness with "perfect equivalence relations," a somewhat related notion but more specifically relevant to the study of F_{pp}-spaces. He answered several questions of Arhangel'skiĭ about this class and came close to a complete characterization, but was beaten to it by E. G. Pytkeev. This is the only case we know of where a result of Zoli has been significantly superseded by someone else. Undeterred, Zoli went on to write a very nice paper [Ba84] on hereditarily strong Σ-spaces, even showing in that paper that an analogue of Pytkeev's F_{pp} characterization does not hold for this class.

3.2. Q-set spaces

Zoli had a long-standing interest in *Q-sets*, i.e., uncountable subsets X of the real line (or separable metric spaces) in which every subset X is a relative G_δ-set. The appropriate generalization to arbitrary spaces is that of a *Q-set space* X,which means that every subset of X is a G_δ-set in X, yet X is not σ-discrete (or equivalently in this situation, not σ-closed discrete). Zoli's first paper to mention Q-set spaces was [Ba78b], where he showed that if an F_{pp}-space was non-metrizable, it was because it contained either the one-point compactification of an uncountable discrete space, or the Alexandrov duplicate of a metric Q-set space (this was the relation to the normal Moore conjecture he refers to in the title).

Zoli's first paper studying a kind of Q-set type of space for itself was with H. Junnila in 1983 [BJ], where the authors consider "totally analytic" spaces, i.e., spaces in which every subset is analytic. In 1980, R. Hansell had shown that under Gödel's Axiom of Constructibility $\mathsf{V} = \mathsf{L}$, a totally analytic space X of character $\leq \omega_1$ is σ-discrete if its product with the irrationals is normal. Also in 1980, G. M. Reed showed that under $\mathsf{V} = \mathsf{L}$, there are no first-countable normal Q-set spaces. The Balogh-Junnila paper shows that the condition about the product with the irrationals in Hansell's theorem may be simply dropped; i.e., under $\mathsf{V} = \mathsf{L}$, every totally analytic space of character $\leq \omega_1$ is σ-discrete. Further, with no character restriction, every totally analytic space (under $\mathsf{V} = \mathsf{L}$) is the countable union of left-separated subspaces.

All of the results mentioned above borrow heavily from W. G. Fleissner's work on the seemingly unrelated problem of collectionwise Hausdorffness in normal first-countable spaces. A key idea in the Balogh-Junnila paper is to use mappings from X into ω^ω to code separations of X into analytic sets (corresponding to Fleissner's coding separations of a discrete set by mappings into $\omega_1 = \chi(X)$). They also use such mappings to define a property formally weaker but actually equivalent to left-separatedness, and show that, if X has underlying set κ and κ is the least such that the theorem fails, then the set of all $\alpha < \kappa$ which witness that this version of left-separatedness at α fails is stationary. This sets the authors up to apply Fleissner's "$\diamondsuit$ for stationary systems," etc., to obtain their results.

Of course, this left open the problem if there could be a totally analytic non-σ-discrete space, or even a Q-set space, in ZFC. Zoli finally settled this [Ba91b] by constructing a ZFC example of a Q-set space of cardinality $\mathfrak{c}$ and character $2^{\mathfrak{c}}$. This was his first use of a technique of M. E. Rudin which he later went on to develop into his amazing example-constructing machine. More on this in Section 3. Later [Ba98a], he saw how to obtain a paracompact Q-set space, and in a handwritten note, unpublished at the time of his death, he obtained a Lindelöf Q-set space.

Zoli had one other paper dealing with Q-sets, with J. Mashburn and P. J. Nyikos [BMN]. It is known that the Pixley-Roy hyperspace $PR(X)$ of a separable metric space X is normal iff X^n is a Q-set for every $n \in \omega$. An X with this property is called a *strong* Q-set. H. Tanaka considered this in the non-separable case and showed that for arbitrary metric X, $PR(X)$ is normal iff every *symmetric* subset of X^n is G_δ in X^n for every n; he called such an X an *almost strong* Q-set. This begs the question: Are almost strong metric Q-sets strong? Zoli and his coauthors give a positive answer, in ZFC, to this question.

