Series involving Arithmetic Functions

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We intend here to collect infinite series, each involving unusual combinations or variations of well-known arithmetic functions. For simplicity's sake, results are often quoted not with full generality but only to illustrate a special case.

Let $\sigma(n)$ denote the sum of all distinct divisors of n, $\kappa(n)$ denote the quotient of n with its greatest square divisor, and $\varphi(n)$ denote the number of positive integers $k \leq n$ satisfying $\gcd(k,n) = 1$. These multiplicative functions are called sum-of-divisors, square-free part, and Euler totient, respectively. It can be shown that the following series are convergent:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{\sigma(n)\varphi(n)} &= \prod_{p}\left(1+\sum_{r=1}^{\infty}\frac{1}{p^{r-1}(p^{r+1}-1)}\right)\\ &= 1.7865764593...,\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{\kappa(n)\varphi(n)} &= \prod_{p}\left(1+\frac{2p}{(p-1)(p^2-1)}\right)\\ &= \frac{\pi^2}{6}\prod_{p}\left(1+\frac{p+1}{p^2(p-1)}\right)\\ &= 3.9655568689... = A\end{aligned}$$

where the product is over all primes p. The former was considered by Silverman [1] while studying the number of generators possessing large order in the group $\mathbb{Z}_j^*$. With regard to the latter, more precise asymptotics can be given [2]:

$$\begin{aligned}\sum_{n\leq N}\frac{1}{\kappa(n)\varphi(n)} &\sim A-\prod_{p}\left(1+\frac{\sqrt{p}+1}{p(p-1)}\right)\cdot\frac{1}{\sqrt{N}}\\ &\sim A-\prod_{p}\left(1+\frac{1}{p(\sqrt{p}-1)}\right)\cdot\frac{1}{\sqrt{N}}\\ &\sim A-\frac{4.9478356259...}{\sqrt{N}}.\end{aligned}$$

Let $d(n)$ denote the number of distinct divisors of n, and $\omega(n)$ denote the number of distinct prime factors of n. The divisor function $d(n)$ is multiplicative; in contrast, $\omega(n)$ is additive. It can be shown that [3, 4]

$$\sum_{n\leq N} d(n)\omega(n) \sim 2N\ln(N)\ln(\ln(N)) + 2B\,N\ln(N)$$

where

$$\begin{aligned}
B &= -\Gamma'(2) + \sum_p \left(\ln\left(1-\frac{1}{p}\right) + \frac{1}{2}\left(1-\frac{1}{p}\right)^2 \sum_{k=1}^{\infty} \frac{k+1}{p^k} \right) \\
&= -(1-\gamma) + \sum_p \left(\ln\left(1-\frac{1}{p}\right) + \frac{1}{p} - \frac{1}{2p^2} \right) \\
&= M - 1 - \frac{1}{2}\sum_p \frac{1}{p^2} = -0.9646264971...
\end{aligned}$$

where M is the Meissel-Merten constant [5] and γ is the Euler-Mascheroni constant [6].

The mean of distinct divisors of n is clearly $\sigma(n)/d(n)$. It can be shown that [7, 8]

$$\sum_{n\leq N} \frac{\sigma(n)}{d(n)} \sim \frac{C}{2\sqrt{\pi}} \frac{N^2}{\sqrt{\ln(N)}}, \qquad \#\left\{ n : \frac{\sigma(n)}{d(n)} \leq x \right\} \sim D\,x\ln(x)$$

where

$$\begin{aligned}
C &= \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{1}{k+1} \left(\sum_{j=0}^{k} \frac{1}{p^j} \right) \frac{1}{p^k} \right) \left(1-\frac{1}{p}\right)^{1/2} \\
&= \prod_p \left(1 + \frac{1}{p-1} \sum_{k=1}^{\infty} \frac{1}{k+1} \frac{p^{k+1}-1}{p^{2k}} \right) \left(1-\frac{1}{p}\right)^{1/2} \\
&= \prod_p \left(1-\frac{1}{p}\right)^{-1/2} p\ln\left(1+\frac{1}{p}\right) = 1.2651951601... \\
&= (0.7138099304...)\sqrt{\pi} = 2(0.3569049652...)\sqrt{\pi},
\end{aligned}$$

$$\begin{aligned}
D &= \prod_p \left(1 + \sum_{k=1}^{\infty} (k+1) \left(\sum_{j=0}^{k} p^j \right)^{-1} \right) \left(1-\frac{1}{p}\right)^2 \\
&= \prod_p \left(1 + (p-1) \sum_{k=1}^{\infty} (k+1) \frac{1}{p^{k+1}-1} \right) \left(1-\frac{1}{p}\right)^2 \\
&= 0.4950461958....
\end{aligned}$$

