Two Asymptotic Series

Steven Finch

December 10, 2003

When enumerating trees [1, 2] or prime divisors [3, 4], the leading term of the corresponding asymptotic series is usually sufficient for practical purposes. Greater accuracy is possible by using several more terms, but the coefficients are not as widely known as one might expect. We briefly provide the formulas required to compute the required constants, as well as some theoretical background.

0.1. Trees. If T_n is the number of non-isomorphic rooted trees with n vertices, then [5]

$$T_n \sim r^{-n} n^{-3/2} \left(0.4399240125... + \frac{0.0441699018...}{n} + \frac{0.2216928059...}{n^2} + \frac{0.8676554908...}{n^3} + \cdots \right)$$

where $r = 0.3383218568...$ is the unique positive root of the equation $F(x, 1) = 0$, where

$$F(x, y) = x \exp\left(y + \sum_{k=2}^{\infty} \frac{T(x^k)}{k} \right) - y$$

and $T(x) = \sum_{n=1}^{\infty} T_n x^n$ is the generating function for $\{T_n\}$. Let us denote the four numerical coefficients by $C_0/(2\sqrt{\pi})$, $C_1/(2\sqrt{\pi})$, $C_2/(2\sqrt{\pi})$ and $C_3/(2\sqrt{\pi})$. Exact formulas for these constants can be written in terms of the partial derivatives

$$F_{i,j} = \frac{\partial^{i+j}}{\partial x^i \partial y^j} F(x, y) \bigg|_{\substack{x=r \\ y=1}}$$

via computer algebra. Note that $F_{0,0} = F_{0,1} = 0$,

$$1 = F_{0,2} = F_{0,3} = F_{0,4} = F_{0,5} = \cdots ,$$

$$0 < F_{1,0} = F_{1,1} = F_{1,2} = F_{1,3} = F_{1,4} = \cdots ,$$

and likewise $F_{i,j} = F_{i,0}$ for all $i \geq 2$, $j \geq 1$. We have

$$C_0 = \sqrt{2\, r\, F_{1,0}},$$

$$C_1 = \{9\,r\,F_{1,0} + r^2\,[-11\,F_{1,0}^2 + 9\,F_{2,0}]\}/\{12\,C_0\},$$

$$\begin{aligned} C_2 \;=\; & \{225\,r\,F_{1,0}^2 + r^2\,[-990\,F_{1,0}^3 + 810\,F_{1,0}\,F_{2,0}] \\ & +r^3\,[769\,F_{1,0}^4 - 990\,F_{1,0}^2\,F_{2,0} - 135\,F_{2,0}^2 + 360\,F_{1,0}\,F_{3,0}]\}/\{576\,F_{1,0}\,C_0\}, \end{aligned}$$

$$\begin{aligned} C_3 \;=\; & \{42525\,r\,F_{1,0}^3 + r^2\,[-571725\,F_{1,0}^4 + 467775\,F_{1,0}^2\,F_{2,0}] \\ & +r^3\,[1211175\,F_{1,0}^5 - 1559250\,F_{1,0}^3\,F_{2,0} - 212625\,F_{1,0}\,F_{2,0}^2 + 567000\,F_{1,0}^2\,F_{3,0}] \\ & +r^4\,[-680863\,F_{1,0}^6 + 1211175\,F_{1,0}^4\,F_{2,0} - 155925\,F_{1,0}^2\,F_{2,0}^2 + 42525\,F_{2,0}^3 \\ & -415800\,F_{1,0}^3\,F_{3,0} - 113400\,F_{1,0}\,F_{2,0}\,F_{3,0} + 113400\,F_{1,0}^2\,F_{4,0}]\}/\{207360\,F_{1,0}^2\,C_0\}. \end{aligned}$$

The associated formula for t_n, the number of non-isomorphic free trees of order n, is [5]

$$t_n \sim r^{-n} n^{-5/2} \left(0.5349496061... + \frac{0.4853877311...}{n} + \frac{2.379745574...}{n^2} + \cdots \right)$$

where r is as before and the first numerical coefficient is simply $C_0^3/(4\sqrt{\pi})$. Exact formulas for the second and third coefficients are

$$\frac{C_0^2(C_0^3 + 30C_1)}{24\sqrt{\pi}}, \qquad \frac{C_0(C_0^6 + 35C_0^3C_1 + 210C_1^2 + 126C_0C_2)}{72\sqrt{\pi}}$$

and we wonder what the next few coefficients might look like.

