

I. N. Bronshtein · K. A. Semendyayev

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I. N. Bronshtein · K. A. Semendyayev

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Preface of the Reprint of the 3rd edition

The book is in such high demand that the publisher decided to reprint the 3rd edition published in 1997.

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Preface

In 1957 one of the two Editors translated the Handbook of Mathematics by I. N. Bronshtein and K. A. Semendyayev from Russian into German. In comparison with the original, there were two additional sections "Calculus of variation" and "Integral equations". Over the years this book has become a standard work also in German speaking countries. For nearly two decades it was an indispensable help for many students, teachers and practitioners of mathematics, although in contents and form it hardly changed at all.

During the same period some newer branches of mathematics have undergone a stormy development. Others have gained considerably in importance or have changed rapidly under the influence of practical needs, not least owing to the development of electronic calculating techniques. Even school mathematics did not stand still in the intervening years, so that new points of view emerged in the more elementary parts of the Handbook.

When all these factors were taken into account, a new edition had to incorporate the following points. New topics had to be included, for example, functional analysis, a section on the foundations of mathematics with the title "Sets, relations, functions" (with due regard to the basic concepts of mathematical logic), measure theory and the Lebesgue-Stieltjes integral, tensor calculus, mathematical methods of operational research (linear, non-linear, and dynamical optimization, graph theory, game theory, etc.), numerical methods and computational techniques.

Some sections had to be enlarged substantially or put on new foundations, for example, probability theory and mathematical statistics, or Fourier analysis and the Laplace transformation.

Several essential supplements were needed, among them a section on matrices within the framework of algebra.

In addition, most sections had to be thoroughly revised to keep up with present-day demands.

It goes without saying that such a huge task could not be carried out by a single person in a reasonable time. The problem had to be tackled by a collective of authors. An agreement was reached between the publishers of the original Russian edition and the publishers of the German translation to the effect that the revised version should be a joint undertaking and the preparation should be entrusted to a collective of authors working in close collaboration with the Soviet authors. This team was recruited largely from the scientific personnel of the section Mathematics at the Karl Marx University of Leipzig.

The editors and authors endeavoured to preserve the diction and presentation of the original, in spite of the incorporation of new material. Nevertheless, it was inevitable that the presentation

as a whole exhibits less homogeneity than the original book. This is due partly to factual matters such as the widely diverging contents of the individual sections or the varying degree of difficulty of the topics.

At the same time, the editors and authors believe that the Handbook can meet the diverse requirements of the numerous potential users, by the level of sophistication appropriate to the theme at hand.

The editors and authors wish to express their thanks to all who have contributed by their advice and helpful criticism to shaping the ultimate form of the new version of the work. Our special thanks are due to the group of advisors at the Technical University of Karl Marx-Stadt under the direction of Professor Schneider, who drew our attention to the diverse mathematical needs of students and graduates in technical disciplines.

Leipzig, December 1978

The Editors

Table of contents

1 Tables and graphical representations	1
1.1 Tables	1
1.1.1 Table of some constants in frequent use	1
1.1.2 Tables of special functions	2
1. The Gamma function (2) – 2. Bessel functions (cylinder functions) (3) – 3. Legendre polynomials (spherical functions) (5) – 4. Elliptic integrals (6) – 5. The Poisson distribution (8) – 6. Normal distribution (9) – 6.1 Density function $\varphi(x)$ of the normed and centred normal distribution (9) – 6.2 Distribution function $\Phi_\sigma(x)$ of the normed and centred normal distribution (10) – 7. Upper 100 α -percent values χ_{α}^2 of the χ^2 -distribution (12) – 8. 100 α -percent values $t_{\alpha m}$ of Student's t -distribution (13) – 9. Upper five-percent values $F_{0.05, m_1, m_2}$ and upper one-percent values $F_{0.01, m_1, m_2}$ (in bold figures) of the F -distribution (14) – 10. Fisher's Z -distribution (18) – 11. Critical numbers for the Wilcoxon test (19) – 12. The Kolmogorov-Smirnov λ -distribution (20)	
1.1.3 Integrals and sums of series	21
1. Table of sums of some numerical series (21) – 1.1 Table of the first few Bernoulli numbers (22) – Table of the first few Euler numbers (22) – 2. Table of the power series expansions of some functions (22) – 3. Tables of indefinite integrals (26) – 4. Table of some definite integrals (56)	
1.2 Graphs of elementary functions	61
1.2.1 Algebraic functions	61
1. Integral rational functions (polynomial functions) (61) – 2. Fractional rational functions (63) – 3. Irrational algebraic functions (66)	
1.2.2 Transcendental functions	68
1. Trigonometric functions and their inverses (68) – 2. Exponential and logarithmic functions (70) – 3. The hyperbolic functions and their inverses (74)	
1.3 Equations and parametric representations of elementary curves	75
1.3.1 Algebraic curves	76
1. Curves of the third order (or cubic curves) (76) – 2. Curves of the fourth order (or quartic curves) (77)	
1.3.2 Cycloids	80
1.3.3 Spirals	83
1.3.4 Catenary and tractrix	85
2 Elementary mathematics	86
2.1 Elementary approximation calculus	86
2.1.1 General considerations	86
1. Representation of numbers in positional systems (86) – 2. Truncation error and rounding rules (87)	
2.1.2 Elementary calculus of errors	88
1. Absolute and relative error (88) – 2. Approximation for the bound of the error of a function (88) – 3. Approximation formulae (89)	
2.1.3 Elementary graphical approximation methods	90
2.2 Combinatorics	91
2.2.1 Basic combinatorial functions	91
1. Factorials and the Gamma function (91) – 2. Binomial coefficients (92) – 3. Multinomial coefficients (94)	
2.2.2 The binomial and multinomial theorems	94
1. The binomial theorem (94) – 2. The multinomial theorem (95)	
2.2.3 Objectives of combinatorics	95
2.2.4 Permutations	96
1. Permutations without repetition (96) – 2. The group of permutations of k elements (96) – 3. Permutations with a fixed point (97) – 4. Permutations with prescribed numbers of cycles (98) – 5. Permutations with repetitions (98)	
2.2.5 Selections	99
1. Selections without repetitions (99) – 2. Selections with repetitions (99)	
2.2.6 Combinations	99
1. Combinations without repetitions (99) – 2. Combinations with repetitions (100)	