The lag-one autocorrelation of $d(n)$ is evident via [9]

$$\sum_{n\leq N} d(n)d(n+1) \sim \frac{6}{\pi^2} N \ln(N)^2;$$

a variation of this includes [10]

$$\sum_{n\leq N} d(n)^2 d(n+1) \sim \frac{1}{\pi^2} \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1-\frac{1}{p}\right)^2 \left(1+\frac{1}{p}\right)^{-1}\right) N \ln(N)^4.$$

Let $r(n)$ denote the number of representations of n as a sum of two squares, counting order and sign (note that $r(n)/4$ is multiplicative). We have [11]

$$\sum_{n\leq N} r(n)^2 d(n+1) \sim 6 \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1-\frac{\chi(p)}{p}\right)^2 \left(1+\frac{1}{p}\right)^{-1}\right) N \ln(N)^2$$

where $\chi(k) = (-4/k)$ is 0 when k is even and $(-1)^{(k-1)/2}$ when k is odd. Also, if $\tau(n)$ denotes the Ramanujan tau function [12], then [13, 14, 15]

$$\sum_{n\leq N} \tau(n)^2 d(n+1) \sim \prod_p \left(1 - \frac{1}{p} + \frac{p^2 - 2p\cos(2\theta_p) + 1}{p^2(p+1)}\right) N^{12} \ln(N)^2$$

where $2\cos(\theta_p) = \tau(p) p^{-11/2}$. Other autocorrelation results include [9]

$$\sum_{n\leq N} \sigma(n)\sigma(n+1) \sim \frac{5}{6} N^3,$$

$$\sum_{n\leq N} \varphi(n)\varphi(n+1) \sim \frac{1}{3} \prod_p \left(1 - \frac{2}{p^2}\right) N^3 = \frac{0.3226340989...}{3} N^3$$

and the latter product is known as the Feller-Tornier constant [16].

Logarithms of arithmetic functions provide some interesting constants [17, 18, 19, 20, 21]:

$$\frac{1}{\ln(2)} \sum_{n\leq N} \ln(d(n)) \sim N \ln(\ln(N)) + E_1 N,$$

$$\sum_{n\leq N} \ln(\varphi(n)) \sim N \ln(N) + E_2 N, \qquad \sum_{n\leq N} \ln(\sigma(n)) \sim N \ln(N) + E_3 N,$$

$$\sum_{n\leq N}{}' \frac{\ln(\varphi(n))}{\ln(\sigma(n))} \sim N + E_4 \frac{N}{\ln(N)}$$

where

$$\begin{aligned}E_1 &= \gamma + \sum_{k=2}^{\infty}\left(\frac{1}{\ln(2)}\ln\left(1+\frac{1}{k}\right)-\frac{1}{k}\right)\sum_p \frac{1}{p^k}\\ &= M + \frac{1}{\ln(2)}\sum_{k=2}^{\infty}\ln\left(1+\frac{1}{k}\right)\sum_p \frac{1}{p^k},\end{aligned}$$

$$E_2 = -1 + \sum_p \frac{1}{p}\ln\left(1-\frac{1}{p}\right) = -1 + \ln(0.5598656169......),$$

$$E_3 = -1 + \sum_p \left(1-\frac{1}{p}\right)\sum_{k=1}^{\infty}\frac{1}{p^k}\ln\left(\frac{p^{k+1}-1}{p^k(p-1)}\right),$$

$$E_4 = \sum_p \left(1-\frac{1}{p}\right)\sum_{k=1}^{\infty}\left(2\ln\left(1-\frac{1}{p}\right)-\ln\left(1-\frac{1}{p^{k+1}}\right)\right)\frac{1}{p^k}$$

and $\sum'$ is interpreted as summation over all n avoiding division by zero. The constant $\exp(1+E_2)$ appeared in [22] as well.