Other varieties of trees examined in [5] include binary trees, identity trees and homeomorphically irreducible trees. Different functional equations apply in each case; for example, we have

$$F(x,y) = x + \frac{1}{2}\left(y^2 + B(x^2)\right) - y$$

for the first variety, where $B(x) = \sum_{n=1}^{\infty} B_n x^n$ is the generating function for the number B_n of non-isomorphic rooted strongly binary trees with n leaves ($B_1 = B_2 = B_3 = 1$, $B_4 = 2$, $B_5 = 3$, ...). One obtains

$$B_n \sim \rho^{-n} n^{-3/2} \left(0.3187766259... + \frac{0.2038317427...}{n} + \frac{0.3682702316...}{n^2} + \frac{1.4768193666...}{n^3} + \cdots \right)$$

with $\rho = 0.4026975036...$ as the radius of convergence. The details are omitted.

An intermediate step to studying $\{T_n\}$ involves the analysis of the series [6, 7]

$$\begin{aligned}T(x) &= \sum_{k=0}^{\infty} c_k (r-x)^{k/2} \\ &= 1-(2.6811281472...)(r-x)^{1/2}+(2.3961493806...)(r-x) \\ &\quad -(1.4507456802...)(r-x)^{3/2}+(1.4447836810...)(r-x)^2 \\ &\quad -(5.1438071207...)(r-x)^{5/2}+\cdots\end{aligned}$$

which is valid as $x \to r^-$, where

$$c_0 = 1, \qquad c_1 = -\sqrt{2\,F_{1,0}}, \qquad c_2 = 2\,F_{1,0}/3,$$

$$c_3 = \left\{11\,F_{1,0}^2 - 9\,F_{2,0}\right\} / \left\{18\,c_1\right\}, \qquad c_4 = \left\{43\,F_{1,0}^2 - 45\,F_{2,0}\right\} /135,$$

$$c_5 = \left\{769\,F_{1,0}^4 - 990\,F_{1,0}^2\,F_{2,0} - 135\,F_{2,0}^2 + 360\,F_{1,0}\,F_{3,0}\right\} / \left\{2160\,F_{1,0}\,c_1\right\}.$$

Note that $c_2 = c_1^2/3$ and $c_4 = (30c_1c_3 - c_1^4)/45$, while c_3 and c_5 cannot be algebraically represented in terms of preceding c_k values.

Likewise, in connection with $\{t_n\}$, we have [6, 7]

$$\begin{aligned}t(x) &= \sum_{k=0}^{\infty} d_k (r-x)^{k/2} \\ &= 0.5657439434... - (4.0484928944...)(r-x) - (6.4243835496...)(r-x)^{3/2} \\ &\quad -(5.5810996983...)(r-x)^2 + (7.3498535571...)(r-x)^{5/2}+\cdots\end{aligned}$$

where

$$d_0 = \tfrac{1}{2}\left(1+T(r^2)\right), \qquad d_1 = 0,$$

$$d_2 = -\tfrac{1}{2}\left(c_1^2 + 2rT'(r^2)\right), \qquad d_3 = c_1c_2,$$

$$d_4 = \tfrac{1}{2}\left(-c_2^2 - 2c_1c_3 + 2r^2T''(r^2) + T'(r^2)\right), \qquad d_5 = -c_2c_3 - c_1c_4$$

and $T'(x)$, $T''(x)$ denote the first and second derivatives of $T(x)$, respectively. The singular part of $t(x)$ (that is, the part corresponding to d_k for odd k) depends just on the coefficients c_j. No analogous simplification of the analytic part of $t(x)$ (d_k for even k) is known.

0.2. Darboux-Pólya Method. Although the asymptotic series for T_n and t_n are evidently new, the underlying method appears (at least implicitly) in the works of Darboux [8, 9] and Pólya [10]. We give the steps of a straightforward algorithm for computing the m^{th} coefficient C_m of the asymptotic series for T_n.

