Apollonian Circles with Integer Curvatures

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Given four mutually tangent circles (one of them internally tangent to the other three), we can inscribe into each of the remaining curvilinear triangles a unique circle. Continuing iteratively in this manner, we obtain what is known as an **Apollonian circle packing**. If the initial four circles possess integer curvatures (reciprocal radii), then all of the circles in the packing possess integer curvatures. Some introductory accounts of this subject include [1, 2, 3, 4]. We examine just two examples, the first starting with curvatures $\{-1, 2, 2, 3\}$ (Figure 1) and the second starting with curvatures $\{-11, 21, 24, 28\}$ (Figure 2). The outer circle is given negative curvature – indicating that the other circles are in its interior – and it is the unique circle with this property.

How are the integer curvatures obtained for each example? Define four 4×4 matrices

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}$$

and consider products $S_{j_1} S_{j_2} \cdots S_{j_n}$ with each $j_k \in \{1, 2, 3, 4\}$ and $j_k \neq j_{k+1}$ for any k. The second generation of circles has curvatures

$$\begin{aligned} (S_4 w)_4 &= 3, \\ (S_3 w)_3 &= 6, \\ (S_2 w)_2 &= 6, \\ (S_1 w)_1 &= 15 \end{aligned}$$

when $w = (-1, 2, 2, 3)$ (the bugeye circle packing) and

$$\begin{aligned} (S_4 w)_4 &= 40, \\ (S_3 w)_3 &= 52, \\ (S_2 w)_2 &= 61, \\ (S_1 w)_1 &= 157 \end{aligned}$$

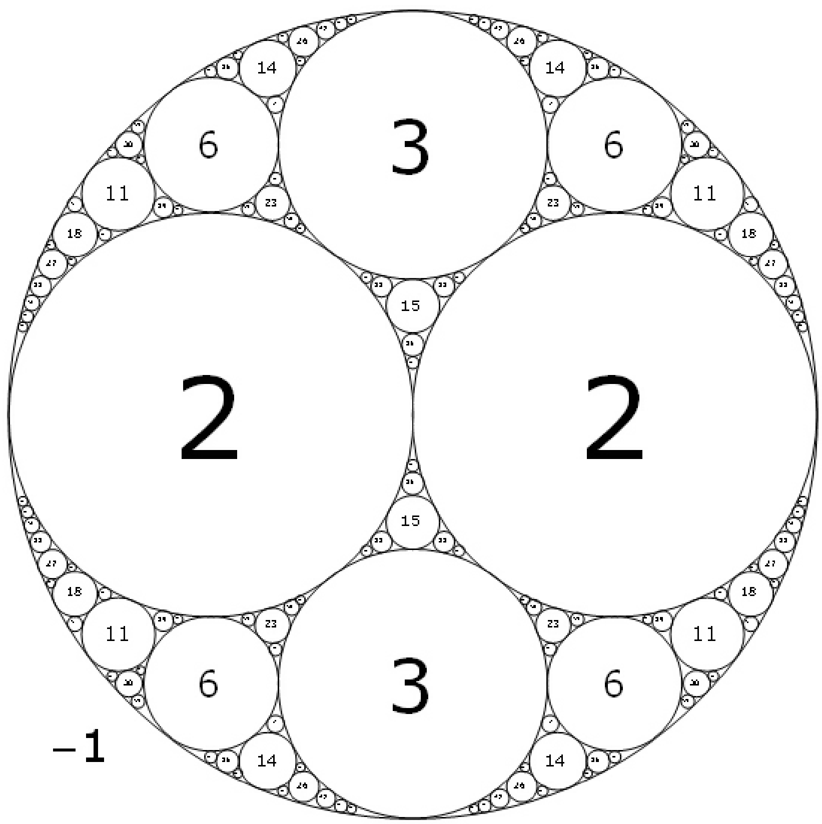

Figure 1: Bugeye circle packing.