Let $a(n)$ denote the number of non-isomorphic abelian groups of order n and $P(k)$ denote the number of unrestricted partitions of k. It can be shown that [23, 24]

$$\sum_{n\le N}{}' \frac{1}{\ln(a(n))} = N\int_{-\infty}^{0}\left(\prod_p\left(1+\sum_{k=2}^{\infty}\frac{P(k)^t-P(k-1)^t}{p^k}\right)-\frac{6}{\pi^2}\right)dt.$$

Let $s(n)$ denote the number of non-isomorphic semisimple rings of order n and $Q(k)$ denote the number of unordered sets of integer pairs (r_j, m_j) for which $k=\sum_j r_j m_j^2$ and $r_j m_j^2 > 0$ for all j. Likewise, we have

$$\sum_{n\le N}{}' \frac{1}{\ln(s(n))} = N\int_{-\infty}^{0}\left(\prod_p\left(1+\sum_{k=2}^{\infty}\frac{Q(k)^t-Q(k-1)^t}{p^k}\right)-\frac{6}{\pi^2}\right)dt.$$

If $n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r}$ is the prime factorization of n, define three additive functions

$$\beta(n)=\sum_{j=1}^{r}p_j,\qquad B(n)=\sum_{j=1}^{r}\alpha_j p_j,\qquad \hat{B}(n)=\sum_{j=1}^{r}p_j^{\alpha_j},$$

the first two of which contrast nicely with the better-known functions

$$\omega(n)=\sum_{j=1}^{r}1,\qquad \Omega(n)=\sum_{j=1}^{r}\alpha_j.$$

While [5]

$$\frac{1}{N}\sum_{n\le N}\omega(n)\sim\ln(\ln(N))+M,\qquad \frac{1}{N}\sum_{n\le N}\Omega(n)\sim\ln(\ln(N))+M+\sum_{p}\frac{1}{p(p-1)}$$

we have [25, 26, 27]

$$\sum_{n\le N}\beta(n)\sim\sum_{n\le N}B(n)\sim\sum_{n\le N}\hat{B}(n)\sim\frac{\pi^2}{12}\frac{N^2}{\ln(N)}.$$

While [28, 29]

$$\begin{aligned}\sum_{n\le N}{}'\frac{1}{\Omega(n)-\omega(n)} &\sim N\int_0^1\left(\prod_p\left(1+\sum_{k=2}^{\infty}\frac{t^{k-1}-t^{k-2}}{p^k}\right)-\frac{6}{\pi^2}\right)\frac{1}{t}dt\\ &\sim N\int_0^1\left(\prod_p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{t-p}\right)-\frac{6}{\pi^2}\right)\frac{1}{t}dt,\end{aligned}$$

we have [30, 31]

$$\begin{aligned}\sum_{n\le N}{}'\frac{1}{B(n)-\beta(n)} &\sim N\int_0^1\left(\prod_p\left(1+\sum_{k=2}^{\infty}\frac{t^{(k-1)p}-t^{(k-2)p}}{p^k}\right)-\frac{6}{\pi^2}\right)\frac{1}{t}dt\\ &\sim N\int_0^1\left(\prod_p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{t^p-p}\right)-\frac{6}{\pi^2}\right)\frac{1}{t}dt.\end{aligned}$$

We also have [30, 32, 33],

$$\sum_{n\le N}{}'\frac{\Omega(n)}{\omega(n)}\sim\sum_{n\le N}{}'\frac{B(n)}{\beta(n)}\sim N,$$

$$\sum_{n\le N}{}'\frac{\hat{B}(n)}{\beta(n)}\sim e^{\gamma}N\ln(\ln(N)),\qquad \sum_{n\le N}{}'\frac{\hat{B}(n)}{B(n)}\sim F\,N$$

where

$$F=\int_1^{\infty}\frac{1}{x}\sum_{j=0}^{\lfloor x\rfloor-1}\frac{\rho(x-\lfloor x\rfloor+j)}{\lfloor x\rfloor-j}dx=\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty}\frac{\rho(y)}{y+k}dy$$

and $\rho(z)$ is Dickman's function [34].

Other constants emerge when arithmetic functions are evaluated not at n, but at quadratic functions of n. For example [20, 35, 36, 37, 38, 39, 40],

$$\sum_{n\leq N} d(n^2+1) \sim \frac{3}{\pi} N \ln(N), \qquad \sum_{n\leq N} \sigma(n^2+1) \sim \frac{5G}{\pi^2} N^3,$$

$$\sum_{n\leq N} r(n^2+1) \sim \frac{8}{\pi} N \ln(N), \qquad \sum_{n\leq N} \varphi(n^2+1) \sim \frac{H}{4} N^3$$

where G is Catalan's constant [41] and

$$H = \prod_{\substack{p\equiv 1 \\ \operatorname{mod} 4}} \left(1 - \frac{2}{p^2}\right) = 0.8948412245...$$

is a modified Feller-Tornier constant that appeared in [42]. As another example [43, 44, 45, 46],

$$\sum_{m,n\leq N} d(m^2+n^2) \sim \frac{\pi}{2G} N^2 \ln(N), \qquad \sum_{m,n\leq N} \sigma(m^2+n^2) \sim I\, N^4$$

