Bessel Function Zeroes

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The Bessel function $J_\nu(x)$ of the first kind

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}, \qquad \nu > -1$$

has infinitely many positive zeros

$$0 < j_{\nu,1} < j_{\nu,2} < j_{\nu,3} < ...$$

as does its derivative $J'_\nu(x)$:

$$0 < j'_{\nu,1} < j'_{\nu,2} < j'_{\nu,3} < ..., \qquad \nu > 0,$$

$$0 = j'_{0,1} < j'_{0,2} < j'_{0,3} < j'_{0,4} < ..., \qquad \nu = 0.$$

See Tables 1 & 2 for the cases $\nu = 0, 1, 2$ and Tables 3 & 4 for the cases $\nu = 1/2, 3/2, 5/2$. These appear in many physical applications that we cannot hope to survey in entirety. We will state only a few properties and several important inequalities. A starting point for research is Watson's monumental treatise [1].

Table 1 *Zeroes of* J_ν *for* $s = 1, 2, 3$ *and integer* ν

$j_{0,s}$	$j_{1,s}$	$j_{2,s}$
2.4048255576...	3.8317059702...	5.1356223018...
5.5200781102...	7.0155866698...	8.4172441403...
8.6537279129...	10.1734681350...	11.6198411721...

Table 2 *Zeroes of* J'_ν *for* $s = 1, 2, 3$ *and integer* ν

$j'_{0,s}$	$j'_{1,s}$	$j'_{2,s}$
0	1.8411837813...	3.0542369282...
3.8317059702...	5.3314427735...	6.7061331941...
7.0155866698...	8.5363163663...	9.9694678230...

Table 3 *Zeroes of J_ν for $s = 1, 2, 3$ and half-integer ν*

$j_{1/2,s}$	$j_{3/2,s}$	$j_{5/2,s}$
π	4.4934094579...	5.7634591968...
2π	7.7252518369...	9.0950113304...
3π	10.9041216594...	12.3229409705...

Table 4 *Zeroes of J'_ν for $s = 1, 2, 3$ and half-integer ν*

$j'_{1/2,s}$	$j'_{3/2,s}$	$j'_{5/2,s}$
1.1655611852...	2.4605355721...	3.6327973198...
4.6042167772...	6.0292923816...	7.3670089715...
7.7898837511...	9.2614019262...	10.6635613904...

Clearly $j_{\nu,s} \to \infty$ as $s \to \infty$ with ν fixed; in fact, $j_{\nu,s+1} - j_{\nu,s} \to \pi$. For $\nu \geq 0$, here is a straightforward lower bound [2, 3]:

$$j_{\nu,s} > \sqrt{(s - \tfrac{1}{4})^2\pi^2 + \nu^2}$$

and, for $\nu > 0$, here are more complicated bounds [4, 5, 6]:

$$\nu + \alpha_s \nu^{1/3} < j_{\nu,s} < \nu + \alpha_s \nu^{1/3} + \frac{3\alpha_s^2}{10}\nu^{-1/3}$$

where $\alpha_s = 2^{-1/3} a_s$ and a_s is the s^{th} positive root of the equation

$$J_{\frac{1}{3}}\left(\tfrac{2}{3}x^{3/2}\right) + J_{-\frac{1}{3}}\left(\tfrac{2}{3}x^{3/2}\right) = 0.$$

For example, $a_1 = 2.3381074104...$ and thus the coefficients of $\nu^{1/3}$ and $\nu^{-1/3}$ for $s = 1$ are 1.8557570814... and 1.0331503036..., respectively. (The left hand side of the equation is the same as $3\,\mathrm{Ai}(-x)/\sqrt{x}$, where Ai is the Airy function.) These bounds are asymptotically precise; more terms in the asymptotic expansion of $j_{\nu,s}$ as $\nu \to \infty$, for any fixed s, can be obtained [7, 8, 9, 10]. Related work includes [11, 12, 13, 14, 15].

Similarly we have

$$\nu + \alpha'_s \nu^{1/3} < j'_{\nu,s} < \nu + \alpha'_s \nu^{1/3} + \frac{3\alpha'^3_s - 1}{10\alpha'_s}\nu^{-1/3}$$

where $\alpha'_s = 2^{-1/3} a'_s$ and a'_s is the s^{th} positive root of the equation

$$J_{\frac{2}{3}}\left(\tfrac{2}{3}x^{3/2}\right) - J_{-\frac{2}{3}}\left(\tfrac{2}{3}x^{3/2}\right) = 0.$$

For example, $a_1' = 1.0187929716...$ and thus the coefficients of $\nu^{1/3}$ and $\nu^{-1/3}$ for $s = 1$ are $0.8086165174...$ and $0.0724901862...$, respectively. (The left hand side of the equation is the same as $3\,\mathrm{Ai}'(-x)/x$.) The zeroes of J_ν and J_ν' are interlaced:

$$... < j_{\nu,s}' < j_{\nu,s} < j_{\nu,s+1}' < j_{\nu,s+1} < ...$$

and further satisfy [16]

$$j_{\nu,s+1}' > \sqrt{j_{\nu,s} j_{\nu,s+1}}.$$

Let $n \geq 0$ be an integer. Every Bessel function $J_{n+1/2}(x)$ is elementary; for example, $\sqrt{x}J_{1/2}(x)$ can be simplified to $\sqrt{2/\pi}\sin(x)$. Consequently $j_{3/2,s}$ is the s^{th} positive root of the equation

$$\sin(x) - x\cos(x) = 0, \quad \text{that is,} \quad \tan(x) = x,$$

and $j_{1/2,s}'$ is the s^{th} positive root of the equation

$$\sin(x) - 2x\cos(x) = 0, \quad \text{that is,} \quad \tan(x) = 2x.$$

Siegel [1, 17, 18] proved that $J_\nu(\xi)$ is transcendental whenever ν is rational and ξ is algebraic. It follows immediately that every zero $j_{\nu,s}$ is transcendental. Further, if μ is rational and $\nu - \mu \neq 0$ is an integer, then $J_\nu(x)$ and $J_\mu(x)$ can never have common zeroes (other than $x = 0$) [19, 20, 21, 22].