Define first $z_{i,j}$ to be 0 if ($i \geq 1$ and $j = 2$) or ($j > 2$), and 1 otherwise. Define $P_{i,j}$ and $A_{i,j}$ via the recursions

$$P_{i,j} = z_{i,j} \frac{F_{i,j} - \sum_{p=1}^{i-1} \sum_{q=0}^{j} \binom{i}{p} \binom{j}{q} A_{p,q} P_{i-p,j-q} - \sum_{q=1}^{j} \binom{j}{q} A_{0,q} P_{i,j-q}}{A_{0,0}},$$

$$A_{i,j} = \frac{F_{i,j+2} - \sum_{p=0}^{i-1} \sum_{q=0}^{j+2} \binom{i}{p} \binom{j+2}{q} A_{p,q} P_{i-p,j-q+2}}{(j+1)(j+2)}$$

with initial conditions $P_{0,2} = 2$ and $P_{0,j} = 0$ for all $j \neq 2$. Let

$$p_k = \frac{P_{k,1}(-r)^k}{k!}, \qquad q_k = \frac{P_{k,0}(-r)^k}{k!}$$

and define b_ℓ via the recursion

$$b_\ell = \frac{-\sum_{k=1}^{\ell-1} b_k b_{\ell-k} + \frac{1}{4} \sum_{k=1}^{\ell} p_k p_{\ell-k+1} - q_{\ell+1}}{2b_0}$$

with initial condition $b_0 = -\sqrt{-q_1}$.

Define next

$$s_i = 2^{i-2} \binom{2i}{i} - \frac{1}{2} \sum_{j=1}^{i-1} \binom{i-1}{j-1} 2^{3(i-j)} s_j - \sum_{k=1}^{i-1} \sum_{j=1}^{i-k} \binom{i-k-1}{j-1} 2^{3(i-j-k)} s_j s_k$$

with initial condition $s_0 = 1$, and the recursion

$$S_{u,v} = \begin{cases} 1 & \text{if } u = v = 0 \\ (-1)^u 2^{1-4u} s_u & \text{if } u \geq 1 \text{ and } v = 0 \\ -\sum_{w=0}^{u} \left(v - \frac{1}{2}\right)^{w+1} S_{u-w,v-1} & \text{if } u \geq 0 \text{ and } v \geq 1. \end{cases}$$

Finally, we have

$$C_m = 2 \sum_{k=0}^{m} b_k S_{m-k,k+1}$$

which completes the algorithm.

Some explanation is clearly needed. We know that $F(x,T(x))=0$. The Weierstrass Preparation Theorem implies that, for (x,y) sufficiently close to $(r,1)$,

$$F(x,y)=A(x,y)\cdot P(x,y)$$

where $A(x,y)$ is analytic, $A(r,1)\neq 0$, and

$$P(x,y)=(y-1)^2+p(x)(y-1)+q(x)$$

where $p(x)$, $q(x)$ are analytic and $p(r)=q(r)=0$. The sequence $\{b_\ell\}$ arises from setting the various coefficients of the polynomial-like approximation $P(x,T(x))$ equal to zero. By Darboux's theorem,

$$T_n\sim(-1)^n r^{-n}\sum_{k=0}^{\infty}b_k\tbinom{k+1/2}{n};$$

hence it remains to compute asymptotic series for half-integer binomial coefficients. We know that [11]

$$\begin{aligned}\tbinom{-1/2}{n} &= \frac{(-1)^n}{\sqrt{\pi n}}\left(1-\frac{1}{8n}+\frac{1}{128n^2}+\frac{5}{1024n^3}-\frac{21}{32768n^4}-\frac{399}{262144n^5}+\cdots\right)\\ &= \frac{(-1)^n}{\sqrt{\pi n}}\sum_{j=0}^{\infty}\frac{S_{j,0}}{n^j}\end{aligned}$$

from which we immediately deduce that

$$\begin{aligned}\tbinom{1/2}{n} &= \frac{(-1)^{n+1}}{2\sqrt{\pi}n^{3/2}}\left(1+\frac{3}{8n}+\frac{25}{128n^2}+\frac{105}{1024n^3}+\frac{1659}{32768n^4}+\frac{6237}{262144n^5}+\cdots\right)\\ &= \frac{1}{2\sqrt{\pi}}\frac{(-1)^n}{n^{3/2}}\sum_{j=0}^{\infty}\frac{2S_{j,1}}{n^j},\end{aligned}$$

$$\begin{aligned}\tbinom{3/2}{n} &= \frac{3(-1)^n}{4\sqrt{\pi}n^{5/2}}\left(1+\frac{15}{8n}+\frac{385}{128n^2}+\frac{4725}{1024n^3}+\frac{228459}{32768n^4}+\frac{2747745}{262144n^5}+\cdots\right)\\ &= \frac{1}{2\sqrt{\pi}}\frac{(-1)^n}{n^{3/2}}\sum_{j=0}^{\infty}\frac{2S_{j,2}}{n^{j+1}},\end{aligned}$$

$$\begin{aligned}\tbinom{5/2}{n} &= \frac{15(-1)^{n+1}}{8\sqrt{\pi}n^{7/2}}\left(1+\frac{35}{8n}+\frac{1785}{128n^2}+\frac{40425}{1024n^3}+\frac{3462459}{32768n^4}+\frac{71996925}{262144n^5}+\cdots\right)\\ &= \frac{1}{2\sqrt{\pi}}\frac{(-1)^n}{n^{3/2}}\sum_{j=0}^{\infty}\frac{2S_{j,3}}{n^{j+2}},\end{aligned}$$

and so forth. The conclusion follows.