where

$$\begin{aligned} I &= \frac{2}{3} \sum_{j=1}^{\infty} \frac{\nu(j)}{j^3} \\ &= \frac{8}{9} \prod_{\substack{p\equiv 1 \\ \operatorname{mod} 4}} \left(1 + \frac{2p+1}{(p+1)(p^2-1)}\right) \prod_{\substack{p\equiv 3 \\ \operatorname{mod} 4}} \left(1 + \frac{1}{(p-1)(p^2+1)}\right) \\ &= 1.03666099... \end{aligned}$$

and $\nu(j)$ denotes the number of solutions of $x^2+y^2=0$ in $\mathbb{Z}_j$, counting order [47, 48].

The average prime factor of n may reasonably be defined in two ways: as an mean of distinct prime factors $\beta(n)/\omega(n)$ or as a mean of all prime factors $B(n)/\Omega(n)$ (with multiplicity). It can be shown that [49]

$$\sum_{n\leq N} \frac{\beta(n)}{\omega(n)} \sim J \frac{N^2}{\ln(N)}, \qquad \sum_{n\leq N} \frac{B(n)}{\Omega(n)} \sim K \frac{N^2}{\ln(N)}$$

for constants $0 < K < J$. Infinite product expressions for J, K are possible but remain undiscovered (as far as is known).

Let $P^+(n)$ denote the largest prime factor of n and $P^-(n)$ denote the smallest prime factor of n. Also let $P^+(1) = P^-(1) = 1$. It follows that [50]

$$\sum_{n\le N} P^+(n) \sim \frac{\pi^2}{12}\frac{N^2}{\ln(N)}, \qquad \sum_{n\le N} P^-(n) \sim \frac{1}{2}\frac{N^2}{\ln(N)}$$

but precise asymptotics for $\sum_{n\le N} P^+(n)/P^-(n)$ and $\sum_{n\le N} 1/P^+(n)$ evidently remain open. By contrast, we have [51, 52, 53, 54]

$$\sum_{n\le N} \frac{P^-(n)}{P^+(n)} \sim \frac{N}{\ln(N)}, \qquad \sum_{n\le N} \frac{1}{P^-(n)} \sim U\,N,$$

$$\sum_{n\le N} \frac{d(n)}{P^-(n)} \sim V\,N\ln(N), \qquad \sum_{n\le N} \frac{\Omega(n)-\omega(n)}{P^-(n)} \sim W\,N$$

$$\sum_{n\le N} \frac{\varphi(n)}{P^-(n)} \sim X\,N^2, \qquad \sum_{n\le N} \frac{1}{n\ln(P^-(n))} \sim Y\ln(N)$$

where

$$U = \sum_p \frac{f(p)}{p^2}, \qquad V = \sum_p \frac{(2p-1)f(p)^2}{p^3},$$

$$W = \sum_p \frac{f(p)}{p}\sum_{\alpha\ge 2}\frac{1}{p^\alpha} + \sum_p \frac{f(p)}{p^2}\sum_{q>p}\sum_{\alpha\ge 2}\frac{1}{q^\alpha},$$

$$X = \frac{3}{\pi^2}\sum_p \frac{1}{p\,(p+1)\tilde{f}(p)}, \qquad Y = \sum_p \frac{f(p)}{p\ln(p)},$$

p and q are primes (of course), and

$$f(k) = \begin{cases} 1 & \text{if } k=2, \\ \displaystyle\prod_{p<k}\left(1-\frac{1}{p}\right) & \text{if } k>2, \end{cases} \qquad \tilde{f}(k) = \begin{cases} 1 & \text{if } k=2, \\ \displaystyle\prod_{p<k}\left(1+\frac{1}{p}\right) & \text{if } k>2. \end{cases}$$

Mertens' formula implies that $\lim_{k\to\infty}\ln(k)f(k) = e^{-\gamma}$ and $\lim_{k\to\infty}\tilde{f}(k)/\ln(k) = 6\pi^{-2}e^{\gamma}$.

