Non-existence of stationary Hadamard states for a black-hole in a box and for the 1+1 massless wave equation to the left of an accelerated mirror

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The question

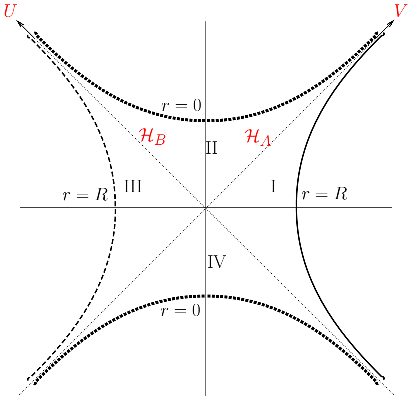
The **very general question**, see Bernard Kay's talk: What is the spacetime of a black hole in a box?

More specifically, here: Does there exist a reasonable description of (thermal) equilibrium between a black hole spacetime and its matter content, in the case where the black hole and its event horizon are "enclosed by a box"?

We **propose** to seek an answer in the semiclassical setting of *quantum field theory in curved spacetimes*, where the spacetime manifold and gravitational field are treated as background, and questions of back-reaction of the matter fields on the metric are initially ignored

Does there exist a reasonable description of thermal equilibrium between a black hole spacetime and its matter content, in the case where the black hole and its event horizon are "enclosed by a box"?

- Black hole in a box $\rightsquigarrow$ A subspacetime of the maximally extended Schwarzschild solution which has a timelike boundary, is invariant under the isometries and contains the future and past singularities. One of two choices, as in picture (A) in Bernard Kay's talk:



Does there exist a reasonable description of equilibrium between a black hole spacetime and its matter content, in the case where the black hole and its event horizon are "enclosed by a box"?

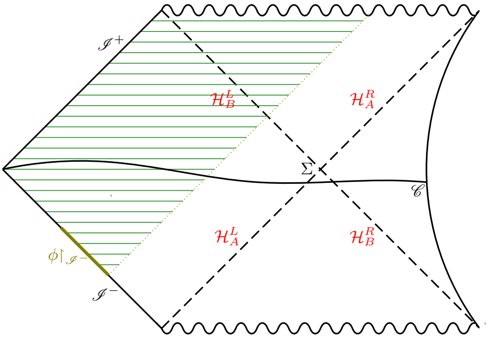
- Matter content $\rightsquigarrow$ A *free* Klein-Gordon quantum field. Boundary conditions must be imposed to ensure predictability. We choose Dirichlet b.c.'s.
- Equilibrium $\rightsquigarrow$ The *state* of the quantum field should be isometry-invariant...
- "Reasonable" $\rightsquigarrow$ ...and sufficiently regular. We take this to mean that its singularity structure should be of Hadamard form. Appropriate generalization to curved spacetimes of the singularity behaviour of the Minkowski vacuum.

Mathematical formulation of our question: Does the portion of Kruskal spacetime to the left of a $r = R > 2M$ hypersurface ("box") in the right exterior region – and, possibly, also to the right of a similar hypersurface in the left exterior region – admit a state of a Klein-Gordon field obeying Dirichlet b.c.'s which is both regular (Hadamard) and isometry-invariant?

In the case of **two** symmetrically placed boxes, no apparent obstruction to the existence of such a state – something like a Hartle-Hawking-Israel state.

In the case of **one** box we conjecture that **no such state can exist**.

Ideas behind the no-go conjecture



Construct a suitable space of real-valued solutions to

$$\begin{cases} (\Box_g + m^2 + \xi R)\phi = 0, \\ \phi\restriction_{\partial M} = 0 \end{cases}$$

and equip it with a symplectic form σ, defined in a natural way by integrating Cauchy data on "Cauchy surfaces" $\mathscr{C}$

Consider subspaces of solutions which "fall entirely through" the B-horizon ("S_B solutions"), or just through left/right portion $\mathcal{H}_B^L/\mathcal{H}_B^R$ of the B-horizon ("S_B^L/S_B^R solutions"). Same with $B \leftrightarrow A$.