3.3. Locally nice spaces

Zoli had a long-standing interest in spaces that are "locally nice," usually in the sense of being locally compact, sometimes also locally connected or a manifold. His earliest paper in this area is "Locally nice spaces under Martin's Axiom" [Ba83]. The following results are probably the most fundamental ones here:

Theorem 3.2. (MA_{ω_1}). *Any locally countable, cardinality ω_1 subset of a countably tight compact space is σ-discrete. (Fremlin improved this to cardinality $< \mathfrak{c}$.)*

Theorem 3.3. *Let X be a locally compact space X of countable tightness. Then the one point compactification of X is countably tight iff X does not contain a perfect pre-image of ω_1.*

Theorem 3.4. *Let X be a locally compact, locally hereditarily Lindelöf, hereditarily collectionwise-Hausdorff space. Then X is paracompact iff X does not contain a perfect pre-image of ω_1.*

The first theorem above is a very useful extension of Z. Szentmiklossy's breakthrough result that there are no locally compact, countably tight S-spaces, while the third has the results of Rudin, G. Gruenhage, Junnila, and D. Lane on paracompactness in perfectly normal manifolds or more generally locally compact spaces as corollaries. Zoli's results here are fundamental structural results which are still finding important uses, e.g., in the recent work of P.

Larson and F. Tall where they solve a long-standing problem of S. Watson by proving that, consistently, all perfectly normal locally compact spaces are paracompact.

In a paper that hadn't appeared at the time of his death, Zoli [Ba02a] obtains a result closely related to Theorem 3.4: Under MA_{ω_1} together with Axiom R, a consequence of PFA, if X is locally compact and hereditarily strongly ω_1- collectionwise-Hausdorff, then X is paracompact iff X does not contain a perfect pre-image of ω_1. The "strongly" assumption is used in part to get hereditarily *ccc* boundaries, enabling Zoli to apply results in his 1983 paper. Of course, Axiom R is used to get by with "ω_1-cwH" instead of full cwH.

In 1986, Zoli published two more papers on the theme of paracompactness in locally nice spaces. In [Ba86a], he answers a question of Gruenhage by showing that normal, locally connected, rimcompact, metalindelöf spaces are paracompact. The paper [Ba86b] starts with answers to questions of Tall and Watson by showing:

Theorem 3.5. *Normal locally compact (or more generally, locally Lindelöf) screenable spaces are paracompact.*

Theorem 3.6. ($\mathsf{V} = \mathsf{L}$). *Normal, locally compact, metalindelöf spaces are paracompact.*

The first result, to be discussed in subsection 3.5, is a pretty partial result on Nagami's famous problem: Whether the statement is true without the "locally compact" assumption (see that subsection for the definition of "screenable"). The second extends Watson's result in which "metacompact" replaces "metalindelöf." Both results essentially follow from Zoli's ZFC result that locally Lindelöf submetalindelöf spaces in which closed discrete collections of points have σ-locally countable expansions are paracompact. The remaining parts of this paper involve results of a similar general flavor when the whole space is assumed to be connected.

Zoli's paper [Ba88b] represents the culmination of a line of research by several authors exploring countably paracompact analogues of Fleissner's deep result that, under $\mathsf{V} = \mathsf{L}$, normal first-countable spaces are cwH, and Watson's extension of this to a result implying that, under $\mathsf{V} = \mathsf{L}$, normal locally compact spaces are cwH. Watson had obtained the countably paracompact analogue of Fleissner's result, and made the natural conjecture that his result on normal locally compact spaces also had a corresponding countably paracompact analogue; i.e., he conjectured that countably paracompact, locally compact spaces are cwH under $\mathsf{V} = \mathsf{L}$. Zoli, who was a master at taking difficult results of others and pushing them well beyond their supposed limits, verified Watson's conjecture by extending the techniques of Fleissner and Watson in a highly non-trivial fashion.

The first of Zoli's "greatest hits" that we get to in this article happens to be in the locally nice theme. The normal Moore space conjecture had been shown to be essentially equivalent to the question whether normal first-countable spaces must be collectionwise normal[3]. It had been known for some time that the normal Moore conjecture was consistently false. Assuming the existence of sufficiently large cardinals, the normal Moore conjecture was finally shown to be consistently true, and hence independent, first by P. Nyikos and K. Kunen, and a bit later using a more flexible technique, by A. Dow, Tall, and W. Weiss. The analogous problem for locally compact spaces was formulated by Watson and worked on over a period of years by Tall, who obtained a number of positive partial results in which typically the character of the space was bounded by some cardinal (e.g., $\aleph_\omega$). It is Tall's work on this problem that apparently prompted Zoli to refer to it as the "Toronto project" in his paper. Here's the result:

Greatest Hit # 1: *It is consistent (modulo sufficiently large cardinals) that all locally compact normal spaces are collectionwise normal* [Ba91a].