The distance between consecutive distinct prime factors of $n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r}$ can be quantified in many ways: for example [55],

$$\frac{1}{r-1}\sum_{j=2}^{r}(p_j - p_{j-1}) = \frac{P^+(n)-P^-(n)}{\omega(n)-1}$$

(whose sum over $n \leq N$ is $\sim \lambda\, N^2/\ln(N)$, where $2\lambda = \sum_{k=2}^{\infty} k^{-2}\omega(k)^{-1} = 0.59737...$) and

$$g(n) = \sum_{j=2}^{r} \frac{1}{p_j - p_{j-1}}$$

(which is perhaps a little artificial). Of course, $g(1) = 0 = g(p)$ for any prime p by the empty sum convention. It can be shown that [56]

$$\begin{aligned}\sum_{n\leq N} g(n) &\sim N \sum_{p_L<p_R} \frac{1}{(p_R - p_L)p_L p_R} \prod_{p_L<p<p_R} \left(1 - \frac{1}{p}\right) \\ &\sim (0.299...)N\end{aligned}$$

where the sum is taken over all pairs of primes $p_L < p_R$ and the product is taken over all primes p strictly between the left prime p_L and the right prime p_R. If no such p exists, then the product is 1 by the empty product convention.

If $1 = \delta_1 < \delta_2 < \ldots < \delta_s = n$ are the consecutive distinct divisors of n, we might examine

$$\frac{1}{s-1} \sum_{j=2}^{s} (\delta_j - \delta_{j-1}) = \frac{n-1}{d(n)-1}$$

(whose sum over $n \leq N$ is $\sim \mu\, N^2/\ln(N)^{1/2}$; the formula for $2\mu = (0.96927...)\pi^{-1/2}$ appears in [17, 57]) and

$$h(n) = \sum_{j=2}^{s} \frac{1}{\delta_j - \delta_{j-1}}.$$

If two positive integers $a < b$ are consecutive divisors of $c_{a,b} = \mathrm{lcm}(a,b)$, let

$$\Delta_{a,b} = \left\{ \frac{d}{\gcd(d, c_{a,b})} : a < d < b \right\}$$

and let $D_{a,b}$ be the largest subset of $\Delta_{a,b}$ such that no element of $D_{a,b}$ is a multiple of another element in $D_{a,b}$. (Clearly $1 \notin \Delta_{a,b}$.) Assuming $D_{a,b} = \{d_1, d_2, \ldots, d_t\}$, we denote by $T(a,b)$ the following expression:

$$1 - \sum_{1\leq i\leq t} \frac{1}{d_i} + \sum_{1\leq i<j\leq t} \frac{1}{\mathrm{lcm}(d_i,d_j)} - \sum_{1\leq i<j<k\leq t} \frac{1}{\mathrm{lcm}(d_i,d_j,d_k)} + \cdots + (-1)^t \frac{1}{\mathrm{lcm}(d_1,d_2,\ldots,d_t)}.$$

It can be shown that [56]

$$\begin{aligned}\sum_{n\leq N} h(n) &\sim N \sum_{a<b} \frac{1}{c_{a,b}(b-a)} T(a,b) \\ &\sim (1.77...)N\end{aligned}$$

where the sum is taken over all pairs of positive integers $a < b$ such that the consecutive divisor requirement is met by a, b.

0.1. Addendum. The following result [58]

$$\begin{aligned}\sum_{n\leq N}\frac{d(n)}{d(n+1)} &\sim \frac{1}{\sqrt{\pi}}\prod_{p}\left(\frac{1}{\sqrt{p(p-1)}}+\sqrt{1-\frac{1}{p}}(p-1)\ln\left(\frac{p}{p-1}\right)\right)\cdot N\sqrt{\ln(N)}\\ &= (0.7578277106...)N\sqrt{\ln(N)}\end{aligned}$$

has a constant similar to that appearing in [57] for $\sum_{n\leq N}1/d(n)$. More logarithmic results include [59, 60, 61]

$$\ln(2)\sum_{n\leq N}{}^{'}\frac{1}{\ln(d(n))}\sim\frac{N}{\ln(\ln(N))}+E_5\frac{N}{\ln(\ln(N))^2},$$

$$\sum_{n\leq N}{}^{'}\frac{1}{\ln(\varphi(n))}\sim\frac{N}{\ln(N)}+E_6\frac{N}{\ln(N)^2},\qquad\sum_{n\leq N}{}^{'}\frac{1}{\ln(\sigma(n))}\sim\frac{N}{\ln(N)}+E_7\frac{N}{\ln(N)^2}$$

where $E_5=1-E_1$, $E_6=-E_2$ and $E_7=-E_3$ (a sign error in [59] has been corrected to give E_5). A numerical estimate $1+E_3=0.4457089175...$ is provided in [62, 63], hence $E_3=0.5542910824....$; also $E_1=0.6394076513...$ and $E_2=-1.5800584938....$