2.3	Finite sequences, sums, products, means	100
2.3.1	Notation for sums and products	100
2.3.2	Finite sequences	101
2.3.3	Some sums of finite sequences	103
2.3.4	Means	103
2.4	Algebra	104
2.4.1	Arithmetical expressions	104
	1. Definition of arithmetical expressions (104) – 2. Interpretation of arithmetical expressions (108) – 3. Equalities between arithmetical expressions (109) – 4. Polynomials (111) – 5. Inequalities between arithmetical expressions (113)	
2.4.2	Algebraic equations	116
	1. Equations (116) – 2. Equivalence transformations (117) – 3. Algebraic equations (118) – 4. General theorems (122) – 5. Systems of algebraic equations (125)	
2.4.3	Some special cases of transcendental equations	126
2.4.4	Linear algebra	127
	1. Vector spaces (127) – 1.1 The concept of a vector space (127) – 1.2 Subspaces (129) – 1.3 Linear dependence (130) – 1.4 Bases and dimension (131) – 1.5 Euclidean vector spaces (133) – 2. Matrices and determinants (135) – 2.1 Definition of a matrix (135) – 2.2 The determinant of a square matrix (135) – 2.3 The rank of a matrix (137) – 2.4 Matrix algebra (138) – 2.5 Special classes of matrices (141) – 3. Systems of linear equations (141) – 3.1 Definition of a system of linear equations, solutions, solution set (141) – 3.2 Existence of solutions of a linear system (142) – 3.3 Calculation of solutions of a linear system (143) – 4. Linear maps (145) – 4.1 Basic concepts (145) – 4.2 Representation of linear maps by matrices (147) – 4.3 Operations on linear maps (148) – 4.4 The inverse operator (149) – 5. Eigenvalues and eigenvectors (149) – 5.1 Eigenvalues and eigenvectors of matrices (149) – 5.2 Propositions on eigenvalues and eigenvectors (150) – 5.3 Applications of the theory of eigenvalues (150)	
2.5	Elementary functions	153
2.5.1	Algebraic functions	153
	1. Integral rational functions (153) – 1.1 Definition of integral rational functions (153) – 1.2 Factorization of integral rational functions (154) – 1.3 Zeros of integral rational functions (154) – 1.4 Behaviour at infinity (155) – 1.5 Particular integral rational functions (155) – 2. Fractional rational functions (155) – 2.1 Definition of fractional rational functions (155) – 2.2 Zeros and poles of rational functions (156) – 2.3 Behaviour of fractional rational functions (156) – 2.4 Particular fractional rational functions (157) – 2.5 Decomposition into partial fractions (158) – 3. Non-rational algebraic functions (160)	
2.5.2	Transcendental functions	161
	1. Trigonometric functions and their inverses (161) – 1.1 Definition of the trigonometric functions (161) – 1.2 Properties of trigonometric functions (162) – 1.3 Relations between the trigonometric functions (163) – 1.4 The general sine function $f(x) = a \sin(bx + c)$ (165) – 1.5 Definition of the inverse trigonometric functions (166) – 1.6 Properties of the inverse trigonometric functions (167) – 1.7 Relations between inverse trigonometric functions (167) – 2. Exponential and logarithmic functions (168) – 2.1 Definitions of the exponential and logarithmic functions (168) – 2.2 Particular exponential and logarithmic functions (168) – 2.3 Properties of exponential and logarithmic functions (169) – 3. Hyperbolic functions and their inverses (169) – 3.1 Definition of hyperbolic functions (169) – 3.2 Properties of the hyperbolic functions (169) – 3.3 Relations between the hyperbolic functions (170) – 3.4 Definition of the inverse hyperbolic functions (171) – 3.5 Properties of the inverse hyperbolic functions (172) – 3.6 Relations between the inverse hyperbolic functions (172)	
2.6	Geometry	173
2.6.1	Plane geometry	173
2.6.2	Three-dimensional geometry	177
	1. Lines and planes in space (177) – 2. Edges, vertices, solid angles (177) – 3. Polyhedra (178) – 4. Bodies bounded by curved surfaces (180)	
2.6.3	Plane trigonometry	183
	1. Solution of triangles (183) – 1.1 Solution of right-angled triangles (183) – 1.2 Solution of triangles (183) – 2. Application to elementary surveying (185)	
2.6.4	Spherical trigonometry	186
	1. Geometry on the sphere (186) – 2. Spherical triangle (187) – 3. Solution of a spherical triangle (188) – 3.1 Solution of the general spherical triangle (188) – 3.2 Solution of a right-angled spherical triangle (190)	
2.6.5	Coordinate systems	190
	1. Coordinate systems in the plane (191) – 1.1 Linear coordinate systems in the plane (191) – 1.2 Curvilinear coordinate systems in the plane (192) – 1.3 Transformation of coordinates in the plane (193) – 2. Coordinate systems in space (194) – 2.1 Linear coordinate systems in space (194) – 2.2 Curvilinear coordinate systems in space (195) – 2.3 Transformation of coordinates in space (196)	
2.6.6	Analytic geometry	198
	1. Analytic geometry of the plane (198) – 2. Analytic geometry of space (206)	