First Crucial Observation: There exists a solution ϕ which

- is an S_B^L solution;
- "leaves no trace" on the A-horizon.

Second Crucial Observation: Due to reflection at the boundary, "initially S_B^R solutions" end up becoming S_A^R solutions!

Third Crucial Observation: One can establish a (symplectic) correspondence between S_B solutions and purely right-moving solutions to the equation $\partial^2\phi/\partial u\partial v = 0$ on $\mathbb{R}^2 \times \Sigma$ – i.e. the massless (!) wave equation in 1+1 dimensions, plus a trivial dependence on the angular variables.

Only classical considerations so far. So we turn to the quantum aspects of the problem.

Kay and Wald (1991): thorough analysis of the properties of states of linear scalar quantum fields in spacetimes with bifurcate Killing horizons. Additional assumption of global hyperbolicity, necessary to even properly formulate the Hadamard regularity condition.

Our black-hole-in-a-box spacetime is *not* globally hyperbolic. But global hyperbolicity still plays a role:
Definition. A quantum state on this spacetime will be called Hadamard when we get a Hadamard state in the usual sense upon localizing in globally hyperbolic subspacetimes which don't "causally intercommunicate" with the boundary. Nice physical interpretation!

$\rightsquigarrow$ Run a "Kay-Wald machine": If our state is isometry-invariant and Hadamard then its two-point function will be uniquely determined on S_B solutions. For two such solutions ϕ_1^B and ϕ_2^B, it will equal

$$-\frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int \frac{f_1(U_1,s)f_2(U_2,s)}{(U_1-U_2-i\varepsilon)^2}\,\mathrm{d}U_1\,\mathrm{d}U_2\,\sqrt{^{(2)}g}\,\mathrm{d}^2 s$$

with $f_i = \phi_i^B\restriction_{\mathcal{H}_B}$ and U_i a smooth (wrt s) choice of affine parameter along the generators of $\mathcal{H}_B$. A similar result for S_A solutions.

Third Crucial Observation can actually be lifted to the quantum level $\Rightarrow$ Our state, restricted to observables associated with S_B or S_A solutions, will correspond to a state of the "massless scalar field in (1+1)-dimensional Minkowski space". But which state? The answer is...

Fourth Crucial Observation: The "usual" vacuum state, restricted to pure right-movers or pure left movers.

But a lot is known about this state! In particular, its restriction to either pure right-movers or pure left-movers:

- Is a pure state
- Has the Reeh-Schlieder property with respect to movers compactly supported to the left or to the right of the bifurcation surface.

So analogous properties will hold for our original state. By a bit of Hilbert space analysis these, together with the Second Crucial Observation, contradict the existence of ϕ as in the First Crucial Observation. **QED**

The Minkowskian analog in 1+1 dimensions

Toy model: Unsurprisingly (but not tautologically), a massless scalar field in the portion of (1+1)-dimensional Minkowski spacetime to the left of an eternally and uniformly decelerated mirror in the right Rindler wedge. Lorentz boosts replace the Schwarzschild isometries.

Model is simple enough that a non-existence theorem can be proved completely rigorously once the infrared pathologies typical of massless low-dimensional theories are dealt with appropriately – a long story. Main ingredients are otherwise exactly the same as in the Kruskal case.

This is the *only* analogous Minkowskian model for which we can prove such a non-existence result by these means: go to higher dimensions and/or introduce a mass, and there is no ϕ as in the First Crucial Observation.

Conclusions

There are a few loose ends in our conjecture for Kruskal in a box – all very technical; in particular one needs good control over the underying mixed hyperbolic PDE problems.

Despite this, we believe these arguments to constitute further evidence that semiclassical descriptions of eternal black holes in boxes must break down at the horizons. One is left with something like 't Hooft's brick-wall model the right exterior region, and having to accept that something less well-understood and inherently quantum-gravitational must occur beyond there.

Thank you for your attention!