More specifically, the statement holds in any model obtained by adding supercompact many Cohen or random reals. The basic argument is forcing and reflection in the spirit of Dow-Tall-Weiss, but it took significant insight to see how to apply it in this situation, and much work to follow the ideas through to get the complete solution to this problem. In his review of this paper, Watson calls this result "one of the finest results of the last few years in general topology."

We should point out that the result actually proven is much more general. E.g., "point-countable type" can replace "locally

[3]A space X is *collectionwise normal* if every closed discrete collection of closed sets can be separated by a pairwise-disjoint collection of open sets.

compact," giving a result that also implies the normal Moore conjecture. Zoli also obtained a countably paracompact analogue of the result which generalized Burke's proof that, modulo large cardinals, it is consistent that countably paracompact Moore spaces are metrizable.

3.4. Base-multiplicity

Zoli's early work included, in particular, results about point-countable (or more generally, point-$\leq \kappa$) bases or point-separating open covers. There are also results from this period on what Zoli later called a "jigsawed base" and "point-separating jigsawed cover." A *jigsawed point-separating cover* is a collection $\mathcal{P}$ of subsets of X such that, for any $x, y \in X$, there is a finite subcollection $\mathcal{F}$ of $\mathcal{P}$ with $x \in \cup\mathcal{F}^o \subset \mathcal{F} \subset X\backslash\{y\}$ (*jigsawed base* is defined similarly). Burke and Michael first used the notion of jigsawed base to obtain a nice proof of Filippov's difficult result that point-countable bases are preserved by perfect maps. They also showed that compact spaces of countable tightness having a point-countable jigsawed base are metrizable. They asked if the result generalized to countably compact, and Zoli showed it did [Ba79c]. (Note: Burke and Michael had it for countably tight.) This improved results of several other topologists by implying that a space having a point-countable point-separating jigsawed cover is metrizable if it is an M-space, and is a σ-space if it is Σ.

Zoli retained an interest in similar "base-multiplicity" topics throughout his professional life. Probably his most interesting work in this area are the results (with S. Davis, W. Just, S. Shelah, and P. Szeptycki) in [BDJSS]. The primary motivation for the results in this paper is an old (circa 1976) question of Heath and Lindgren: Does every first-countable space with a weakly uniform base have a (possibly different) point-countable base? Recall that a base $\mathcal{B}$ is *weakly uniform* if the intersection of any infinite subcollection of $\mathcal{B}$ is either empty or a singleton.

Old partial results of Davis, Reed, and M. Wage say that there is a counterexample under $\mathsf{MA}(\omega_2)$, though the answer is positive in ZFC if there are not more than $\aleph_1$-many isolated points. These results already suggest that some interesting combinatorics are at the heart of this problem. Much later, Arhangel'skiĭ, Just, E.

Reznichenko, and Szeptycki showed that, under CH, every first-countable space with a weakly uniform base and no more than $\aleph_\omega$-many isolated points has a point-countable base.

In [BDJSS], the authors finish off the problem, obtaining a consistent positive answer with no restriction on the number of isolated points. They introduce the axiom CECA, which is equivalent to GCH plus a bit of $\square_\lambda$ for singular λ and thus follows from $\mathsf{V} = \mathsf{L}$, and show that the following holds:

Theorem 3.7. (CECA). *A space X has a point-countable base if it is first-countable and has a base $\mathcal{B}$ such that, for every infinite subset A of X, some finite subset of A is included in only finitely many members of $\mathcal{B}$.*

A weak uniform base could reasonably be called 2-*in-finite*, since any *two* distinct points are in only *finitely* many members of the base. This suggests notions of n-in-finite, ω-in-countable, etc. Note that the stated base condition is weaker than n-in-finite for any fixed n; it is sometimes called $< \omega$*-in-finite*.

The topological result follows from the following combinatorial result:

Theorem 3.8. (CECA). *Suppose $\mathcal{A} = \{A_\alpha\}_{\alpha<\lambda}$ is such that every infinite subcollection $\mathcal{B}$ of $\mathcal{A}$ has a finite subset $\mathcal{F}$ such that $|\cap\mathcal{F}| < \omega$. Then there are $A'_\alpha \in [A_\alpha]^{\leq\omega}$ such that $\{A_\alpha \backslash A'_\alpha\}_{\alpha<\lambda}$ is point-finite.*

The above is one of many results in this paper that say that under certain conditions, a collection of sets can be made point-countable (say) by shaving off a small number of points from each. These are strong combinatorial results which should have many other applications in topology.

