Airy Function Zeroes

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On the negative real axis ($x < 0$), the Airy function

$$\mathrm{Ai}(x) = \frac{1}{3}(-x)^{1/2}\left[J_{-\frac{1}{3}}\left(\tfrac{2}{3}(-x)^{3/2}\right) + J_{\frac{1}{3}}\left(\tfrac{2}{3}(-x)^{3/2}\right)\right]$$

has an oscillatory behavior similar to that of the Bessel function $J_v(x)$ [1]. Note the special values [2]

$$\mathrm{Ai}(0) = \frac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})} = 0.3550280538..., \qquad \mathrm{Ai}'(0) = -\frac{1}{3^{\frac{1}{3}}\Gamma(\frac{1}{3})} = -0.2588194037...$$

and the integral representations

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + x\,t\right)dt, \qquad \mathrm{Ai}'(x) = -\frac{1}{\pi}\int_0^\infty t\sin\left(\frac{1}{3}t^3 + x\,t\right)dt.$$

Let $0 < a_1 < a_2 < \ldots$ be the zeroes of $\mathrm{Ai}(-x)$ and $0 < a_1' < a_2' < \ldots$ be the zeroes of $\mathrm{Ai}'(-x)$. See Table 1 for the first several terms of both sequences. We saw these values when bounding the zeroes of $J_v(x)$ [1] and we will see them again when estimating the L_1-norm of Brownian motion [3]. In the present essay, our focus is on two applications to physics.

Table 1 *Negatives of zeroes of* Ai *and* Ai$'$ *for* $n = 1, 2, 3, 4, 5$

a_n	a_n'
2.3381074104...	1.0187929716...
4.0879494441...	3.2481975821...
5.5205598280...	4.8200992111...
6.7867080900...	6.1633073556...
7.9441335871...	7.3721772550...

0.1. Quantum Mechanics of Falling. Consider a quantum mechanical (QM) particle in free fall, that is, on the positive x-axis with linear potential x. The time-independent Schrödinger equation becomes

$$\frac{d^2 f}{dx^2} + (\lambda - x) f = 0, \qquad \lim_{x \to \infty} f(x) = 0.$$

If a Dirichlet condition $f(0) = 0$ is imposed (elastic reflection), then the eigenvalues λ are the Airy function zeroes $\{a_n\}_{n=1}^{\infty}$ [4, 5, 6, 7, 8, 9]. If instead a Neumann condition $f'(0) = 0$ is imposed, then the eigenvalues λ are the derivative zeroes $\{a_n'\}_{n=1}^{\infty}$ [10, 11].

What is the physical significance of these results? The eigenfunctions f contain information about the behavior of the particle, for example, the probability densities of position and momentum. Admissible solutions to the time-independent Schrödinger equation exist only if the total energy of the particle is quantized, that is, restricted to a discrete set of eigenvalues λ. (This counterintuitive fact is akin to Bohr's model of the hydrogen atom possessing discrete shells for the electron to occupy, as indicated by spectroscopy.) Different boundary conditions or different potentials, of course, lead to different allowed energy levels.

Consider rather a QM particle on the whole x-axis with the potential $|x|$. Then the eigenvalues corresponding to even eigenfunctions come from $\{a_n'\}$ and the eigenvalues corresponding to odd eigenfunctions come from $\{a_n\}$ [10, 11]. A listing of the eigenvalues λ consists of the interlaced zeroes of Ai' and Ai. It is remarkable that the Airy function zeroes occur here, in the QM analog of the simplest of all classical physics problems.

0.2. Van der Pol's Equation. For constant $\mu > 0$, all solutions of van der Pol's equation

$$\frac{d^2 g}{dt^2} + \mu(g^2 - 1)\frac{dg}{dt} + g = 0,$$

other than the trivial solution $g = 0$, tend to a unique periodic limit cycle as $t \to \infty$. The proof of this theorem is due to Liénard [12]. We are interested in how the magnitude $A(\mu)$ and the period $T(\mu)$ of the limit cycle vary with increasing μ.