3 Analysis	215
3.1 Differential and integral calculus of functions of one and several variables	215
3.1.1 Real numbers	215
1. System of axioms for the real numbers (215) – 2. The natural numbers \mathbb{N} , integers \mathbb{Z} , and rational numbers \mathbb{Q} (217) – 3. Absolute value, elementary inequalities (218)	
3.1.2 Point sets in \mathbb{R}^n	219
3.1.3 Sequences	222
1. Real sequence (222) – 1.1 Boundedness, convergence, examples (222) – 1.2 Theorems concerning sequences (223) – 1.3 Definite divergence (224) – 2. Point sequences (224)	
3.1.4 Real functions	225
1. Functions of a real variable (225) – 1.1 Definition, graphical representation, boundedness (225) – 1.2 Limits of a function of one variable (227) – 1.3 Calculation of limits (229) – 1.4 Continuous functions of one variable (230) – 1.5 Discontinuities, order of magnitude of functions (232) – 1.6 Theorems on continuous functions in a closed interval (234) – 1.7 Special types of functions (235) – 2. Functions of several real variables (236) – 2.1 Definition, graphical representation, boundedness (236) – 2.2 Limits of functions of several variables (238) – 2.3 Continuous functions of several variables (238)	
3.1.5 Differentiation of functions of a real variable	240
1. Definition and geometrical interpretation of the first derivative, examples (240) – 2. Higher derivatives (242) – 3. Theorems on differentiable functions (243) – 4. Monotonic and convex functions (245) – 5. Relative extrema and points of inflection (246) – 6. Elementary discussion of curves (248)	
3.1.6 Differentiation of functions of several variables	249
1. Partial derivatives; geometrical interpretation (249) – 2. Total derivative, total differential, directional derivative and gradient (251) – 3. Theorems on differentiable functions of several variables (253) – 4. Differentiable mappings from \mathbb{R}^n into \mathbb{R}^m ; Jacobian determinants; implicit functions; solubility theorems (254) – 5. Substitution of variables in differential expressions (258) – 6. Relative extrema for functions of several variables (259)	
3.1.7 Integral calculus for functions of one variable	263
1. Definite integrals (263) – 2. Properties of the definite integral (264) – 3. The indefinite integral (266) – 4. Properties of indefinite integrals (268) – 5. Integration of rational functions (270) – 6. Integration of other classes of functions (274) – 6.1 Integration of certain algebraic functions (274) – 6.2 Integration of transcendental functions (277) – 7. Improper integrals (280) – 8. Geometrical and physical applications of the definite integral (288)	
3.1.8 Line integrals	291
1. Line integrals of the first kind (291) – 2. Existence and calculation of a line integral of the first kind (292) – 3. Line integrals of the second kind (293) – 4. Properties and calculation of line integrals of the second kind (294) – 5. Independence of the path of a line integral (296) – 6. Geometrical and physical applications of line integrals (298)	
3.1.9 Parameter integrals	299
1. Definition of a parameter integral (299) – 2. Properties of parameter integrals (299) – 3. Improper parameter integrals (301) – 4. Examples of parameter integrals (303)	
3.1.10 Integrals over plane domains	305
1. Definition of the double integral and elementary properties (305) – 2. Calculation of a double integral (306) – 3. Transformation of variables in double integrals (307) – 4. Geometrical and physical application of the double integral (309)	
3.1.11 Integrals over spatial domains	310
1. Definition of the triple integral and elementary properties (310) – 2. Calculation of triple integrals (311) – 3. Transformation of variables in triple integrals (312) – 4. Geometrical and physical applications of triple integrals (313)	
3.1.12 Surface integrals	315
1. Area of a smooth surface (315) – 2. Surface integrals of the first and the second kind (316) – 3. Geometrical and physical applications of the surface integral (320)	
3.1.13 Integral theorems and supplements	322
1. Gauss's integral theorem (322) – 2. Green's formulae (322) – 3. Stokes' integral theorem (323) – 4. Improper line, double, surface, and triple integrals (324) – 5. Multi-dimensional parameter integrals (325)	
3.1.14 Infinite series. Sequences of functions	328
1. Basic concepts (328) – 2. Tests for convergence or divergence of series with non-negative terms (329) – 3. Series with arbitrary terms. Absolute convergence (332) – 4. Sequences and series of functions (334) – 5. Power series (338) – 6. Analytic functions. Taylor series. Expansion of elementary functions in power series (342)	
3.1.15 Infinite products	347
3.2 Calculus of variations and optimal processes	349
3.2.1 Calculus of variations	349
1. Formulation of the problems, examples, and basic concepts (349) – 2. The Euler-Lagrange theory (351) – 3. The Hamilton-Jacobi theory (363) – 4. The inverse problem of the calculus of variations (364) – 5. Numerical methods (366) – 6. Methods of functional analysis (371)	