The Dedekind totient ψ enjoys close parallels with the Euler totient φ:

$$\psi(n)=n\prod_{p|n}\left(1+\frac{1}{p}\right),\qquad\varphi(n)=n\prod_{p|n}\left(1-\frac{1}{p}\right);$$

$$\sum_{n\leq N}\psi(n)\sim\underbrace{\frac{1}{2}\prod_{p}\left(1+\frac{1}{p^2}\right)}_{15/(2\pi^2)}\cdot N^2,\qquad\sum_{n\leq N}\varphi(n)\sim\underbrace{\frac{1}{2}\prod_{p}\left(1-\frac{1}{p^2}\right)}_{3/\pi^2}\cdot N^2;$$

$$\sum_{n\leq N}\frac{1}{\psi(n)}\sim\underbrace{\prod_{p}\left(1-\frac{1}{p(p-1)}\right)}_{C_{\mathrm{Artin}}}\cdot\left(\ln(N)+\gamma+\sum_{p}\frac{\ln(p)}{p^2+p+1}\right),$$

$$\sum_{n\leq N}\frac{1}{\varphi(n)}\sim\underbrace{\prod_{p}\left(1+\frac{1}{p(p-1)}\right)}_{315\zeta(3)/(2\pi^4)}\cdot\left(\ln(N)+\gamma-\sum_{p}\frac{\ln(p)}{p^2-p+1}\right).$$

Further results include [64]

$$\sum_{n\leq N}\frac{\varphi(n)}{\psi(n)}\sim\prod_{p}\left(1-\frac{2}{p(p+1)}\right)\cdot N,$$

$$\sum_{n\leq N}\psi(n)^2\sim\tfrac{1}{3}\prod_{p}\left(1+\tfrac{2}{p^2}+\tfrac{1}{p^3}\right)\cdot N^3,\qquad \sum_{n\leq N}\varphi(n)^2\sim\tfrac{1}{3}\prod_{p}\left(1-\tfrac{2}{p^2}+\tfrac{1}{p^3}\right)\cdot N^3.$$

The first of the three products appears in [65] with regard to cube roots of nullity mod n, and in [66] with regard to strongly carefree couples. Asymptotics for $\sum_{n\leq N}\varphi(n)^\ell$ were found by Chowla [67], where ℓ is any positive integer. His formula naturally carries over to $\sum_{n\leq N}\psi(n)^\ell$. It is known that the Riemann hypothesis is true if and only if [68, 69]

$$\varphi\left(\prod_{k=1}^{n}p_k\right)<e^{-\gamma}\left(\prod_{k=1}^{n}p_k\right)/\ln\left(\ln\left(\prod_{k=1}^{n}p_k\right)\right),$$

$$\psi\left(\prod_{k=1}^{n}p_k\right)>\frac{6\,e^{\gamma}}{\pi^2}\left(\prod_{k=1}^{n}p_k\right)\cdot\ln\left(\ln\left(\prod_{k=1}^{n}p_k\right)\right)$$

for all $n\geq 3$, where $p_1=2$, $p_2=3$, $p_3=5$, $\ldots$ is the sequence of all primes. A related inequality, due to Robin, appears in [70].

Two open problems given earlier were, in fact, solved by van de Lune [71]:

$$\sum_{n\leq N}\frac{P^+(n)}{P^-(n)}\sim Z\frac{N^2}{\ln(N)}$$

where

$$Z=\frac{\pi^2}{12}\sum_{p}\left(\frac{1}{p^3}\prod_{q<p}\left(1-\frac{1}{q^2}\right)\right)$$

and Erdös, Ivić & Pomerance [72]:

$$\sum_{n\leq N}\frac{1}{P^+(n)}\sim N\int_{2}^{N}\rho\left(\frac{\ln(N)}{\ln(t)}\right)\frac{1}{t^2}dt.$$

Rongen [73] proved that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N}\frac{\ln(n)}{\ln(P^+(n))}=e^{\gamma}$$

and variations of this include [71]

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N}\frac{\ln(P^+(n))}{\ln(n)}=\lambda=\lim_{N\to\infty}\frac{1}{N\ln(N)}\sum_{n\leq N}\ln(P^+(n))$$

where $\lambda = 0.6243299885...$ is the Golomb-Dickman constant [34]. A simple, precise estimate of

$$\sum_{n \leq N} \frac{1}{\ln(P^+(n))}$$

evidently has not yet been found.

Let $k(n)$ denote the smallest prime not dividing n and $\ell(n)$ denote the smallest integer > 1 not dividing n. Their respective average values are [74, 75, 76]

$$\lim_{N\to\infty} \frac{1}{N} \sum_{3\leq n\leq N} k(n) = \sum_{p} (p-1)/\prod_{q<p} q = 2.9200509773...,$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{3\leq n\leq N} \ell(n) = \sum_{j\geq 2} \left(\frac{1}{\operatorname{lcm}\{1,2,\ldots,j-1\}} - \frac{1}{\operatorname{lcm}\{1,2,\ldots,j\}} \right) j = 2.7877804561....$$

Compare these to the quadratic nonresidue constants at the end of [5].

