Bohr's Inequality

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This essay complements an earlier one [1] on uncertainty inequalities. Let $B_{n,r}$ denote the open n-dimensional ball of radius r centered at the origin. Assume that $f:\mathbb{R}^n\to\mathbb{R}$ is integrable and that its Fourier transform:

$$\hat{f}(\xi)=\int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x}f(x)\,dx$$

satisfies $\hat{f}(\xi)=0$ for all $\xi\in B_{n,r}$. Note that, to be consistent with the partial differential equations literature, we omit the factor 2π from the exponent (compare with [1]). Assume also that both f and its gradient ∇f are continuous and bounded on $\mathbb{R}^n$. In the case $n=1$, Bohr [2, 3, 4] proved that

$$r\sup_{x\in\mathbb{R}}|f(x)|\leq\frac{\pi}{2}\sup_{x\in\mathbb{R}}|f'(x)|.$$

The constant $\pi/2$ is clearly best possible, for examine the periodic function $f(x)=-r|x|+\pi/2$ for $|x|\leq\pi/r$ (of period $2\pi/r$). See [0.1] for more discussion of this example. In the case $n=2$, Rüssmann [5] and Hörmander & Bernhardsson [6] calculated that the best constant in the inequality

$$r\sup_{x\in\mathbb{R}^2}|f(x)|\leq C\sup_{x\in\mathbb{R}^2}\|\nabla f(x)\|$$

is $C=2.9038872827...$ (the indicated vector norm is Euclidean). They succeeded in reducing the computation of C to the following one-dimensional optimization problem:

$$C=\min\int\limits_0^\infty|g(y)|\,dy$$

where the minimum is taken over all integrable functions $g:\mathbb{R}\to\mathbb{R}$ satisfying $g(0)=1$, $g(y)=g(-y)$ for all y, and $\hat{g}(\eta)=0$ for all $|\eta|\geq 1$. In fact, g can be extended to an entire analytic function of exponential type 1 on the complex plane; the zeroes of g are all real and simple.

The constants $\pi/2$ and C also appear in connection with solving the linear operator equation $PX - XQ = Y$, where $P : \mathbb{H} \to \mathbb{H}$, $Q : \mathbb{K} \to \mathbb{K}$ and $Y : \mathbb{H} \to \mathbb{K}$ are bounded operators on Hilbert spaces $\mathbb{H}$ and $\mathbb{K}$. If the spectra $\sigma(P)$, $\sigma(Q)$ of P, Q are disjoint subsets of $\mathbb{C}$, then the equation $PX - XQ = Y$ possesses a unique solution X. Let $\delta = \inf_{\lambda \in \overline{\sigma(P)}, \mu \in \overline{\sigma(Q)}} |\lambda - \mu|$, the separation between closed sets containing the spectra. The norm of the transformation $Y \mapsto X$ can be bounded by $(\pi/2)/\delta$ if P, Q are self-adjoint and C/δ if instead P, Q are normal. Bhatia, Davis & Koosis [7, 8, 9] wrote that there is "no substantial evidence" for expecting these two constants to be best possible here, but added that they cannot be far off. A related problem involves perturbation bounds for spectral subspaces.

What can be said if higher-order derivatives of f are continuous and bounded on $\mathbb{R}^n$? In the case $n = 1$, Favard [10] proved that

$$r^m \sup_{x \in \mathbb{R}} |f(x)| \leq K_m \sup_{x \in \mathbb{R}} |f^{(m)}(x)|$$

for each positive integer m, where the constants [11]

$$1 = K_0 < K_2 = \frac{\pi^2}{8} < K_4 < \ldots < \frac{4}{\pi} < \ldots < K_5 < K_3 = \frac{\pi^3}{24} < K_1 = \frac{\pi}{2}$$

are all best possible. This is called the **Bohr-Favard inequality**. The case $n \geq 2$ remains open.

What can be said if we assume instead that $\hat{f}(\xi) = 0$ for all $\xi \notin \bar{B}_{n,r}$? (That is, we assume the support of f is completely contained within the closed r-ball, the opposite of before.) In the case $n = 1$, Bernstein [12, 13] proved that

$$\sup_{x \in \mathbb{R}} |f^{(m)}(x)| \leq r^m \sup_{x \in \mathbb{R}} |f(x)|$$

for each positive integer m, where the constant 1 is best possible. Such functions f are said to be **band-limited** and, like g, can be extended to an entire function of exponential type r. The generalization

$$\sup_{x \in \mathbb{R}^n} \|\nabla f(x)\| \leq r \sup_{x \in \mathbb{R}^n} |f(x)|$$

for $n \geq 2$ (when $m = 1$) and higher-order analogs (when $m > 1$) were apparently first found by Nikolskii [14, 15].

0.1. Tempered Distributions. Let f denote the periodic triangular wave function mentioned earlier. It is not true that f is integrable on $\mathbb{R}$: strictly speaking, its Fourier transform is undefined (although signal processing engineers would describe $\hat{f}$

as a weighted sequence of equidistant Dirac impulses at $\xi = \pm r, \pm 2r, \pm 3r, \ldots$). We can circumvent this difficulty by defining a family of rapidly decreasing test functions

$$\varphi_k(x) = e^{-x^2/k^2}, \qquad k = 1, 2, 3, \ldots$$

and then taking

$$\hat{f}(\xi) = \lim_{k\to\infty} \int_{-\infty}^{\infty} e^{-i\xi x} \varphi_k(x) \, f(x) \, dx.$$

What allows us, however, to conclude that $\hat{f}$ is independent of the choice of test functions $\{\varphi_k\}_{k=1}^{\infty}$, $\varphi_k \to 1$ as $k \to \infty$?

Here is a little background. The space of all infinitely differentiable functions φ such that $\varphi^{(j)}(x) = O(|x|^{-n})$ as $x \to \pm\infty$, for any $j \geq 0$ and $n \geq 1$, is called the **Schwarz space** $\mathcal{S}$. A **tempered distribution** is a continuous linear functional T on $\mathcal{S}$ and its (generalized) Fourier transform is defined by

$$\hat{T}(\varphi) = T(\hat{\varphi});$$

this induces an automorphism $\mathcal{S}' \to \mathcal{S}'$ of the dual space $\mathcal{S}'$ of $\mathcal{S}$. Consider now the example

$$F(\varphi) = \int_{-\infty}^{\infty} \varphi(x) \, f(x) \, dx,$$

where f is the periodic triangular wave, and let $f_k = \varphi_k \, f$. Clearly

$$\hat{F}(\varphi) = \int_{-\infty}^{\infty} \hat{\varphi}(x) \, f(x) \, dx = \int_{-\infty}^{\infty} \hat{\varphi}(x) \left(\lim_{k\to\infty} f_k(x) \right) dx$$

while

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \varphi(\xi) \, \hat{f}_k(\xi) \, d\xi = \lim_{k\to\infty} \int_{-\infty}^{\infty} \hat{\varphi}(x) \, f_k(x) \, dx$$

follows by interchanging the order of integration. Since $|\hat{\varphi} \, f_k| \leq |\hat{\varphi} \, f|$ and $\hat{\varphi} \, f$ is integrable on $\mathbb{R}$, the limit may be brought inside the integral by Lebesgue's dominated convergence theorem. Hence, just as f and F are regarded as the same, we may identify $\hat{f}$ and $\hat{F}$.

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