In [BG01], Zoli generalized the classical result that compact spaces with a point-countable base are metrizable by showing the same holds for an ω-in-countable base[4] . With his undergraduate student J. Griesmer, he showed [BGri] that this fails under CH for jigsawed bases, but is true in ZFC if the jigsawed base is $< \omega$-in-countable instead of just ω-in-countable. The countably compact case, which turns out to depend on set-theory, is also discussed in these papers.

[4]This result of the paper was due entirely to Zoli.

Getting back to the classical notion of point-countable base, Zoli became very interested in the following question: Is it consistent that a first-countable space X must have a point-countable base if every subspace of cardinality $\leq \omega_1$ does? A positive answer (which must be a consistency result depending on large cardinal axioms, as any non-reflecting stationary set in ω_2 is a counterexample) would consistently solve "the" point-countable base problem of P. J. Collins, Reed, and A. Roscoe.

Zoli obtained some interesting partial results, in two papers that have appeared since his death. In [Ba03a], he shows that for spaces of density $\leq \omega_1$, the answer to the above question is "yes," in ZFC. A corollary is Dow's reflection theorem that a compact space is metrizable if every $\leq \omega_1$-sized subspace is metrizable. Again, a non-reflecting stationary set gives (via a ladder space) a counterexample to the locally compact analogue of Dow's theorem. Zoli obtains a (consistent modulo large cardinals) locally compact analogue in [Ba02a]: under Axiom R, a locally compact space is metrizable if every subspace of cardinality $\leq \omega_1$ has a point-countable base.

3.5 Dowker spaces

A classical homotopy extension theorem of K. Borsuk had as part of the hypothesis that $X \times [0,1]$ is normal. But it was not known at the time if normality of X was sufficient to imply normality of $X \times [0,1]$. In 1951, C. H. Dowker characterized those normal spaces X whose product with the unit interval $[0,1]$ is not normal as precisely those normal spaces which are not countably paracompact. He asked if such spaces, soon to be called Dowker spaces, exist. In 1971, Rudin constructed a Dowker space. But this was far from the end of the matter, because it turned out that the Dowker pathology was present in many natural topological problems. Thus, it was important to search for "nice" Dowker spaces. Rudin's example failed to be nice in many ways. In particular, it was not "small" in the sense of cardinality or weight (which were $\aleph_\omega^\omega$), or character (which was $\aleph_\omega$). Many Dowker spaces that were small, and/or "nice" in other ways, were constructed assuming various axioms beyond ZFC. E.g., Rudin herself constructed a Dowker manifold (non-metrizable, of course) assuming CH. But for decades the only known ZFC Dowker space was still Rudin's 1971 example.

So Zoli's 1996 example of an entirely new ZFC Dowker space was very exciting and certainly deserves the "greatest hit" label.

Greatest Hit # 2: *A σ-discrete Dowker space of cardinality $\mathfrak{c}$ in* ZFC [Ba96].

J. E. Vaughan, in his review of Zoli's paper, calls this result "a milestone in set-theoretic topology." Indeed it was, not only for its properties stated above, or just that it was the first new ZFC Dowker space in a quarter century, but also even more for the technique, which Zoli subsequently applied, in highly non-trivial fashion, to obtain solutions of long-standing problems of K. Nagami and K. Morita, which we also are calling greatest hits.

Greatest Hit # 3 : *Solution to Nagami's problem: There is a normal screenable non-paracompact space* [Ba98b].

R. H. Bing defined a space to be *screenable* if every open cover has a σ-disjoint open refinement. In 1955, Nagami explicitly asked the natural question whether normal screenable spaces are paracompact. It is easily seen that normal, countably paracompact, screenable spaces are paracompact, so a counterexample, if it exists, must be a Dowker space. In 1983, Rudin obtained an example under $\diamondsuit^{++}$. In 1998, Zoli finally settled the problem with an example in ZFC. Applying a result of Rudin to Zoli's example shows that there is also a normal screenable space which is not even collectionwise normal.

Greatest Hit #4 : *Morita conjectures established: X is metrizable iff its product with every Morita P-space is normal* [Ba01b].