0.3. Addendum I. Philippe Flajolet maintained that the preceding discussion tends to "hide the facts" and provided thoughtful comments. Briefly, the equation $F(x,T(x))=0$ can be rearranged as $T(x)=\xi\exp(T(x))$ with

$$\xi(x)=x\exp\left(\sum_{k=2}^{\infty}\frac{T(x^k)}{k}\right).$$

The inverse function of $y\exp(-y)$ is the well-known Cayley tree function τ, an elementary variant of the Lambert W function:

$$\tau(z)=\sum_{n=1}^{\infty}n^{n-1}\frac{z^n}{n!}$$

on the complex plane. In a small disk around the origin, therefore, $T(z)=\tau(\xi(z))$. From here, singularities are easily accessed, making a full asymptotic expansion possible. Writing such conceptual remarks were, in Flajolet's words, an "enjoyable intermezzo" for him despite limited time. These eventually found their way into his treatise [12] with Sedgewick. For completeness, we mention that $C_0=1.5594900203...$ for rooted trees (as presented in [12]) and that the corresponding coefficient is 1.1300337163... for binary trees

0.4. Prime Divisors. If $\omega(n)$ is the number of distinct prime divisors of n, and $\Omega(n)$ is the total number (including multiplicity) of prime divisors of n, then

$$\mathrm{E}_n(\omega)\sim\ln(\ln(n))+0.2614972128...+\sum_{k=1}^{\infty}\left(-1+\sum_{j=0}^{k-1}\frac{\gamma_j}{j!}\right)\frac{(k-1)!}{\ln(n)^k},$$

$$\mathrm{Var}_n(\omega)\sim\ln(\ln(n))-1.8356842740...+\frac{1.0879488865...}{\ln(n)}+\frac{3.3231293098...}{\ln(n)^2}+\cdots,$$

$$\mathrm{E}_n(\Omega)\sim\ln(\ln(n))+1.0346538818...+\sum_{k=1}^{\infty}\left(-1+\sum_{j=0}^{k-1}\frac{\gamma_j}{j!}\right)\frac{(k-1)!}{\ln(n)^k},$$

$$\mathrm{Var}_n(\Omega)\sim\ln(\ln(n))+0.7647848097...-\frac{2.8767219464...}{\ln(n)}-\frac{4.9035933594...}{\ln(n)^2}+\cdots,$$

where

$$\mathrm{E}_n(X)=\frac{1}{n}\sum_{i=1}^{n}X(i),\qquad\mathrm{Var}_n(X)=\mathrm{E}_n(X^2)-\mathrm{E}_n(X)^2$$

and γ_j is the j^{th} Stieltjes constant [13]. The leading numerical terms in each of the four expansions are [4, 14]

$$\lambda = \gamma_0 + \sum_p \left[\ln\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right] = \gamma_0 + \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \ln(\zeta(k)),$$

$$\lambda - \sum_p \frac{1}{p^2} - \frac{\pi^2}{6} = \lambda - \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln(\zeta(2k)) - \frac{\pi^2}{6},$$

$$\Lambda = \gamma_0 + \sum_p \left[\ln\left(1 - \frac{1}{p}\right) + \frac{1}{p-1}\right] = \gamma_0 + \sum_{k=2}^{\infty} \frac{\varphi(k)}{k} \ln(\zeta(k)),$$

$$\Lambda + \sum_p \frac{1}{(p-1)^2} - \frac{\pi^2}{6} = \Lambda + \sum_{k=2}^{\infty} \frac{\varphi_2(k) - \varphi(k)}{k} \ln(\zeta(k)) - \frac{\pi^2}{6},$$

respectively, where $\zeta(x)$ is the Riemann zeta function, $\mu(k)$ is the Möbius mu function, $\varphi(k)$ is the Euler totient function, and the function $\varphi_\ell(k)$ is defined by

$$\frac{\varphi_\ell(k)}{k^\ell} = \prod_{p|k} \left(1 - \frac{1}{p^\ell}\right), \qquad \frac{\zeta(s-\ell)}{\zeta(s)} = \sum_{k=1}^{\infty} \frac{\varphi_\ell(k)}{k^s}$$

(in particular, $\varphi = \varphi_1$).