3.2.2	Optimal processes	372
	1. Basic concepts (372) – 2. Continuous optimal processes (373) – 3. Discrete systems (382) – 4. Numerical methods (383)	
3.3	Differential equations	385
3.3.1	Ordinary differential equations	385
	1. Explanations. Existence and uniqueness theorems for ordinary differential equations and systems (385) – 2. Differential equations of the first order (387) – 2.1 Explicit differential equations of the first order (387) – 2.2 Implicit differential equations of the first order (392) – 2.3 General approximation methods for the solution of differential equations of the first order (398) – 3. Linear differential equations and linear systems (399) – 3.1 General theory for linear differential equations (399) – 3.2 Linear differential equations with constant coefficients (402) – 3.3 Linear systems of differential equations (404) – 3.4 Linear differential equations of the second order (407) – 4. General non-linear differential equations (417) – 5. Stability (418) – 6. The operational method for the solution of ordinary differential equations (419) – 7. Boundary-value and eigenvalue problems (421) – 7.1 Boundary-value problems. The Green's function (421) – 7.2 Eigenvalue problems (425)	
3.3.2	Partial differential equations	427
	1. Fundamental concepts and special methods of solution (427) – 2. Partial differential equations of the first order (431) – 2.1 The initial value problem (432) – 2.2 Complete integrals (436) – 2.3 Contact transformations. Canonical equations and canonical transformations (438) – 3. Partial differential equations of the second order (442) – 3.1 Classification. Characteristics. Well-posed problems (442) – 3.2 General methods for the construction of solutions (447) – 3.3 Hyperbolic differential equations (453) – 3.4 Elliptic differential equations (460) – 3.5 Parabolic differential equations (469)	
3.4	Complex numbers. Functions of a complex variable	471
3.4.1	General remarks	471
3.4.2	Complex numbers. The Riemann sphere. Domains	471
	1. Definition of the complex numbers. The field of complex numbers (471) – 2. Conjugate complex numbers. Absolute value of a complex number (472) – 3. Geometrical interpretation of the complex numbers and their addition (473) – 4. Trigonometric and exponential form of complex numbers and their multiplication and division (474) – 5. Powers, roots (474) – 5.1 Natural number exponent n (474) – 5.2 Negative integer exponent n (474) – 5.3 Rational exponent n (474) – 5.4 Arbitrary real exponent $n = \epsilon$ (476) – 6. The Riemann sphere. Domains. Jordan curves (476)	
3.4.3	Complex functions of a complex variable	477
3.4.4	The most important elementary functions	479
	1. Elementary algebraic functions (479) – 1.1 Polynomial functions (479) – 1.2 Rational functions (479) – 1.3 Irrational algebraic functions (479) – 2. Elementary transcendental functions (479) – 2.1 The exponential function (479) – 2.2 The natural logarithm (479) – 2.3 The general power (482) – 2.4 Trigonometric functions and hyperbolic functions (482)	
3.4.5	Analytic functions	483
	1. Derivative (483) – 2. The Cauchy-Riemann differential equations (483) – 3. Analytic functions (484)	
3.4.6	Complex curvilinear integrals	484
	1. Integral of a complex function (484) – 2. Independence of the path (486) – 3. Indefinite integrals (486) – 4. The fundamental theorem of the integral calculus (486) – 5. Cauchy's integral formulae (487)	
3.4.7	Series expansions of analytic functions	487
	1. Sequences and series (487) – 2. Function series. Power series (489) – 3. Taylor series (491) – 4. Laurent series (491) – 5. Classification of singular points (491) – 6. The behaviour of analytic functions at infinity (492)	
3.4.8	Residues and their application	492
	1. Residues (492) – 2. The residue theorem (493) – 3. Application to the calculation of definite integrals (494)	
3.4.9	Analytic continuation	494
	1. Principle of analytic continuation (494) – 2. The Schwarz principle of reflection (495)	
3.4.10	Inverse functions. Riemann surfaces	495
	1. One-sheeted functions. Inverse functions (495) – 2. The Riemann surface of the function $z = \sqrt[n]{w}$ (496) – 3. The Riemann surface of $z = \ln w$ (497) – 4. Poles, zeros and branch points (497)	
3.4.11	Conformal mapping	498
	1. The concept of conformal mapping (498) – 2. Some simple conformal mappings (499)	
4	Special chapters	501
4.1	Sets, relations, functions	501
4.1.1	Basic concepts of mathematical logic	501
	1. Expressions of propositional logic (501) – 2. Equivalence of logical expressions (503) – 3. Predicative expressions (504)	