In 1976, Morita stated three basic conjectures about normality in products. The first one, that $X \times Y$ is normal for all normal Y iff X is discrete, was solved in the affirmative by Rudin in 1978. The second conjecture states that X is metrizable iff $X \times Y$ is normal for every space Y such that $Y \times M$ is normal for every metric space M (such Y are called Morita P-spaces). The third conjecture, which is implied by the second, is that X is metrizable and σ-(closed locally compact) iff $X \times Y$ is normal for every countably paracompact normal space Y. K. Chiba, T. Przymusinski, and Rudin showed that the second conjecture (and hence all) is true if, for each uncountable cardinal κ, there is a space X whose product with every metric space is normal, such that X has an open cover which is increasing in type ω_1 but has no refinement by at most κ-many closed sets. Such examples were constructed by A. Beslagic and Rudin in 1985 under $\mathsf{V} = \mathsf{L}$. But there was no ZFC solution to the problem until Zoli, using another version of his Dowker space technique, constructed spaces in ZFC having the same properties as those constructed by Beslagic and Rudin under $\mathsf{V} = \mathsf{L}$. This is an outstanding achievement which finally settles the conjectures of Morita in the affirmative in ZFC.

The last three major results above are based on the same fundamental technique which, as Zoli mentions in his Dowker space paper, goes back to a reflection-type technique Rudin used to solve a problem of Dowker on the existence of a normal non-cwH simplicial complex. Zoli revamped Rudin's technique through the use of elementary submodels (and similar objects such as what he termed "control pairs") and then proceeded to extend it, first to obtain Q-set spaces in ZFC, then his new Dowker space, followed by the solutions to the problems of Nagami and Morita, among others. But each of these problems had its own special difficulties which needed major new insights to add to the basic technique. After all, we all knew and at least thought we understood Zoli's Dowker space, but only Zoli himself was able to use the technique to solve Nagami's problem a couple of years later, and years after that settle Morita's conjectures. At a certain point, the technique changed from being an extension of a technique of Rudin to being Zoli's technique. Just where that point is may be open to interpretation, but we would place it no later than his ZFC Dowker space. At that point, at least, we say the technique became Zoli's technique.

Since this technique is so important, and will surely be a major part of Zoli's mathematical legacy, we give a rough description of it here, using the normal screenable space construction as an example. The set for the space is $\mathfrak{c} \times \omega$, with $\mathfrak{c} \times n$, $n < \omega$, the countable open cover which witnesses non-countable paracompactness. The topology is made to be normal and screenable in a kind of random way by inductively adding open sets to the topology in $2^{\mathfrak{c}}$ steps. Since the space has cardinality $\mathfrak{c}$, there are $2^{\mathfrak{c}}$ many potential pairs of disjoint closed sets and $2^{\mathfrak{c}}$ many potential open covers, and if any one of these potential objects ends up being a real pair of closed sets or open cover, it does so at some stage prior to $2^{\mathfrak{c}}$, and is "taken care of" by adding an open separation for disjoint closed sets or a screening of an open cover. Just how these open separations or screenings are introduced is determined by a listing in type $\mathfrak{c}$ of what Zoli calls "control pairs," which are essentially pairs of countable elementary submodels M, N with $M \in N$, together with a function which diagonalizes over N. In an induction of length $\mathfrak{c}$ at a given stage γ of the main induction of length $2^{\mathfrak{c}}$, Zoli looks at each $\{\beta\} \times \omega \subset X$ and uses the β^{th} control pair to decide in which member of the open separation or screening (depending on what it is that is given at stage γ) each point (β, k) is to be put. So the space ends up being normal and screenable simply because it is made to be in the induction. The difficult part of the argument is the proof that the space fails to be countably paracompact; this is where the control pairs come in. The randomization has been controlled just enough by the diagonalizations through these control pairs so that one can prove (with considerable difficulty, however!) that no closed shrinking of the $\mathfrak{c} \times n$'s has been introduced. A rough idea is the following. Suppose $\{H_n\}_{n\in\omega}$ is a closed shrinking of $\{\mathfrak{c} \times n\}_{n\in\omega}$. There is a countable set Z of stages in which, for some $n \in \omega$, the pair $\{H_n, X \setminus (\mathfrak{c} \times n)\}$ is a pair of disjoint closed sets that is being considered for an open separation. But the reflection argument finds a $\beta \in \mathfrak{c}$ such that $\{\beta\} \times \omega$ is not split by the open sets added at the stages in Z, contradicting that for some n, both H_n and $X \setminus (\mathfrak{c} \times n)$ meet $\{\beta\} \times \omega$.

As Zoli himself points out in [Ba03b], the idea of adding open sets at each step to guarantee normality, etc., as above is not apparent in his original Dowker space construction, which used instead some simplifying tricks due to Watson. However, he left a set of notes (see [Ba03d]) entitled "A natural Dowker space," in which he presents his basic Dowker space in this more "natural" way. This should be very useful for researchers hoping to apply Zoli's method to their own problems.