The second numerical coefficient in $\mathrm{Var}_n(\omega)$ is

$$\gamma_0 - 1 + 2\sum_p \frac{\ln(p)}{p(p-1)} = \gamma_0 - 1 + 2\sum_{k=2}^{\infty} \mu(k) \frac{\zeta'(k)}{\zeta(k)}$$

and the second numerical coefficient in $\mathrm{Var}_n(\Omega)$ is

$$\gamma_0 - 1 - 2\sum_p \frac{\ln(p)}{(p-1)^2} = \gamma_0 - 1 + 2\sum_{k=2}^{\infty} \varphi(k) \frac{\zeta'(k)}{\zeta(k)},$$

where $\zeta'(x)$ is the derivative of the zeta function. This result, as well as the result for means, appears in [14, 15, 16] but apparently with errors. Knuth [17] revisited Diaconis' original computations; this essay closely follows [17]. Finally, the third numerical coefficient in $\mathrm{Var}_n(\omega)$ is

$$-\gamma_1 - (\gamma_0 - 1)\left(\gamma_0 + 2\sum_p \frac{\ln(p)}{p(p-1)}\right) + 2\sum_p \frac{(2p-1)\ln(p)^2}{2p(p-1)^2}$$

and the third numerical coefficient in $\mathrm{Var}_n(\Omega)$ is

$$-\gamma_1 - (\gamma_0 - 1)\left(\gamma_0 - 2\sum_p \frac{\ln(p)}{(p-1)^2}\right) - 2\sum_p \frac{p\ln(p)^2}{(p-1)^3};$$

this result is new and awaits confirmation.

For completeness' sake, we record the values of six relevant prime series [4, 14, 18]:

$$t = \sum_p \frac{1}{p^2} = 0.4522474200..., \qquad T = \sum_p \frac{1}{(p-1)^2} = 1.3750649947...,$$

$$u = \sum_p \frac{\ln(p)}{p(p-1)} = 0.7553666108..., \qquad U = \sum_p \frac{\ln(p)}{(p-1)^2} = 1.2269688056...,$$

$$v = \sum_p \frac{(2p-1)\ln(p)^2}{2p(p-1)^2} = 1.1837806913..., \qquad V = \sum_p \frac{p\ln(p)^2}{(p-1)^3} = 2.0914802823....$$

0.5. Selberg-Delange Method. The theory here is much deeper than what was discussed earlier. It starts with asymptotic formulas for the generating functions [19, 20, 21]

$$\frac{1}{N}\sum_{n=1}^{N} z^{\omega(n)} = \ln(N)^{z-1}\left(a_0(z) + \frac{a_1(z)}{\ln(N)} + \frac{a_2(z)}{\ln(N)^2} + \cdots \frac{a_r(z)}{\ln(N)^r} + O\left(\frac{1}{\ln(N)^{r+1}}\right)\right),$$

$$\frac{1}{N}\sum_{n=1}^{N} z^{\Omega(n)} = \ln(N)^{z-1}\left(A_0(z) + \frac{A_1(z)}{\ln(N)} + \frac{A_2(z)}{\ln(N)^2} + \cdots \frac{A_r(z)}{\ln(N)^r} + O\left(\frac{1}{\ln(N)^{r+1}}\right)\right),$$

where if

$$\frac{s-1}{s}\prod_p \left(1 - \frac{1}{p^s}\right)^{z-1}\left(1 + \frac{z}{p^s - 1}\right) = \sum_{k=0}^{\infty} b_k(z)(s-1)^k = b(z),$$

$$\frac{s-1}{s}\prod_p \left(1 - \frac{1}{p^s}\right)^{z-1}\left(1 - \frac{z}{p^s}\right)^{-1} = \sum_{k=0}^{\infty} B_k(z)(s-1)^k = B(z),$$

then

$$a_j(z) = \frac{b_j(z)}{\Gamma(z-j)}, \qquad A_j(z) = \frac{B_j(z)}{\Gamma(z-j)}.$$

Let us focus on $\omega(n)$ for the sake of definiteness. Delange's formula expresses that, asymptotically, if n is uniformly distributed on $\{1, 2, ..., N\}$, then the distribution of $\omega(n)$ is the convolution of a Poisson random variable with mean $\ln(\ln(N))$ and another random variable X whose generating function is

$$\mathrm{E}(z^X) \sim a_0(z) + \frac{a_1(z)}{\ln(N)} + \frac{a_2(z)}{\ln(N)^2} + \cdots$$