4.1.2	Fundamental concepts of set theory	505
	1. Sets and elements (505) – 2. Subsets (505) – 3. Particular constructions of sets (506)	
4.1.3	Operations on sets and systems of sets	506
	1. Union and intersection of sets (506) – 2. Difference, symmetric difference, and complements of sets (507) – 3. Euler-Venn diagrams (508) – 4. The Cartesian product of sets (508) – 5. Union and intersection of systems of sets (509)	
4.1.4	Relations, functions, operations	510
	1. Relations (510) – 2. Equivalence relations (511) – 3. Order relations (511) – 4. Further order-theoretical concepts (513) – 5. Correspondences, functions, and mappings (513) – 6. Sequences and families of sets (514) – 7. Operations and algebras (515)	
4.1.5	Cardinality	515
	1. Equivalence of sets (515) – 2. Countable and uncountable sets (516)	
4.2	Vector analysis	516
4.2.1	Vector algebra	516
	1. Fundamental concepts (516) – 2. Multiplication by a scalar and addition (517) – 3. Multiplication of vectors (518) – 4. Geometrical applications of vector algebra (521)	
4.2.2	Vector calculus	522
	1. Vector functions of a scalar variable (522) – 2. Fields (524) – 3. Gradient of a scalar field (528) – 4. Curvilinear integral and potential in a vector field (530) – 5. Surface integrals in vector fields (532) – 6. Divergence of a vector field (535) – 7. Rotation of a vector field (537) – 8. Laplace operator and vector gradient (538) – 9. Calculation of composite expressions (nabla calculus) (539) – 10. Integral theorems (541) – 11. Determination of a vector field from its sources and sinks (543) – 12. Dyads (545)	
4.3	Differential geometry	550
4.3.1	Plane curves	551
	1. Possible definitions of a plane curve (551) – 2. Local elements of a plane curve (551) – 3. Special points (554) – 4. Asymptotes (557) – 5. Evolute and involute (558) – 6. Envelope of a family of curves (559)	
4.3.2	Space curves	559
	1. Possible definitions of a space curve (559) – 2. Local elements of a space curve (559) – 3. Fundamental theorem of curve theory (561)	
4.3.3	Surfaces	562
	1. Possible definitions of a surface (562) – 2. Tangent plane and normal to a surface (563) – 3. Metrical properties of surfaces (564) – 4. Curvature properties of surfaces (566) – 5. The fundamental theorem of surface theory (569) – 6. Geodesics on a surface (570)	
4.4	Fourier series, Fourier integrals, and the Laplace transformation	571
4.4.1	Fourier series	571
	1. General considerations (571) – 2. Table of some Fourier expansions (573) – 3. Numerical harmonic analysis (579)	
4.4.2	Fourier integrals	581
	1. General considerations (581) – 2. Table of Fourier transforms (583)	
4.4.3	The Laplace transformation	592
	1. General considerations (592) – 2. Application of the Laplace transformation to initial-value problems in ordinary differential equations (593) – 3. Table of the reverse transformation of rational image functions (595)	
5	Probability theory and mathematical statistics	598
5.1	Probability theory	598
5.1.1	Random events and their probabilities	598
	1. Random events (598) – 2. The axioms of probability theory (599) – 3. Probabilities in the classical case (600) – 4. Conditional probabilities (601) – 5. The theorem on the total probability. Bayes' formula (602)	
5.1.2	Random variables	603
	1. Discrete random variable (603) – 1.1 The indicator of an event (604) – 1.2 The binomial distribution (604) – 1.3 The hypergeometric distribution (605) – 1.4 The Poisson distribution (605) – 2. Absolutely continuous random variables (606) – 2.1 The rectangular distribution (606) – 2.2 The normal (Gaussian) distribution (607) – 2.3 The exponential distribution (607) – 2.4 The Weibull distribution (607)	
5.1.3	The moments of a distribution	608
5.1.4	Random vectors	610
	1. Discrete random vectors (611) – 2. Absolutely continuous random vectors (611) – 3. Marginal distributions (612) – 4. The moments of a multi-dimensional random variable (613) – 5. Conditional distributions (614) – 6. The independence of random variables (615) – 7. Theoretical regression quantities (615) – 7.1 Regression curves (616) – 7.2 Regression lines (616) – 8. Functions of random variables (616)	
5.1.5	Limit theorems	617
	1. The laws of large numbers (617) – 2. The limit theorem of de Moivre-Laplace (618) – 2.1 The local limit theorem (618) – 2.2 The integral limit theorem (618) – 3. The central limit theorem (619)	