Let $\alpha = a_1 = 2.3381074104...$ for convenience. The work of Haag [13], Dorodnicyn [14] and others [15, 16, 17, 18, 19, 20] gives that

$$\begin{aligned} A(\mu) &= 2 + \frac{1}{3}\alpha\mu^{-4/3} - \frac{16}{27}\mu^{-2}\ln(\mu) \\ &\quad + \frac{1}{9}\left(3\beta + 2\ln(2) - 8\ln(3) - 1\right)\mu^{-2} + O\left(\mu^{-8/3}\right) \end{aligned}$$

$$\begin{aligned} T(\mu) &= (3 - 2\ln(2))\mu + 3\alpha\mu^{-1/3} - \frac{2}{3}\mu^{-1}\ln(\mu) \\ &\quad + \left(3\beta + \ln(2) - \ln(3\pi) - 2\ln\left(\mathrm{Ai}'(-\alpha)\right) - 1\right)\mu^{-1} + O\left(\mu^{-4/3}\ln(\mu)\right) \end{aligned}$$

as $\mu \to \infty$, where $\beta = 0.17234...$ is defined as follows. The function $-\operatorname{Ai}'(x)/\operatorname{Ai}(x)$ maps the interval $(-\alpha, \infty)$ onto $(-\infty, \infty)$ in a one-to-one fashion; let $z(x)$ denote its inverse. Define $Q(x) = x^2 - z(x)$ and

$$P(x) = \exp\left(-\int_0^x \frac{1}{Q(u)^2}du\right).$$

Then the expression

$$\frac{1}{P(x)}\int_x^\infty P(v)\left\{\frac{v}{Q(v)} - \frac{v^3}{3Q(v)^2} - \frac{2v}{3(v^2+\alpha/2)} + \frac{\ln(v^2+\alpha/2)}{3Q(v)^2}\right\}dv$$

approaches β as $x \to -\infty$. Hence, for example, we have the asymptotic expression

$$T(\mu) \sim (1.613705...)\mu + (7.014322...)\mu^{-1/3} - (0.666666...)\mu^{-1}\ln(\mu) - (1.3232...)\mu^{-1}.$$

The final coefficient for $T(\mu)$ is sometimes written as $3\beta + 3\ln(2) - \ln(3) - 1 - 2\iota$ or as $\beta + 3\ln(2) - \ln(3) - 3/2 - 2\delta$, where

$$\iota = \ln(2) + \tfrac{1}{2}\ln(\pi) + \ln\left(\operatorname{Ai}'(-\alpha)\right) = 0.9105654320..., \qquad \delta = -\beta + \iota - \tfrac{1}{4} = 0.4882....$$

Two additional terms in the series for $A(\mu)$ were determined by Bavinck & Grasman [20, 21]; we omit these for reasons of space. Early textbooks [22, 23] often repeat errors originating in [14]; the final two coefficients for $T(\mu)$ are mistakenly given as $-22/9$ and $+0.0087$.

A relevant theory of special functions arose in [24, 25, 26]. For example, the Haag function $\operatorname{Hg}(x)$ is defined to be what we call $-z(-x)$; thus $\operatorname{Hg}(0) = a_1'$, $\lim_{x\to\infty}\operatorname{Hg}(x) = a_1$ and

$$\frac{d}{dx}\operatorname{Hg}(x) = \frac{1}{x^2 + \operatorname{Hg}(x)}, \qquad \lim_{x\to-\infty}\frac{\operatorname{Hg}(x)}{x^2} = -1.$$

The Dorodnicyn function $\operatorname{Dn}(x)$ satisfies

$$\frac{d}{dx}\operatorname{Dn}(x) = -\frac{\operatorname{Dn}(x)}{\left(x^2+\operatorname{Hg}(x)\right)^2} + \frac{x}{x^2+\operatorname{Hg}(x)}, \qquad \lim_{x\to-\infty}\operatorname{Dn}(x) = -\frac{1}{2}$$

as well as

$$\lim_{x\to\infty}\left(\operatorname{Dn}(x) - \ln(x)\right) = -\frac{3}{2}\beta - \frac{1}{4} = -0.50851....$$

Clearly Hg has a unique zero at $-3^{1/3}\Gamma(2/3)/\Gamma(1/3) = -0.7290111329...$; a similar exact expression for the unique zero $0.8452...$ of Dn isn't known.

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