Thus the mean of $\omega(n)$ will be $\ln(\ln(N))$ plus the mean of X, and the variance will be $\ln(\ln(N))$ plus the variance of X. We have

$$\mathrm{E}(X) \sim a_0'(1) + \frac{a_1'(1)}{\ln(N)} + \frac{a_2'(1)}{\ln(N)^2} + \cdots,$$

$$\mathrm{E}(X(X-1)) \sim a_0''(1) + \frac{a_1''(1)}{\ln(N)} + \frac{a_2''(1)}{\ln(N)^2} + \cdots,$$

hence

$$\mathrm{Var}(X) \sim c_0 + \frac{c_1}{\ln(N)} + \frac{c_2}{\ln(N)^2} + \cdots$$

where

$$c_j = a_j''(1) + a_j'(1) - \sum_{i=0}^{j} a_i'(1) a_{j-i}'(1).$$

The corresponding coefficients for $\Omega(n)$ will be denoted by C_0, C_1, C_2, ... and satisfy similar relations.

To obtain the mean, note that setting $z = 1$ in the formula for $b(z)$ gives

$$\frac{s-1}{s}\zeta(s) = \sum_{k=0}^{\infty} b_k(1)(s-1)^k.$$

Replacing s by $s+1$, we have

$$\left(\sum_{i=0}^{\infty}(-1)^i s^i\right)\left(1 + \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\gamma_j s^{j+1}\right) = \frac{s}{s+1}\zeta(s+1) = \sum_{k=0}^{\infty} b_k(1) s^k$$

thus

$$b_0(1) = 1, \qquad b_1(1) = \gamma_0 - 1, \qquad b_2(1) = -(\gamma_1 + \gamma_0 - 1).$$

Since

$$a_0'(1) = b_0'(1) + \gamma_0 b_0(1) = \lambda \qquad \text{(to be proved shortly)},$$

$$a_k'(1) = (-1)^{k-1}(k-1)! b_k(1), \qquad k \geq 1,$$

the result follows. This argument also applies verbatim to $B(z)$, but with λ replaced by Λ.

To obtain the variance, differentiate $b(z)$ and set $z = 1$:

$$\begin{aligned} b'(1) &= b(1) \sum_p \left[\ln\left(1 - \frac{1}{p^s}\right) + \frac{1}{p^s} \right] \\ &= \left\{ 1 + (\gamma_0 - 1)(s-1) - (\gamma_1 + \gamma_0 - 1)(s-1)^2 + \cdots \right\} \\ &\quad \cdot \left\{ (\lambda - \gamma_0) + u(s-1) - v(s-1)^2 + \cdots \right\} \end{aligned}$$

thus

$$b_0'(1) = \lambda - \gamma_0, \qquad b_1'(1) = (\gamma_0 - 1)(\lambda - \gamma_0) + u,$$
$$b_2'(1) = -v + (\gamma_0 - 1)u - (\gamma_1 + \gamma_0 - 1)(\lambda - \gamma_0).$$

Also

$$\begin{aligned} b''(1) &= b'(1) \sum_p \left[\ln\left(1 - \frac{1}{p^s}\right) + \frac{1}{p^s} \right] - b(1) \sum_p \frac{1}{p^{2s}} \\ &= \{(\lambda - \gamma_0) + \cdots\} \{(\lambda - \gamma_0) + \cdots\} - \{1 + \cdots\} \{t + \cdots\} \end{aligned}$$

therefore $b_0''(1) = (\lambda - \gamma_0)^2 - t$. Since

$$a_0''(1) = b_0''(1) + 2\gamma_0 b_0'(1) + \left(\gamma_0^2 - \tfrac{\pi^2}{6}\right) b_0(1) = \lambda^2 - t - \tfrac{\pi^2}{6},$$

$$a_k''(1) = 2(-1)^{k-1}(k-1)! \left(b_k'(1) + \left(\gamma_0 - \sum_{j=1}^{k-1} \tfrac{1}{j} \right) b_k(1) \right), \qquad k \geq 1,$$

the formulas for c_0, c_1, c_2 follow.