The paper [Ba01a] is another important one which used the technique. In 1971, Rudin asked if there is a realcompact Dowker space, and in 1972, Hodel asked if every collection-wise normal metalindelöf space must be paracompact. The latter question was repeated by Burke and Watson. Zoli finishes off these questions by constructing, in ZFC, a hereditarily collection-wise normal, hereditarily realcompact Dowker space. The construction is quite similar to his normal screenable non-paracompact space construction, though the reflection techniques in the proof of non-countable paracompactness (always the hard part) are a bit different.

3.6 Miscellaneous

The name of this category does not imply a value judgment of any sort. In fact, two of his greatest hits are here! We begin our discussion with these.

Greatest Hit #5: *Solution to the Moore-Mrowka problem: The Proper Forcing Axiom implies that compact countably tight spaces are sequential* [Ba89a].

In an AMS Notices article in 1964, R. C. Moore and S. Mrowka asked if every countably tight compact Hausdorff space is sequential. In other words, if the topology of a compact Hausdorff space X is determined by its countable subsets (in the sense that a subset A is closed iff A contains all limit points of its countable subsets), must the topology of X in fact be determined by its convergent sequences?

This natural and important problem received quite a bit of attention. Nyikos called it "Classic Problem VI" in his 1977 list of major open problems in set-theoretic topology. A. V. Arhangel'skiĭ gives a thorough discussion of it in a 1978 survey paper, where he puts it as Number 1 in an extensive list of open problems, and indicates that he believes there should be a ZFC counterexample. In 1976, A. J. Ostaszewski and V. V. Fedorcuk each constructed counterexamples under $\diamondsuit$. Then not very much happened until the power of proper forcing became widely known. In 1986, Fremlin and Nyikos had obtained some related results using Fremlin's write-up of a proper forcing method due to S. Todorčević. Nyikos also showed that $\mathsf{MA} + \neg\mathsf{CH}$ is not sufficient to solve the problem, and that the Proper Forcing Axiom (PFA) implies a positive answer for hereditarily normal spaces. Then Zoli completed the solution, showing that the answer is positive under PFA. The result followed as a corollary to a more general statement which had some of the results of Fremlin and Nyikos as other corollaries; in particular, under PFA, countably compact regular T_1-spaces are either compact or contain a closed pre-image of the space of countable ordinals, and also countably tight, initially ω_1-compact, regular T_1-spaces are compact. In his 1991 survey of compact spaces over the previous 8-10 years, D. B. Shakhmatov called Zoli's result "the main advance in the theory of compact spaces during the covered period."

Greatest Hit # 6: *Every open cover of a monotonically normal space X has a σ-disjoint (partial) refinement V by open sets such that $X \setminus \bigcup V$ is the union of a discrete family of closed subspaces each homeomorphic to some stationary subset of a regular uncountable cardinal (the cardinal may vary with the subspaces)* [BR92].

The class of monotonically normal spaces was introduced by Heath, D. J. Lutzer, and P. Zenor in 1973 as a common generalization of ordered spaces and metrizable (or more generally, "stratifiable") spaces. Balogh and Rudin proved the deep and powerful result stated above, a very important corollary of which is R. Engelking and Lutzer's theorem, which says that an ordered space is paracompact iff it does not contain a closed copy of a stationary subset of a regular uncountable cardinal, extends to the class of monotonically normal spaces. The Balogh-Rudin result answered almost every question in the literature having to do with covering properties of monotonically normal spaces. The proof is lengthy and complicated – just what one would expect, given these authors!

Zoli was a big user of elementary submodels. We have mentioned that they play a very important role in his Dowker space constructions. He also proposed in many lectures using elementary submodels to prove covering property results. What got him started on this was an elementary submodel technique for proving the Jiang-Rudin theorem that strict p-spaces are submetacompact. He wrote a paper [Ba02b] in which he illustrates his ideas by presenting the strict-p argument along with another covering property result.

Zoli obtained many more significant results but there is no space here to mention them all. We content ourselves with discussing just one more very nice paper. An interesting problem of M. Katetov, which dates back to 1951, is whether every normal T_2-space X in which the Baire and Borel algebras in X coincide must be perfectly normal. (If a space is perfectly normal, they must coincide.) In [Ba88a], Zoli obtained several examples giving a negative answer to the problem under CH, and another one based on a consistent construction due to A. Miller of a subset M of the real line in which every subset is Baire but not every subset is G_δ. It is not known if a counterexample to Katetov's question exists in ZFC. In the same paper, Zoli solves in ZFC a 1965 problem of K. A. Ross and K. Stromberg by constructing a normal locally compact space in which there exists a closed Baire set which is not a zero-set.

