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ON A CONJECTURE OF ERDŐS AND STEWART

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ABSTRACT. For any $k \geq 1$, let p_k be the kth prime number. In this paper, we confirm a conjecture of Erdős and Stewart concerning all the solutions of the diophantine equation $n! + 1 = p_k^a p_{k+1}^b$, when $p_{k-1} \leq n < p_k$.

1. Introduction

For any $k \geq 1$ let p_k be the kth prime number. From [3], we found out that Erdős and Stewart conjectured that the only solutions of the equation

$$n! + 1 = p_k^a p_{k+1}^b \quad \text{for some } a \geq 0, b \geq 0 \text{ and } p_{k-1} \leq n < p_k \tag{1}$$

are obtained for $n \leq 5$.

In this paper, we prove the following

Theorem. *Equation* (1) *has no solutions for* $n \geq 6$.

One can check that equation (1) has no solutions for $5 < n \leq 11$. From now on, we work with a potential solution of (1) with $n \geq 12$.

2. An elementary lemma

The following elementary result turns out to be helpful when searching for the values of n.

Lemma. *In equation* (1), *one has* $ab \neq 0$.

Proof of the Lemma. Assume that this is not so and write

$$n! + 1 = p^a \quad \text{for some } p \in \{p_k, p_{k+1}\}. \tag{2}$$

Let $a = 2^i a_1$ where $a_1 \geq 1$ is odd. Then,

$$\mathrm{ord}_2(n!) = \mathrm{ord}_2(p^a - 1) \leq \max(\mathrm{ord}_2(p \pm 1)) + i \leq \log_2(p_{k+1} + 1) + \log_2(a). \tag{3}$$

From equation (2), we know that

$$n^a < p^a = n! + 1 < n^n, \tag{4}$$

therefore $a < n$. Since the interval $[n+1, 2n]$ contains at least two primes for $n \geq 12$, we get $p_{k+1} + 1 \leq 2n$. Hence, inequality (3) implies

$$\mathrm{ord}_2(n!) < \log_2(2n) + \log_2(n) = 2\log_2(n) + 1. \tag{5}$$

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From Lemma 1 in [1], we know that

$$\mathrm{ord}_2(n!) \geq n - \log_2(n+1). \tag{6}$$

From inequalities (5) and (6), we get

$$n - \log_2(n+1) < 2\log_2(n) + 1, \tag{7}$$

which implies $n \leq 11$. This contradicts the assumption on $n \geq 12$.

3. A linear form in logarithms and a bound on n

Write

$$n! = p_k^a p_{k+1}^b - 1 = p_{k+1}^b \left(p_k^a - \left(\frac{1}{p_{k+1}} \right)^b \right). \tag{8}$$

We find an upper bound for $\mathrm{ord}_2(n!)$. We apply Théoréme 4 in [1] with the choices

$$p = 2, \quad D = 1, \quad g = 1,$$
$$\alpha_1 = p_k,\ \alpha_2 = \frac{1}{p_{k+1}},\ b_1 = a,\ b_2 = b,$$
$$A_1 = p_k,\ A_2 = p_{k+1}$$

and

$$\mu = 15, \quad \nu = 10, \quad c(\mu, \nu) = 18.$$

From the result in [1], it follows that

$$\mathrm{ord}_2(n!) \leq \frac{36}{(\log 2)^4} (\max\{\log b' + \log\log 2 + 0.4, 15\log 2\})^2 \log p_k \log p_{k+1}, \tag{9}$$

where

$$b' = \frac{a}{\log p_{k+1}} + \frac{b}{\log p_k}. \tag{10}$$

We now find a bound on b' in terms on n. Since

$$p_k^a p_{k+1}^b = n! + 1 < n^n,$$

it follows that

$$a \log p_k + b \log p_{k+1} = \log p_k^a p_{k+1}^b < \log n^n = n \log n. \tag{11}$$

Hence,

$$b' = \frac{a}{\log p_{k+1}} + \frac{b}{\log p_k} = \frac{a \log p_k + b \log p_{k+1}}{\log p_k \log p_{k+1}} < \frac{n \log n}{\log p_k \log p_{k+1}} < \frac{n}{\log n}. \tag{12}$$

Since the interval $[n+1, 2n]$ contains at least two primes, it follows that $p_k < p_{k+1} < 2n$. Inequality (9) now implies

(13)

$$\mathrm{ord}_2(n!) < \frac{36}{(\log 2)^4} \left(\max\left\{ \log\left(\frac{n}{\log n} \right) + \log\log 2 + 0.4, 15\log 2 \right\} \right)^2 \log^2(2n).$$

When

$$\log\left(\frac{n}{\log n} \right) + \log\log 2 + 0.4 \leq 15 \log 2,$$

we get $n < 409\ 506$. When

$$\log\left(\frac{n}{\log n}\right) + \log\log 2 + 0.4 > 15\log 2,$$

we get, by inequalities (6) and (13), that

$$n - \log_2(n+1) < \frac{36}{(\log 2)^4}\left(\log\left(\frac{n}{\log n}\right) + \log\log 2 + 0.4\right)^2 \log^2(2n), \tag{14}$$

which implies $n < 7\ 242\ 116$. The conclusion is that $n < p_k < p_{k+1} < 7.5 \cdot 10^6$.

4. The remaining computations

For the remaining computations, we used the following result due to Erdős and Obláth (see [2]).

Theorem EO. *The equation*

$$x^p \pm y^p = n! \tag{15}$$

has no solutions such that $p > 2$ *is prime and* $\gcd(x, y) = 1$.

Case 1. $n > 193$.

The idea here was to check, computationally, that if n leads to a solution of (1), then $a \equiv b \equiv 0 \pmod 3$. Once we prove this, the impossibility of (1) follows from Theorem EO for $p = 3$.

Assume, for example, that (1) has a solution such that either $3 \nmid a$ or $3 \nmid b$. Write

(16)

$$n! + 1 = Ax^3 \quad \text{where } A = p_k^{\delta_1} p_{k+1}^{\delta_2} \text{ for some } \delta_1,\ \delta_2 \in \{0, 1, 2\} \text{ with } (\delta_1, \delta_2) \neq (0, 0).$$

Let $q \leq 193$ be a prime congruent to 1 modulo 3. Equation (1) implies that $Ax^3 \equiv 1 \pmod q$ for every such q. It now follows that A is a cubic residue modulo q for every $q \leq 193$ which is congruent to 1 modulo 3. Since a number y is a cubic residue modulo q if and only if y^2 is a cubic residue modulo q, it follows that we need to identify only those numbers A of the form

$$A = p_k \quad \text{or} \quad A = p_k p_{k+1} \quad \text{or} \quad A = p_k^2 p_{k+1} \tag{17}$$

in the range $193 < p_k < p_{k+1} < 7.5 \cdot 10^6$ which are cubic residues with respect to every prime $q \leq 193$ which is congruent to 1 modulo 3. Achim Flammenkamp wrote a computer program which checked in a few minutes that there are no such A's. Hence, $n \leq 193$.

Case 2. $n \leq 193$.

By the Lemma, we know that if n leads to a solution of (1), then $ab > 0$. Achim Flammenkamp wrote another computer program which checked in less than a second that in this range $n! + 1 \not\equiv 0 \pmod{p_k p_{k+1}}$.

The Theorem is therefore proved.

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