In the same way, to obtain the variance for $\Omega(n)$, differentiate $B(z)$ and set $z = 1$:

$$\begin{aligned} B'(1) &= B(1) \sum_p \left[\ln\left(1 - \frac{1}{p^s}\right) + \frac{1}{p^s - 1} \right] \\ &= \left\{ 1 + (\gamma_0 - 1)(s-1) - (\gamma_1 + \gamma_0 - 1)(s-1)^2 + \cdots \right\} \\ &\quad \cdot \left\{ (\Lambda - \gamma_0) - U(s-1) + V(s-1)^2 + \cdots \right\} \end{aligned}$$

thus

$$B_0'(1) = \Lambda - \gamma_0, \qquad B_1'(1) = (\gamma_0 - 1)(\Lambda - \gamma_0) - U,$$
$$B_2'(1) = V - (\gamma_0 - 1)U - (\gamma_1 + \gamma_0 - 1)(\Lambda - \gamma_0).$$

Also

$$\begin{aligned} B''(1) &= B'(1) \sum_p \left[\ln\left(1 - \frac{1}{p^s}\right) + \frac{1}{p^s - 1} \right] + B(1) \sum_p \frac{1}{(p^s - 1)^2} \\ &= \{(\Lambda - \gamma_0) + \cdots\} \{(\Lambda - \gamma_0) + \cdots\} + \{1 + \cdots\} \{T + \cdots\} \end{aligned}$$

therefore $B_0''(1) = (\Lambda - \gamma_0)^2 + T$. We have $A_0''(1) = \Lambda^2 + T - \frac{\pi^2}{6}$ and a formula for $A_k''(1)$, $k \geq 1$, identical to that for $a_k''(1)$ earlier; hence the formulas for C_0, C_1, C_2 follow. It is interesting that higher-order terms for $\mathrm{E}_n(\omega)$ and $\mathrm{E}_n(\Omega)$ coincide, but differ for $\mathrm{Var}_n(\omega)$ and $\mathrm{Var}_n(\Omega)$.

We conclude with an unsolved problem. The expressions

$$\sum_{n=1}^{N} 2^{\omega(n)}, \qquad \sum_{n=1}^{N} 3^{\omega(n)}, \qquad \sum_{n=1}^{N} 2^{\Omega(n)}$$

were mentioned in [22]. Tenenbaum [23] has computed that

$$\sum_{n=1}^{N} 3^{\Omega(n)} = N^{\theta} g\left(\tfrac{\ln(N)}{\ln(2)}\right) + O(N \ln(N)^3)$$

where $\theta = \ln(3)/\ln(2) = 1.5849625007...$ [24] and $g(x)$ is a fractal-like function of period 1 that oscillates between two positive constants. In fact,

$$g(x) = \frac{3}{2} \sum_{\substack{m \geq 1 \\ \gcd(m,6)=1}} \left(\frac{3^{\Omega(m)}}{m^{\theta}} \cdot \sum_{k \geq 0} 3^{-(\theta-1)k - \left\{x - \frac{\ln(m)}{\ln(2)} - \theta k\right\}} \right)$$

where $\{y\} = y - \lfloor y \rfloor$ for all real numbers y, and

$$3.74... = \lim_{x \to 1^-} g(x) = \inf_x g(x) < \sup_x g(x) = \lim_{x \to 0^+} g(x) = 4.74...$$

It would be good to someday know these bounds to higher precision.

0.6. Addendum II. Let $P(n)$ be the largest prime factor of n. The average of $P(n)$ satisfies [25]

$$\frac{1}{N} \sum_{n \leq N} P(n) \sim \sum_{k=0}^{\infty} k! \frac{N}{\ln(N)^{k+1}} \xi_k$$

as $N \to \infty$, where

$$\xi_k = \frac{1}{2^{k+1}} \sum_{j=0}^{k} \frac{2^j (-1)^j}{j!} \zeta^{(j)}(2)$$

and the median $M(N)$ of $\{P(n) : n \leq N\}$ satisfies [26]

$$M(N) \sim \exp\left(\frac{\gamma - 1}{\sqrt{e}}\right) N^{1/\sqrt{e}} = (0.7738078734...) N^{0.6065306597...}$$

(actually, more terms in the asymptotic expansion of $M(N)$ are possible). Clearly the median value grows substantially slower than the mean value. The mode (most frequent value) grows even more slowly [27]. Another interesting asymptotic expansion [26] refines de la Vallée Poussin's average for fractional parts of a large integer N divided by each prime $p \leq N$:

$$\sum_{p \leq N} \left\{ \frac{N}{p} \right\} \sim \sum_{k=0}^{\infty} k! \frac{N}{\ln(N)^{k+1}} \eta_k$$

as $N \to \infty$, , where

$$\eta_k = 1 - \sum_{j=0}^{k} \frac{\gamma_j}{j!}.$$

Alternative proofs regarding $\mathrm{E}_n(\omega)$ and $\mathrm{E}_n(\Omega)$ also appear in [26], but not $\mathrm{Var}_n(\omega)$ nor $\mathrm{Var}_n(\Omega)$.

