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The 18th Century Chinese Discovery of the Catalan Numbers

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A Chinese academic got there first in spotting the significance of the Catalan numbers.

Consider the Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, \ldots\} = \{1, 1, 2, 5, 14, 42, 132, 429, \ldots\}$ generated most easily by the expression for the general $(r+1)$th term

$$c_r = \frac{1}{r+1}\binom{2r}{r}, \qquad r = 0, 1, 2, \ldots, \tag{1}$$

with which most of us are conversant. One might well suppose that the sequence originated with Eugène Catalan, for it is in an 1838 paper on mathematical aspects of (triangulated) polygon division that (1) is first given in this form and from which the sequence was to take its name. Readers of *Mathematical Spectrum* may have seen the equation

$$c_{n-1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!} 2^{n-1}, \qquad n > 1, \tag{2}$$

in an article by Vun and Belcher (reference 1) in which the Catalan numbers are shown to arise in three different ways, one of these being the polygon dissection problem. In fact Leonhard Euler had already identified the numbers $c_1, c_2, c_3, \ldots$, in this geometrical setting in the middle of the previous century (see references 2 and 3). A readable account of the problem is available in H. Dörrie's book *100 Great Problems of Elementary Mathematics: Their History and Solution* (Dover Publications, New York, 1965) as Problem No. 7. Many people will, however, not have access to a 1988 paper by Luo (reference 4) who asserts that the full sequence was known prior to this by Antu Ming (c.1692–1763), a Chinese scholar with a wide variety of scientific and mathematical interests. Since Luo's article is published in Chinese this fact is not common knowledge in the Western World. I wish here, therefore, to set down some of the historical details he provides and to call attention to a novel formulation of the numbers therein which is due to Ming.

At the beginning of the 18th century a French Jesuit, Pierre Jartoux, brought with him to China the three expansions

$$\begin{aligned}
\pi &= 3\Big\{1 + \frac{1^2}{4 \cdot 3!} + \frac{(1 \cdot 3)^2}{4^2 \cdot 5!} \\
&\qquad + \frac{(1 \cdot 3 \cdot 5)^2}{4^3 \cdot 7!} + \cdots\Big\}, \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \\
\text{versin}(x) &= 1 - \cos(x) \\
&= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots,
\end{aligned}$$

all of them without proof. The second and third (valid for all values of x) are immediately recognisable as Maclaurin series and are credited to James Gregory (1638–1675), while the first is attributed to Isaac Newton (1642–1726). Ming saw these results, but, suspecting that Western mathematicians would be unwilling to share their derivations, he set about obtaining them for himself. In this he succeeded, using a mixture of arithmetic and (imported) Euclidean geometry as his basic tools, and he went on to derive six further expansions of other trigonometric functions. Applying recurrence methods systematically, his algorithms have the striking feature of program and calculation, and he unquestionably laid a foundation for the operation of infinite series in China which constitutes an important contribution to the country's historical development in mathematics. His methods were, of course, not rigorous by today's standards, and the question of what was actually meant by a proof there at that time is discussed by Jami (reference 5) in relation to Ming's work. Luo states that Ming discovered the Catalan numbers through his geometric models, and he highlights one or two associated representations of the function $\sin(m\alpha)$ as power series in $\sin(\alpha)$ in which they appear. Ming dealt with the values $m = 2, 3, 4, 5, 10, 10^2, 10^3, 10^4$, and found that the series for $\sin(3\alpha)$ and $\sin(5\alpha)$ terminated whilst his other expansions were infinite. We see, for example, how the full Catalan sequence occurs in each of the following two results:

$$\begin{aligned}
\sin(2\alpha) &= 2\Big\{\sin(\alpha) - \sum_{n=1}^{\infty}\Big[\frac{c_{n-1}}{2^{2n-1}}\Big]\sin^{2n+1}(\alpha)\Big\} \\
&= 2\sin(\alpha) - \sin^3(\alpha) - \tfrac{1}{4}\sin^5(\alpha) \\
&\qquad - \tfrac{1}{8}\sin^7(\alpha) - \cdots,
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
\sin(4\alpha) &= 2\Big\{2\sin(\alpha) - 5\sin^3(\alpha) \\
&\qquad + \sum_{n=1}^{\infty}\Big[\frac{8c_{n-1} - c_n}{4^n}\Big]\sin^{2n+3}(\alpha)\Big\} \\
&= 4\sin(\alpha) - 10\sin^3(\alpha) + \tfrac{7}{2}\sin^5(\alpha) \\
&\qquad + \tfrac{3}{4}\sin^7(\alpha) + \cdots.
\end{aligned} \tag{4}$$

We do not concern ourselves with the convergence properties of these series. It is readily seen from (1) that (3) and (4) agree with an alternative formulation of $\sin(m\alpha)$ (m integer) which is said to be Euler's, namely,