4. Zoli's publications

[Ba76] Z. Balogh, *Relative compactness and recent common generalizations of metric and locally compact spaces*, General topology and its relations to modern analysis and algebra, IV (Proc. Fourth Prague Topological Sympos., Prague, 1976), Part B, 37–44.

[Ba78a] Z. Balogh, *Relative compactness and recent common generalizations of metric and locally compact spaces*, Fund. Math. **100** (1978), 165–177.

[Ba78b] Z. Balogh, *On the heredity of being a paracompact M-space and its relation to the normal Moore space conjecture*, Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978), 137–142.

[Ba78c] Z. Balogh, *On recent common generalizations of metrizable and compact T_2 spaces*, Proceedings of the Meeting on General Topology (Univ. Trieste, Trieste, 1978), 41–46.

[Ba79a] Z. Balogh, *Metrization theorems concerning relative compactness*, General Topology Appl. **10** (1979), 107–119.

[Ba79b] Z. Balogh, *On the structure of spaces which are paracompact p-spaces hereditarily*, Acta Math. Acad. Sci. Hungar. **33** (1979), 361–368.

[Ba79c] Z. Balogh, *Relative countable compactness*, Uspekhi Mat. Nauk **34** (1979), 139–143.

[Ba81a] Z. Balogh, *On the metrizability of spaces which are paracompact p-spaces hereditarily*, Period. Math. Hungar. **12** (1981), 83–86.

[Ba81b] Z. Balogh, *On the metrizability of F_{pp}-spaces and its relationship to the normal Moore space conjecture*, Fund. Math. **113** (1981), 45–58.

[Ba83] Z. Balogh, *Locally nice spaces under Martin's axiom*, Comment. Math. Univ. Carolin. **24** (1983), 63–87.

[Ba84] Z. Balogh, *On hereditarily strong Σ-spaces*, Topology Appl. **17** (1984), 199–215.

[Ba85] Z. Balogh, *Topological spaces with point-networks*, Proc. Amer. Math. Soc. **94** (1985), 497–501.

[Ba86a] Z. Balogh, *Paracompactness in normal, locally connected, rim-compact spaces*, Topology Appl. **22** (1986), 1–6.

[Ba86b] Z. Balogh, *Paracompactness in locally Lindelöf spaces*, Canad. J. Math. **38** (1986), 719–727.

[Ba88a] Z. Balogh, *On two problems concerning Baire sets in normal spaces*, Proc. Amer. Math. Soc. **103** (1988), 939–945.

[Ba88b] Z. Balogh, *Locally compact, countably paracompact spaces in the constructible universe*, Topology Appl. **30** (1988), 19–26.

[Ba89] Z. Balogh, *On compact Hausdorff spaces of countable tightness*, Proc. Amer. Math. Soc. **105** (1989), 755–764.

[Ba91a] Z. Balogh, *On collectionwise normality of locally compact, normal spaces*, Trans. Amer. Math. Soc. **323** (1991), 389–411.

[Ba91b] Z. Balogh, *There is a Q-set space in ZFC*, Proc. Amer. Math. Soc. **113** (1991), 557–561.

[Ba96] Z. Balogh, *A small Dowker space in* ZFC, Proc. Amer. Math. Soc. **124** (1996), 2555–2560.

[Ba98a] Z. Balogh, *There is a paracompact Q-set space in* ZFC, Proc. Amer. Math. Soc. **126** (1998), 1827–1833.

[Ba98b] Z. Balogh, *A normal screenable nonparacompact space in* ZFC, Proc. Amer. Math. Soc. **126** (1998), 1835–1844.

[Ba98c] Z. Balogh, *Nonsrinking open covers and K. Morita's third conjecture*, Topology Appl. **84** (1998), 185–198.

[Ba01a] Z. Balogh, *Dowker spaces and paracompactness questions*, Topology Appl., **114** (2001), 49–60.

[Ba01b] Z. Balogh, *Nonshrinking open covers and K. Morita's duality conjectures*, Topology Appl. **115** (2001), 333–341.

[Ba02a] Z. Balogh, *Locally nice spaces and Axiom R*, Topology Appl. **125** (2002), 335–341.