0.7. Acknowledgements. I am grateful to Donald Knuth for his detailed letter [17], on which my discussion of the Selberg-Delange method is based, and Philippe Flajolet for his supportive guidance. Persi Diaconis kindly provided [15] and Pascal Sebah computed the constants v and V.

References

[1] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A000055 and A000081.

[2] S. R. Finch, Otter's tree enumeration constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 295–316.

[3] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A001221, A001222, A013939, A022559, and A069811.

[4] S. R. Finch, Meissel-Mertens constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 94–98.

[5] J. M. Plotkin and J. W. Rosenthal, How to obtain an asymptotic expansion of a sequence from an analytic identity satisfied by its generating function, *J. Austral. Math. Soc. Ser. A* 56 (1994) 131–143; MR1250997 (94k:05015).

[6] M. Drmota and B. Gittenberger, The distribution of nodes of given degree in random trees, *J. Graph Theory* 31 (1999) 227–253; http://dmg.tuwien.ac.at/drmota/litalt.html; MR1688949 (2000f:05069).

[7] M. Drmota, Combinatorics and asymptotics on trees, *Cubo* 6 (2004) 105–136; http://dmg.tuwien.ac.at/drmota/litalt.html; MR2092045 (2005h:05100).

[8] G. Darboux, Mémoire sur l'approximation des fonctions de très grands nombres, *Journal de Mathématiques* 4 (1878) 5–56, 377–416.

[9] G. Szegö, *Orthogonal Polynomials*, 2$^{\text{nd}}$ ed., Amer. Math. Soc., 1975, pp. 205–206; MR0106295 (21 #5029).

[10] G. Pólya and R. C. Read, *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds*, Springer-Verlag, 1987; MR0884155 (89f:05013).

[11] D. E. Knuth, I. Vardi and R. Richberg, The asymptotic expansion of the middle binomial coefficient, *Amer. Math. Monthly* 97 (1990) 626-630.

[12] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, 2009, pp. 475–480; http://algo.inria.fr/flajolet/Publications/AnaCombi/; MR2483235 (2010h:05005).

[13] S. R. Finch, Stieltjes constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 166–171.

[14] D. E. Knuth, *Selected Papers on Analysis of Algorithms*, CSLI, 2000, pp. 338-339; MR1762319 (2001c:68066).

[15] P. Diaconis, Asymptotic expansions for the mean and variance of the number of prime factors of a number n, Dept. of Statistics Tech. Report 96, Stanford Univ., 1976.

[16] P. Diaconis, G. H. Hardy and probability??? *Bull. London Math. Soc.* 34 (2002) 385–402; MR1897417 (2003d:01023).

[17] D. E. Knuth, Asymptotics for $\mathrm{E}_n(\Omega)$ and $\mathrm{Var}_n(\Omega)$, unpublished note (2003).

[18] P. Sebah, Computing $\sum \frac{p}{(p-1)^3} \log^2 p$ and $\sum \frac{2p-1}{2p(p-1)^2} \log^2 p$, unpublished note (2003).

[19] A. Selberg, Note on a paper by L. G. Sathe, *J. Indian Math. Soc.* 18 (1954) 83-87; MR0067143 (16,676a).

[20] B. Saffari, Sur quelques applications de la "méthode de l'hyperbole" de Dirichlet à la théorie des nombres premiers, *Enseign. Math.* 14 (1970) 205–224; MR0268138 (42 #3037).

[21] H. Delange, Sur des formules de Atle Selberg, *Acta Arith.* 19 (1971) 105-146 (errata insert); MR0289432 (44 #6623).

[22] S. R. Finch, Hafner-Sarnak-McCurley constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 110–112.

[23] G. Tenenbaum, Asymptotics for the sum of $3^{\Omega(n)}$, unpublished note (2001).

[24] S. R. Finch, Stolarsky-Harborth constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 145–151.

[25] E. Naslund, The average largest prime factor, *Integers* 13 (2013) A81; MR3167928.

[26] E. Naslund, The median largest prime factor, *J. Number Theory* 141 (2014) 109–118; arXiv:1207.0232; MR3195391.

[27] N. McNew, Popular values of the largest prime divisor function, arXiv:1504.05985.