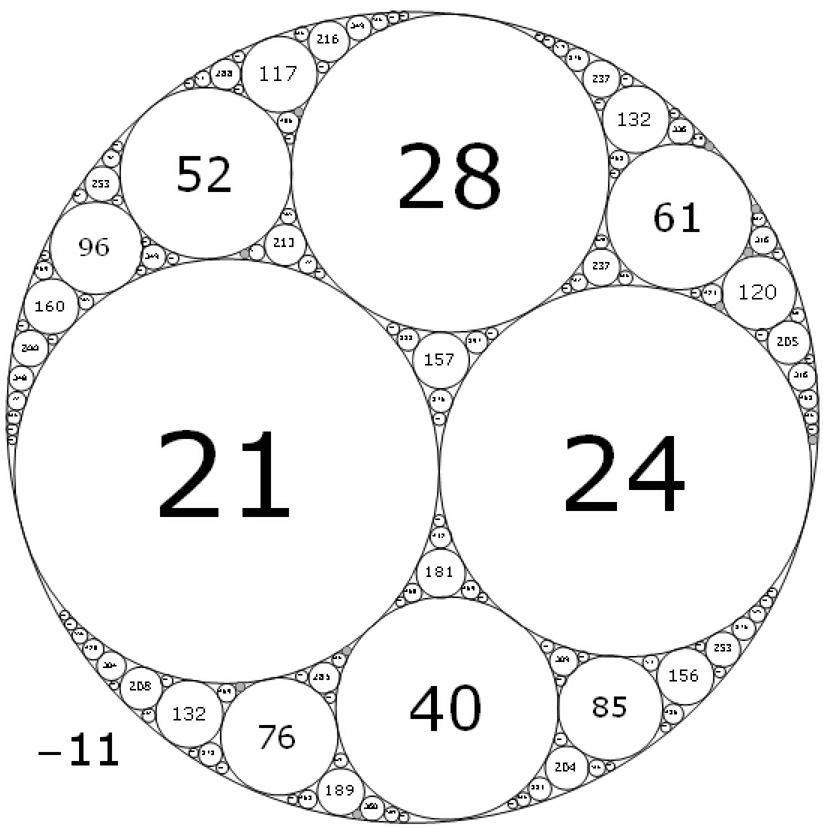

Figure 2: Nickel-dime-quarter packing.

when $w = (-11, 21, 24, 28)$ (the nickel-dime-quarter packing). The third generation of circles has curvatures

$$\begin{array}{lll} (S_1S_4w)_1 = 15, & (S_2S_4w)_2 = 6, & (S_3S_4w)_3 = 6, \\ (S_1S_3w)_1 = 23, & (S_2S_3w)_2 = 14, & (S_4S_3w)_4 = 11, \\ (S_1S_2w)_1 = 23, & (S_3S_2w)_3 = 14, & (S_4S_2w)_4 = 11, \\ (S_2S_1w)_2 = 38, & (S_3S_1w)_3 = 38, & (S_4S_1w)_4 = 35 \end{array}$$

when $w = (-1, 2, 2, 3)$ and

$$\begin{array}{lll} (S_1S_4w)_1 = 181, & (S_2S_4w)_2 = 85, & (S_3S_4w)_3 = 76, \\ (S_1S_3w)_1 = 213, & (S_2S_3w)_2 = 117, & (S_4S_3w)_4 = 96, \\ (S_1S_2w)_1 = 237, & (S_3S_2w)_3 = 132, & (S_4S_2w)_4 = 120, \\ (S_2S_1w)_2 = 397, & (S_3S_1w)_3 = 388, & (S_4S_1w)_4 = 376 \end{array}$$

when $w = (-11, 21, 24, 28)$. The fourth generation of circles for the latter starts with $(S_4S_3S_4w)_4 = 132$, which is the first duplicate; the next two terms are $(S_4S_2S_4w)_4 = 156$ and $(S_3S_4S_3w)_3 = 160$. Arranging all the curvatures in order (with multiplicities), we have [5]

$$\begin{array}{l} 2, 2, 3, 3, 6, 6, 6, 6, 11, 11, 11, 11, 14, 14, 14, 14, 15, 15, 18, 18, 18, 18, 23, 23, 23, 23, 26, 26, 26, 26, \\ 27, 27, 27, 27, 30, 30, 30, 30, 35, 35, 35, 35, 35, 35, 38, 38, 38, 38, 38, 38, 38, 38, 39, 39, 39, 39, \\ 42, 42, 42, 42, 47, 47, 47, 47, 50, 50, 50, 50, 51, 51, 51, 51, 54, 54, 54, 54, 59, 59, 59, 59, 59, 59, 59, 59, \ldots \end{array}$$

when $w = (-1, 2, 2, 3)$ and

$$\begin{array}{l} 21, 24, 28, 40, 52, 61, 76, 85, 96, 117, 120, 132, 132, 156, 157, 160, 181, 189, \\ 204, 205, 208, 213, 216, 237, 237, 244, 253, 253, 285, 288, 304, 309, 316, 316, \ldots \end{array}$$

when $w = (-11, 21, 24, 28)$. A theorem due to Kontorovich & Oh [6] provides the growth rate for these sequences:

$$\nu(x) \sim c \cdot x^{\delta}$$

as $x \to \infty$, where $\nu(x)$ is the number of circles in the packing with curvature less than x, the exponent $\delta = 1.30568...$ has been discussed [7], and the coefficients

$$c = \begin{cases} 0.402... & \text{if } w = (-1, 2, 2, 3), \\ 0.0176... & \text{if } w = (-11, 21, 24, 28) \end{cases}$$

were estimated by Fuchs & Sanden [8]. (The values 0.201... in [2] and 0.0458... in [3] are apparently mistaken.) Expressions for c exist [9, 10, 11], but are not suitably practical to allow numerical calculations.