[Ba02b] Z. Balogh, *Covering properties via elementary submodels*, Topology Proc. **26** (2001/2002), 37–43.

[Ba03a] Z. Balogh, *Reflecting point-countable families*, Proc. Amer. Math. Soc. **131** (2003), 1289–1296.

[Ba03b] Z. Balogh, *Dowker spaces and their constructions*, Encyclopedia of General Topology, North-Holland Publishing Company.

[Ba03c] Z. Balogh, *On density and number of G_δ-points in somewhat Lindelöf spaces*. To appear in Topology Proceedings.

[Ba03d] Z. Balogh, *A natural Dowker space*. To appear in Topology Proceedings.

[Ba??] Z. Balogh, *A Lindelof Q-set space of cardinality* $\mathfrak{c}$, handwritten notes.

[BB87] Z. Balogh and H. Bennett, *Total paracompactness of real GO-spaces*, Proc. Amer. Math. Soc. **101** (1987), 753–760.

[BB89a] Z. Balogh and H. Bennett, *Conditions which imply metrizability in manifolds*, Houston J. Math. **15** (1989), 153–162.

[BB89b] Z. Balogh and H. Bennett, *On two classes of sets containing all Baire sets and all co-analytic sets*, Topology Appl. **33** (1989), 247–264.

[BB89c] Z. Balogh and H. Bennett, *Quasi-developable manifolds*, Topology Proc. **14** (1989), 201–212.

[BBBGLM] Z. Balogh, H. Bennett, D. Burke, G. Gruenhage, D. Lutzer, and J. Mashburn, *OIF spaces*, Questions Answers Gen. Topology **18** (2000), 129–141.

[BBM] Z. Balogh, H. Bennett, and C. Martin, *On the observability of ergodic flows on abelian groups with characteristic functions*, Proceedings of the Guilford College Sesquicentennial Topology Conference, 1988, (Greensboro, NC, 1988) 31–35.

[BBu92] Z. Balogh and D. K. Burke, *A total ladder system space by ccc forcing*, Topology Appl. **44** (1992), 37–44.

[BBu94] Z. Balogh and D. K. Burke, *On $\nearrow$-normal spaces*, Topology Appl. **57** (1994), 71–85.

[BBD] Z. Balogh, D. K. Burke, and S. W. Davis *A ZFC nonseparable Lindelöf symmetrizable Hausdorff space* , C. R. Acad. Bulgare Sci. **42** (1989), 11–12.

[BDJSS] Z. Balogh, S. W. Davis, W. Just, S. Shelah, and P. J. Szeptycki, *Strongly almost disjoint sets and weakly uniform bases*, Trans. Amer. Math. Soc. **352** (2000), 4971–4987.

[BDFN] Z. Balogh, A. Dow, D. H. Fremlin, and P. J. Nyikos, *Countable tightness and proper forcing*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), 295–298.

[BGri] Z. Balogh and J. Griesmer, *On the multiplicity of jigsawed bases in compact and countably compact spaces*, Topology Appl. **130** (2003), 65–73.

[BG91] Z. Balogh and G. Gruenhage, *On a problem of C. H. Dowker*, J. Symbolic Logic **56** (1991), 1284–1289.

[BG01] Z. Balogh and G. Gruenhage, *Base multiplicity in compact and generalized compact spaces*, Topology Appl. **115** (2001), 139–151.

[BG02] Z. Balogh and G. Gruenhage, *When the collection of ϵ-balls is locally finite*, Topology Appl. **124** (2002), no. 3, 445–450.

[BG04] Z. Balogh and G. Gruenhage, *Two more perfectly normal non-metrizable manifolds*. To appear in Topology Appl.

[BGT] Z. Balogh, G. Gruenhage, and V. Tkacuk, *Additivity of metrizability and related properties*, Topology Appl. **84** (1998), 91–103.

[BJ] Z. Balogh and H. Junnila, *Totally analytic spaces under $V = L$*, Proc. Amer. Math. Soc. **87** (1983), 519–527.

[BMN] Z. Balogh, J. Mashburn, and P. Nyikos *Countable covers of spaces by migrant sets*, Topology Proc. **14** (1989), 7–23.

[BR] Z. Balogh and M. E. Rudin,*Monotone normality*, Topology Appl. **47** (1992), 115–127.

[VPB] T. Vertse, K. F. Pál, and Z. Balogh, *GAMOW, A program for calculating the resonant state solution of the radial Schrödinger equation in an arbitrary optical potential*, Computer Physics Communications **27** (1982), 309–322.

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