Aissen's Convex Set Function

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Let D be a bounded open convex set in the plane and let C denote the boundary of D. For each $p \in D$ and $q \in C$, let h_{pq} be the Euclidean distance from p to the support line (tangent line) to D at q. Let ds_q denote the line element at q. It is known that [1, 2]

$$\text{arclength of } C = \int\limits_C ds_q,$$

$$\text{area of } D = \frac{1}{2}\int\limits_C h_{pq}\, ds_q \qquad \text{(independent of } p\text{)},$$

$$r(D) = \text{inradius of } D = \max_{p\in D}\min_{q\in C} h_{pq}$$

where r is the radius of the largest disk contained by D [3]. The boundary of such a disk is called an incircle; its center is called an incenter. Aissen [1, 2] studied the function

$$B(D) = \min_{p\in D}\int\limits_C h_{pq}^{-1}\, ds_q$$

and deduced that the optimizing point p corresponds to an incenter of D if D is a triangle, parallelogram, regular polygon or ellipse. (We are careful to say "an incenter" rather than "the incenter": a suitably elongated parallelogram has infinitely incircles, all of the same radius. In contrast, the incenter for an arbitrary triangle is unique.) This is a remarkable feature of B. It is natural to wonder whether the same is true for an arbitrary convex set.

The simplest counterexample is a trapezoid with vertices $(\pm 1, 1)$, $(\pm 3, -1)$, for which the optimizing point p has x-coordinate 0 (by symmetry) but y-coordinate > 0. More generally, examine the trapezoid with vertices $\left(\pm\left(\sqrt{2}-1+t\right), 1\right)$, $\left(\pm\left(\sqrt{2}+1+t\right), -1\right)$ where $t \geq 0$ is fixed. The integral within B becomes a sum of four ratios:

$$2\left(\frac{\sqrt{2}-1+t}{1-y} + \frac{\sqrt{2}+1+t}{1+y} + \frac{2}{\sqrt{2}+t+x-y} + \frac{2}{\sqrt{2}+t-x-y}\right)$$

each of the form sidelength/distance. As an instance, the rightmost side has equation

$$v - \frac{1}{\sqrt{2}} = -u + \left(\frac{1}{\sqrt{2}} + t\right)$$

in the uv-plane, that is, $u + v - \sqrt{2} - t = 0$. The distance from the point (x, y) to the line is

$$\frac{\left|x + y - \sqrt{2} - t\right|}{\sqrt{1^2 + 1^2}} = \frac{\sqrt{2} + t - x - y}{\sqrt{2}}$$

and the sidelength is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. Forming a ratio gives the final term in the sum. Differentiating the sum with respect to x, we see that $x = 0$ is necessary for minimization. The derivative with respect to y is more complicated. In the special case $t = 0$, each of the trapezoidal sides is tangent to the unit circle, thus $y = 0$. If instead $t = 2 - \sqrt{2}$, then the inradius is still 1 but $y \approx 0.116257$ is the unique positive zero of the quartic $y^4 + 8y^3 - 25y^2 + 20y - 2$. If instead $t = 3 - \sqrt{2}$, we have $y \approx 0.130385$ (increasing). If instead $t = 4 - \sqrt{2}$, we have $y \approx 0.110399$ (decreasing). As $t \to \infty$, we have $y \to 0^+$. Aissen's optimizing point appears not to be associated with the trapezoidal incenter except at the extremes $t = 0$, $t = \infty$.

Another counterexample – the half-disk $0 \leq v \leq \sqrt{1 - u^2}$ – comes from [1, 2]. Again $x = 0$ follows by symmetry. The integral within B here becomes

$$\frac{2}{y} + \frac{2\arcsin(y) + \pi}{\sqrt{1 - y^2}}$$

and is minimized when $y = 0.5432763603... > 1/2$. The value of B itself is 8.7915361561.... Such values play a role in estimating hard physical quantities like torsional rigidity P in terms of area A [4]. For the half-disk, P turns out to be known exactly and the lower bound [5]

$$0.2975567820... = \frac{\pi}{2} - \frac{4}{\pi} = P \geq A^2 B^{-1} = \frac{(\pi/2)^2}{8.7915361561...} \approx 0.280$$

is excellent.

Returning to geometry, let d_{pq} simply be the Euclidean distance from p to q. Clearly

$$R(D) = \text{circumradius of } D = \min_{p \in D} \max_{q \in C} d_{pq}$$

where R is the radius of the smallest disk containing D [3]. The boundary of such a disk is called a circumcircle; its center is called a circumcenter. The circumcenter for an arbitrary convex set is unique. We wonder if a "dual" to Aissen's function can be defined and what its interplay with the circumcenter for various D might be.

REFERENCES

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