5.2	Mathematical statistics	620
5.2.1	Samples 1. The histogram and the sample distribution function (620) – 2. Sample functions (622) – 3. Some distributions important in statistics (622)	620
5.2.2	The estimation of parameters 1. Properties of point estimators (623) – 2. Methods of obtaining estimators (624) – 2.1 The method of moments (624) – 2.2 The maximum-likelihood method (625) – 3. Confidence estimation (626) – 3.1 The confidence estimation of an unknown probability on the basis of a large sample (627) – 3.2 The confidence estimation of a from an $N(a, \sigma)$ -normally distributed population with an unknown σ (627) – 3.3 The confidence estimation of σ from an $N(a, \sigma)$ -normally distributed population with an unknown a (627) – 3.4 Confidence intervals of asymptotically normally distributed estimators (628)	623
5.2.3	Testing of hypotheses 1. Statement of the problem (628) – 2. The general theory (628) – 3. The t -test (629) – 4. The F -test (629) – 5. The Wilcoxon test (630) – 6. The χ^2 -test of fit (631) – 7. The case of additional parameters (632) – 8. The Kolmogorov-Smirnov test of fit (633)	628
5.2.4	Correlation and regression 1. The estimation of correlation and regression coefficients from samples (633) – 2. Testing the hypothesis $\rho = 0$ in the case of normally distributed populations (634) – 3. A general regression problem (634)	633
6	Linear optimization	636
6.1	The problem of linear optimization and the simplex algorithm	636
6.1.1	General statement of the problem, the geometric interpretation and solution of problems in two variables	636
6.1.2	Canonical form. Representation of a vertex in the simplex tableau 1. The simplex tableau (640) – 2. Vertex property and the role of the basis inverse (641) – 3. Vertices and basis solutions (642)	639
6.1.3	The simplex algorithm for optimization with a given initial tableau 1. Test for minimality (643) – 2. Passage to a new tableau when the minimality test fails (643)	642
6.1.4	Obtaining an initial vertex 1. The method of artificial variables (646) – 2. Solution of the auxiliary problem (647) – 3. Passage from the optimal tableau of the auxiliary problem to a starting tableau of the original problem (647)	646
6.1.5	The case of degeneracy and its treatment in the simplex algorithm 1. Definition of the lexicographic ordering of vectors (648) – 2. Supplement to the simplex tableau (649) – 3. Supplements to the simplex algorithm (649)	648
6.1.6	Duality in linear optimization 1. Duality theorems (650) – 2. The dual simplex algorithm (651)	650
6.1.7	Revised algorithms. Posterior change in the problem 1. The revised simplex algorithm (652) – 2. The revised dual simplex algorithm (655) – 3. Obtaining an initial vertex (655) – 4. Modification of the problem after optimization (655) – 4.1 General statement of the problem (655) – 4.2 Use of another object function (656) – 4.3 Use of other right-hand sides (656) – 4.4 Taking into account a further inequality as a constraint (656) – 4.5 Introducing a new variable (657)	652
6.1.8	Decomposition of large optimization problems	657
6.2	The transportation problem and the transportation algorithm	658
6.2.1	The linear transportation problem	658
6.2.2	Obtaining an initial solution	660
6.2.3	The transportation algorithm	662
6.3	Typical applications of linear optimization	665
6.3.1	Use of capacity	665
6.3.2.	Mixtures	665
6.3.3	Sharing out, subdivision of plans, assignments	666
6.3.4	Cutting, shift planning, covering	667
6.4	Parametric linear optimization	668
6.4.1	Statement of the problem	668
6.4.2	Solution procedure for the type “one-parameter object function”	668
7	Numerical mathematics and computation techniques	673
7.1	Numerical mathematics	673
7.1.1	Errors and their detection	673
7.1.2	Numerical methods 1. The solution of systems of linear equations (675) – 1.1 Direct methods (Gaussian elimination) (675) – 1.2 Iterative methods (679) – 2. Linear eigenvalue problems (681) – 2.1 Direct methods (681) – 2.2 Iterative methods (683) – 3. Non-linear equations (684) – 4. Non-linear systems of equations (687) – 5. Approximation (689) – 5.1 The linear approximation problem	675