Let $\mathbb{Z}_m$ be the additive group of residue classes modulo m. The number of subgroups of $\mathbb{Z}_m$ is $d(m)$ and each subgroup is cyclic. The number $s(m,n)$ of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ satisfies [77, 78, 79, 80]

$$s(m,n) = \sum_{a|m,b|n} \gcd(a,b),$$

$$\sum_{m,n\leq x} s(m,n) \sim x^2 \left(A_3 \ln(x)^3 + A_2 \ln(x)^2 + A_1 \ln(x) + A_0 \right)$$

where

$$A_3 = \frac{1}{3\zeta(2)} = \frac{2}{\pi^2}, \qquad A_2 = \frac{1}{\zeta(2)} \left(3\gamma - 1 - \frac{\zeta'(2)}{\zeta(2)} \right),$$

$$A_1 = \frac{1}{\zeta(2)} \left(8\gamma^2 - 6\gamma - 2\gamma_1 + 1 - 2(3\gamma-1)\frac{\zeta'(2)}{\zeta(2)} + 2\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2 - \frac{\zeta''(2)}{\zeta(2)} \right)$$

and the number $c(m,n)$ of cyclic subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ satisfies

$$c(m,n) = \sum_{\substack{a|m,b|n,\\ \gcd\left(\frac{m}{a},\frac{n}{b}\right)=1}} \gcd(a,b),$$

$$\sum_{m,n\leq x} c(m,n) \sim x^2 \left(B_3 \ln(x)^3 + B_2 \ln(x)^2 + B_1 \ln(x) + B_0 \right)$$

where

$$B_3 = \frac{1}{3\zeta(2)^2} = \frac{12}{\pi^4}, \qquad B_2 = \frac{1}{\zeta(2)^2} \left(3\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)} \right),$$

$$B_1 = \frac{1}{\zeta(2)^2}\left(8\gamma^2 - 6\gamma - 2\gamma_1 + 1 - 4(3\gamma - 1)\frac{\zeta'(2)}{\zeta(2)} + 6\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2 - 2\frac{\zeta''(2)}{\zeta(2)}\right).$$

The expressions for A_0, B_0 are complicated and not helpful for numerical evaluation; γ_1 is the first Stieltjes constant [81]. In particular,

$$\sum_{n\le x} s(n,n) \sim \frac{5\pi^2}{24}x^2, \qquad \sum_{n\le x} c(n,n) \sim \frac{5}{4}x^2;$$

analogously,

$$\sum_{n\le x} s(n,n,n) \sim \frac{1}{3}x^3\left[H(3)\left(\ln(x) + 2\gamma - 1\right) + H'(3)\right]$$

where

$$H(z) = \zeta^2(z)\prod_p\left(1 + \frac{2}{p^{z-1}} + \frac{2}{p^z} + \frac{1}{p^{2z-1}}\right), \qquad \operatorname{Re}(z) > 2.$$

Of related interest are series $\sum_{n\le x} t(n)$ and $\sum_{m,n\le x} t(m\,n)$, where $t(n)$ is the number of squares dividing n [82, 83]. More examples appear in [84, 85]; cases when the underlying Dirichlet series is a product of zeta function expressions give rise to asymptotic expansions with exact coefficients (found via residues).

Recall the earlier series $\sum_{n\le N}\sigma(n)/d(n)$ with growth rate $N^2\ln(N)^{-1/2}$; a related series

$$\sum_{n\le N}\frac{\sigma(n)}{\varphi(n)} \sim (3.6174...)N$$

appears without comment in [86], with cryptic reference to [87]. It would be good to learn more about this result.

Let $P_2^+(n)$ denote the second largest prime factor of n if it exists, otherwise set $P_2^+(n) = \infty$. The asymptotic behavior of $P_2^+(n)$ is completely different from that of $P^+(n)$ [88, 89]:

$$\begin{aligned}\sum_{n\le N}\frac{1}{P_2^+(n)} &\sim \left(\sum_{m=1}^{\infty}\frac{1}{m}\sum_{p\ge P^+(m)}\frac{1}{p^2}\right)\frac{N}{\ln(N)} \\ &\sim \left(\sum_p\frac{1}{p^2}\prod_{q\le p}\left(1-\frac{1}{q}\right)^{-1}\right)\frac{N}{\ln(N)} \sim (1.254...)\,\frac{N}{\ln(N)}.\end{aligned}$$