Rather than counting all circles with curvature $< x$, we might instead restrict attention to the n^{th} generation (which has $4 \cdot 3^{n-2}$ members) and determine the average curvature as a function of n. Most circles born at a large generation n possess curvature $\sim \exp(\gamma n)$, where $\gamma = 0.9149...$ is the Lyapunov exponent associated with random products $S_{j_1} S_{j_2} \cdots S_{j_n}$. The logarithm of curvature, divided by n, is asymptotically normal with mean γ and variance $\sim \alpha/n$, where $\alpha = 0.065....$ We hope to elaborate on this alternative approach later [3, 12].

Kissing Primes. The primes appearing in the preceding sequences (curvatures with multiplicities) are [5]

$$2, 2, 3, 3, 11, 11, 11, 11, 23, 23, 23, 23, 47, 47, 47, 47, 59, 59, 59, 59, 59, 59, 59, 59, 71, 71, 71, 71,$$
$$83, 83, 83, 83, 83, 83, 83, 83, 107, 107, 107, 107, 107, 107, 107, 107,$$
$$131, 131, 131, 131, 131, 131, 131, 131, 131, 131, 131, 131,$$
$$167, 167, 167, 167, 167, 167, 167, 167, 167, 167, 167, 167, ...$$

when $w = (-1, 2, 2, 3)$ and

$$61, 157, 181, 349, 373, 397, 421, 541, 661, 709, 733, 829, 853, 877, ...$$

when $w = (-11, 21, 24, 28)$. Each term corresponds to a circle C of prime curvature $a(C)$. Define a weighted prime count

$$\psi(x) = \sum_{\substack{a(C)<x, \\ a(C) \text{ prime}}} \ln(a(C))$$

then it is conjectured that

$$\psi(x) \sim G \cdot \nu(x)$$

as $x \to \infty$, where the coefficient $G = 0.9159655941...$ is Catalan's constant [13]. It is remarkable that the coefficient is independent of the packing.

Assume that an unordered pair of tangent circles C, C' are both of prime curvature p, p'. The two primes are said to be **kissing primes** (for the packing under consideration). We have pairs (with multiplicities)

$$(2, 2), (2, 3), (2, 3), (2, 3), (2, 3), (2, 11), (2, 11), (2, 11), (2, 11), (2, 23), (2, 23), (2, 23), (2, 23),$$
$$(3, 23), (3, 23), (3, 23), (3, 23), (3, 47), (3, 47), (3, 47), (3, 47), (2, 59), (2, 59), (2, 59), (2, 59), ...$$

when $w = (-1, 2, 2, 3)$ and

$$(157, 397), (61, 421), (61, 1069), (157, 1093), (181, 1213), ...$$

when $w = (-11, 21, 24, 28)$. Define a weighted prime count

$$\psi^{(2)}(x) = \sum_{\substack{a(C),a(C')<x,\\ C,C' \text{ tangent},\\ a(C),a(C') \text{ prime}}} \ln(a(C)) \cdot \ln(a(C'))$$

then it is conjectured that

$$\psi^{(2)}(x) \sim H \cdot \nu(x)$$

as $x \to \infty$, where the coefficient

$$H = G^2 \cdot 2 \prod_{p\equiv 3 \bmod 4} \left(1 - \frac{2}{p(p-1)^2}\right) = G^2(1.6493376890...) = 3(0.4612609086...)$$

is again independent of the packing. These estimates improve upon the values 1.646... in [8] and 0.0460... in [3].

The number of circles of prime curvature $< x$ is asymptotically $\psi(x)/\ln(x)$, hence $\sim G\cdot\nu(x)/\ln(x)$ by the first-order conjecture. For the number of kissing prime circles both with curvatures $< x$, the relationship with $\psi^{(2)}(x)/\ln(x)^2$ is less clear. This would be good to clarify someday. Interestingly, Catalan's constant also appears in [1], although in an unrelated manner.

Recent progress on this subject is described in [14, 15, 16, 17, 18, 19].

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