in a Hilbert space (689) – 5.2 Chebyshev approximation (693) – 6. Interpolation (694) – 6.1 Interpolation polynomials (694) – 6.2 Spline interpolation (698) – 7. Numerical quadrature (701) – 8. Approximate differentiation (707) – 9. Differential equations (708) – 9.1 Initial value problems in ordinary differential equations (708) – 9.2 Boundary-value problems for ordinary differential equations (713) – 9.3 Difference methods for the solution of boundary-value problems on the Poisson equation in the plane (715)	
7.1.3 Realization of numerical models in digital computer systems 718	
1. Criteria for the choice of a method (718) – 2. Methods of control (719) – 3. The presentation of functions (719)	
7.1.4 Nomography and slide rules 722	
1. Relations between two variables; function ladders (or scales) (722) – 2. Slide rules (723) – 3. Alignments and plane nets (724)	
7.1.5 Processing empirical data 726	
1. The method of least squares (727) – 1.1 The smoothing of direct observations (727) – 1.2 Smoothing with straight lines $\hat{y} = ax + b$ (727) – 1.3 Smoothing parabola $\hat{y} = ax^2 + bx + c$ (728) – 2. Further smoothing principles (729)	
7.2 Computing technique and data processing 730	
7.2.1 Electronic digital computers (data processing systems) 730	
1. Introductory remarks (730) – 2. The presentation of information and the storage unit of an electronic digital computer (730) – 3. Transfer channels (731) – 4. The programme (731) – 5. The programming (732) – 6. The steering of electronic digital computers (733) – 7. The mathematical equipment (programme library) of an electronic digital computer (734) – 8. Carrying out work on an electronic digital computer (734)	
7.2.2 Analogue computers 735	
1. The principle of the technique of computing by analogy (735) – 2. Computing elements of an analogue computer (737) – 3. The fundamental programming of systems of ordinary differential equations (737) – 4. Quantitative programming (738)	
8 Analysis 741	
8.1 Functional analysis 741	
8.1.1 Spaces 741	
1. Metric spaces (741) – 1.1 Definitions and examples (741) – 1.2 Convergence in a metric space (742) – 1.3 Closed and open sets (742) – 1.4 Separability (743) – 1.5 Compactness (744) – 2. Normed spaces (745) – 3. Banach spaces (748) – 4. Hilbert spaces (751)	
8.1.2 Operators and functionals 755	
1. General concepts (755) – 2. Linear functionals (758) – 2.1 Extension of linear functionals, corollaries (758) – 2.2 Linear functionals in special spaces (759) – 2.3 Spaces of operators and functionals (761) – 3. Linear operators with special properties (766) – 4. The spectrum (773) – 5. The Sobolev spaces $W_p^1(\Omega)$ and the embedding theorems (774)	
8.1.3 General existence theorems on the solution of operator equations 777	
1. The Banach fixed point theorem and applications (778) – 2. The Schauder fixed point theorem (780)	
8.1.4 The equations $f - \mu Kf = g$ and $\lambda f - Kf = g$ 781	
1. The equation $f - \mu Kf = g$ in a Hilbert space or a Banach space with a basis (782) – 2. The Equation $\lambda f - Kf = g$ in an arbitrary Banach space B (784) – 3. Special results when K is normal or symmetric (785) – 4. Spectral representation of self-adjoint operators in a Hilbert space (787)	
8.1.5 Approximation methods 789	
1. On the Banach fixed point theorem (789) – 2. Newton's method (791) – 3. The Ritz and Trefftz methods (795) – 4. The Galerkin method (799)	
8.2 Measure theory and the Lebesgue-Stieltjes integral 800	
8.2.1 Content and measure 800	
8.2.2 Content and measure in n -dimensional Euclidean space \mathbb{R}^n 801	
1. The Peano-Jordan content (801) – 2. Extension to the Lebesgue measure (802) – 3. The Lebesgue-Stieltjes measure (803)	
8.2.3 Measurable functions 803	
8.2.4 The Lebesgue-Stieltjes integral 804	
1. Definition of the integral (804) – 2. Summable functions (805) – 3. Rules of integration (805) – 4. Limit theorems (806) – 5. The indefinite Lebesgue-Stieltjes integral (806)	
8.2.5 The Stieltjes integral for functions of one variable 807	
8.3 Tensor calculus 808	
8.3.1 Tensor algebra 808	
1. Basic concepts (808) – 2. Algebraic operations with tensors (810) – 3. Special tensors (812) – 4. Tensor equations (813)	
8.3.2 Tensor analysis 813	
1. Tensor functions of a scalar variable (813) – 2. Tensor fields (814) – 3. Covariant differentiation (815) – 4. Vector analysis in curvilinear coordinates (817) – 5. Alternating differential forms and vector analysis (817)	