Let $P_3^+(n)$ denote the third largest prime factor of n if it exists, otherwise set $P_3^+(n) = \infty$. Interestingly, the same constant occurs [88, 89]:

$$\sum_{n\le N}\frac{1}{P_3^+(n)} \sim (1.254...)\frac{N\ln(\ln(N))}{\ln(N)}$$

but the growth rate is faster. A well-known constant $\sum 1/p^2 = 0.4522474200...$ from [5] appears in [90], stemming (almost surely) from the reciprocal sum of a uniformly drawn prime factor of n, for each n. The growth rate $N/\ln(\ln(N))$ is faster still.

Here is a comparatively neglected topic: for a random integer n between 1 and N, since

$$\lim_{N\to\infty} \mathrm{P}\left(P^+(n) \le n^x\right) = \rho\left(\frac{1}{x}\right)$$

for $0 < x \le 1$, the median value of x satisfies $\rho(1/x) = 1/2$, that is, $x = 1/\sqrt{e} = 0.6065306597....$ The mode (peak of density) is $1/2$; see Figure 1. Define the second-order Dickman function $\rho_2(x)$ by [88]

$$x\rho_2'(x) + \rho_2(x-1) = \rho(x-1) \quad \text{for } x > 1, \qquad \rho_2(x) = 1 \quad \text{for } 0 \le x \le 1$$

then the corresponding median value satisfies $\rho_2(1/x) = 1/2$, that is, $x = 0.2117211464...$ [91]. An early approximation (0.24) appeared long ago [92]; medians are more robust estimators of centrality than means (being less sensitive to data outliers). The mode here is 0.2350396459...; see Figure 2. Likewise, the third-order Dickman function $\rho_3(x)$ is [88]

$$x\rho_3'(x) + \rho_3(x-1) = \rho_2(x-1) \quad \text{for } x > 1, \qquad \rho_3(x) = 1 \quad \text{for } 0 \le x \le 1$$

and the corresponding median value satisfies $\rho_3(1/x) = 1/2$, that is, $x = 0.0758437231...$ [91]. We hope to report on [93, 94] later.

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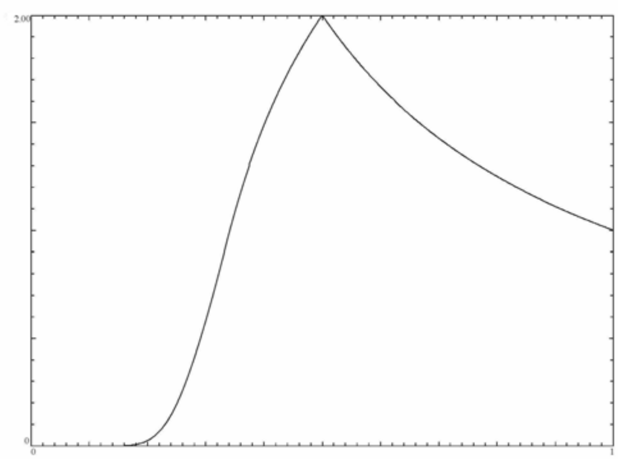

Figure 1: Plot of $d/dx\,\rho_1(1/x)$ when $0 < x < 1$; for random n, the density of x such that n^x is the largest prime factor of n. Image courtesy of David Broadhurst.

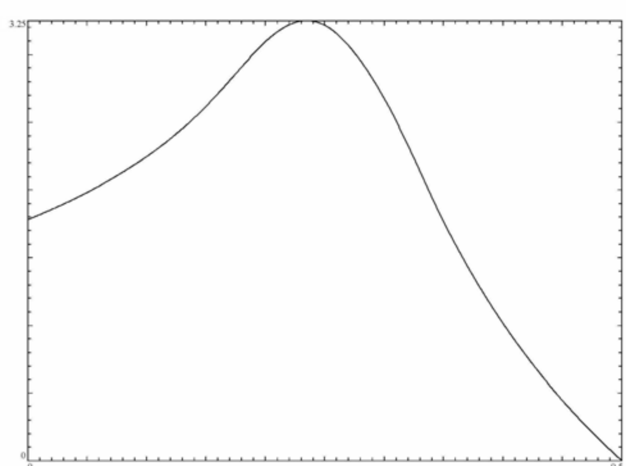

Figure 2: Plot of $d/dx\,\rho_2(1/x)$ when $0 < x < 1/2$; for random n, the density of x such that n^x is the second-largest prime factor of n. Image courtesy of David Broadhurst.

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