8.4	Integral equations	826
8.4.1	General concepts	826
8.4.2	Simple integral equations that can be reduced to ordinary differential equations by differentiation	827
8.4.3	Integral equations that can be solved by differentiation	828
8.4.4	The Abel integral equation	829
8.4.5	Integral equations with product kernels	831
8.4.6	The Neumann (stepwise) approximation	836
8.4.7	The Fredholm method of solution	841
8.4.8	The Nyström approximation method for the solution of Fredholm integral equations of the second kind	844
8.4.9	The Fredholm alternative for integral equations of the second kind. Symmetric kernels	846
8.4.10	The operator method in the theory of integral equations	847
8.4.11	The Schmidt series	854
9	Mathematical methods of operational research	857
9.1	Integral linear optimization	857
9.1.1	Statement of the problem and geometrical interpretation	857
9.1.2	Gomory's cut method	858
	1. Purely integral linear optimization problems (858) – 2. Mixed integral linear optimization problems (859)	
9.1.3	Branching procedures	860
9.1.4	Comparison of the procedures	862
9.2	Non-linear optimization	863
9.2.1	Survey and special types of problem	863
	1. General non-linear optimization problems in \mathbb{R}^n ; convex optimization (863) – 2. Linear quotient optimization (863) – 3. Quadratic optimization (864) – 3.1 Wolfe's procedure (864) – 3.2 The Hildreth-d'Esopo iteration procedure (866) – 3.3 The problem of linear complementarity, Lemke's procedure (868)	
9.2.2	Convex optimization	868
	1. Fundamental theoretical results (868) – 2. Free optimization problems for unimodal functions (871) – 2.1 Direct search for a minimum (871) – 2.2 Descent procedures (872) – 2.3 Methods with conjugate directions (873) – 3. Gradient procedures for problems with constraints (874) – 3.1 Basic concepts (874) – 3.2 Procedure with an optimal useful direction (875) – 3.3 The method of projected gradients (877) – 4. The method of intersecting planes (879) – 5. Transforming a problem with constraints into a free one (881)	
9.3	Dynamic optimization	882
9.3.1	Model structure and basic concepts in the deterministic case	882
	1. Introductory example and Bellman's principle (882) – 2. Stationary processes (884) – 3. Forwards and backwards solution (884)	
9.3.2	Theory of Bellman's functional equations	885
	1. Statement of the problem and classification (885) – 2. Existence and uniqueness theorems for the Types I and II (885) – 3. Monotonicity Type III (886) – 4. Fundamental remarks about practical solutions (886)	
9.3.3	Examples of deterministic dynamic optimization	887
	1. The problem of storage (887) – 2. Sharing-out problem (888) – 3. Determination of the rank in a net plan (889)	
9.3.4	Stochastic dynamic modes	889
	1. Generalization of the deterministic model (889) – 2. The stochastic model and the role of Bellman's principle (889) – 3. Example: A storage problem (890) – 3.1 The model (890) – 3.2 The functional equation and the (s, S) -policy (891)	
9.4	Graph theory	891
9.4.1	Basic concepts of the theory of directed graphs	891
9.4.2	The technique of net plans (longest paths in net plans)	892
	1. Monotonic numbering and Ford's algorithm (892) – 2. Finding the critical path (893) – 3. Deadlines and buffer times for the processes (895) – 4. Programme evaluation and review technique (PERT) (895)	
9.4.3	Shortest paths in graphs	896
	1. Algorithms (896) – 2. Example (897)	
9.5	Theory of games	898
9.5.1	Statement of the problem and classification	898
9.5.2	Matrix games	899
	1. Definitions and theoretical results (899) – 2. Solution through linear optimization (900) – 3. Solution by iteration or relaxation (901)	

9.6	Combinatorial optimization problems	902
9.6.1	Characterization and typical examples	902
9.6.2	The Hungarian method for the solution of assignment problems	903
9.6.3	Branch-and-bound algorithms	907
	1. The basic idea (907) – 2. An example: Deployment of discrete means (907) – 3. Application to the use of machines (910)	
10	Mathematical information processing	911
10.1	Basic concepts	911
10.2	Automata	912
10.2.1	Abstract deterministic automata	912
10.2.2	The synthesis of automata	918
10.2.3	The realization of automata	920
10.2.4	Non-deterministic and stochastic automata	921
10.3	Algorithms	924
10.3.1	Basic concepts	924
10.3.2	Turing machines	924
10.3.3	Computing automata	927
10.4	Elementary switch algebra	929
10.4.1	Connection with the calculus of propositions	929
	1. Series-parallel switchworks (929) – 2. The switch function (929) – 3. Logical description of switchworks (930) – 4. Analysis and synthesis (930)	
10.4.2	Optimal normal forms	932
	1. Basic concepts (932) – 2. Procedure for determining the prime conjunctions (932) – 2.1 The Karnaugh table (932) – 2.2 McCluskey's method (933) – 2.3 Quine's method (933) – 2.4 Nelson's method (934) – 3. Minimal normal forms (934)	
10.4.3	Switchworks with incompletely given working conditions	935
10.5	Simulation and statistical planning and optimization of experiments	936
10.5.1	Simulation	936
	1. General remarks (936) – 2. The Monte Carlo method (936) – 3. Advantages and disadvantages of simulation (938) – 4. Examples of applications (938) – 4.1 Sequential optimizations (938) – 4.2 Servicing models (939) – 4.3 Game-theoretical models (939) – 4.4 The investigation of model sensitivity (940) – 5. Remarks about adaptation (940)	
10.5.2	Statistical design and optimization of experiments	941
	1. Choice of the influencing and object quantities (941) – 2. Design of experiments (941) – 2.1 Factorial designs (942) – 2.2 Composite rotatable experiment designs of the second order (943) – 3. Carrying out the optimization (944) – 3.1 Optimization according to Box-Wilson (944) – 3.2 The ridge-line analysis according to Hoerl (944)	
Bibliography		947
Index		953