Series of the form [1, 23]

$$\sum_{s=1}^{\infty} \frac{1}{j_{\nu,s}^2} = \frac{1}{4(\nu+1)}, \qquad \sum_{s=1}^{\infty} \frac{1}{j_{\nu,s}^4} = \frac{1}{16(\nu+1)^2(v+2)}$$

possess well-known special cases. If $\nu = 1/2$, then $j_{\nu,s} = \pi s$ and

$$\sum_{s=1}^{\infty} \frac{1}{s^2} = \frac{\pi^2}{6}, \qquad \sum_{s=1}^{\infty} \frac{1}{s^4} = \frac{\pi^4}{90}$$

as given in [24]. We also have

$$\sum_{s=1}^{\infty} \frac{1}{j_{0,s}^2} = \frac{1}{4}, \qquad \sum_{s=1}^{\infty} \frac{1}{j_{3/2,s}^2} = \frac{1}{10}$$

and the latter series appears in [25]. Other identities can be found in [26, 27].

We need three more tables before continuing. Define

$$P_\nu(x) = \frac{d}{dx}\left(x^{1-\nu}J_\nu(x)\right) = x^{-\nu}\left((1-\nu)J_\nu(x) + xJ_\nu'(x)\right)$$

$$Q_\nu(x) = J_\nu(x)I_{\nu+1}(x) + I_\nu(x)J_{\nu+1}(x)$$

where $I_\nu(x)$ is the modified Bessel function of the first kind:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k} = i^{-\nu} J_\nu(ix).$$

Let $p_{\nu,s}$ and $q_{\nu,1}$ denote the s^{th} smallest positive zeroes of $P_\nu(x)$ and $Q_\nu(x)$. It is clear that $p_{1,s} = j'_{1,s}$ for all s. See Tables 5 & 6.

Table 5 *Zeroes of* P_ν *for* $s = 1, 2, 3$

$p_{1,s}$	$p_{3/2,s}$	$p_{2,s}$
1.8411837813...	2.0815759778...	2.2999103302...
5.3314427735...	5.9403699905...	6.5414028262...
8.5363163663...	9.2058401429...	9.8647278383...

Table 6 *Zeroes of* Q_ν *for* $s = 1, 2, 3$

$q_{0,s}$	$q_{1/2,s}$	$q_{1,s}$
3.1962206165...	3.9266023120...	4.6108998790...
6.3064370476...	7.0685827456...	7.7992738008...
9.4394991378...	10.2101761228...	10.958067191...

Finally, we offer an application. Table 7 gives the vibration modes of an idealized timpani (or kettledrum). In contrast, the frequency ratios for overtones of an idealized guitar string are all integers [28].

Table 7 *Frequency ratios for the first five overtones of a fixed circular membrane*

ν	s	$j_{\nu,s}/\pi$	$j_{\nu,s}/j_{0,1}$
0	1	0.7654797495...	1
1	1	1.2196698912...	1.5933405056...
2	1	1.6347193503...	2.1355487866...
0	2	1.7570954350...	2.2954172674...
3	1	2.0308686069...	2.6530664045...
1	2	2.2331305943...	2.9172954551...

0.1. Membrane and Plate Inequalities.

Let $n \geq 2$. Let $\Omega \subseteq \mathbb{R}^n$ be a connected bounded open set of volume $|\Omega|$, and assume that its boundary $\partial\Omega$ is smooth. Define the **Laplacian** and **bi-Laplacian (biharmonic)** operators

$$\triangle f = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial^2 x_k}, \qquad \triangle^2 f = \triangle(\triangle f)$$

for smooth functions $f : \Omega \to \mathbb{R}$. We will briefly consider four famous eigenvalue problems (i.e., isoperimetric inequalities) that occur in structural dynamics for which Bessel function zeroes play a role [29, 30].

The **fixed (fastened) membrane** problem involves the Laplacian with **Dirichlet** boundary conditions:

$$-\triangle u = \lambda u \qquad \text{in } \Omega$$

$$u = 0 \qquad \text{on } \partial\Omega.$$

We seek the smallest eigenvalue $\lambda_1(\Omega)$, that is, the fundamental frequency of vibration. When is $\lambda_1(\Omega)$ minimal? The **Rayleigh-Faber-Krahn** inequality provides that [31]

$$\lambda_1(\Omega) \geq \left(\frac{\omega_n}{|\Omega|}\right)^{2/n} j^2_{\frac{n}{2}-1,1}$$

with equality if and only if Ω is a ball. Here $\omega_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit ball in $\mathbb{R}^n$. Only the case $n = 2$ was mentioned in [32]. For example, $j^2_{0,1} = 5.7831859629...$

The **free membrane** problem involves the Laplacian with **Neumann** boundary conditions:

$$-\triangle v = \mu v \qquad \text{in } \Omega$$

$$\frac{\partial v}{\partial n} = 0 \qquad \text{on } \partial\Omega,$$

where $\partial v/\partial n$ denotes the outward normal derivative of v. Since $\mu_1(\Omega) = 0$, we seek the next-to-smallest eigenvalue $\mu_2(\Omega)$. When is $\mu_2(\Omega)$ maximal? The **Szegö-Weinberger** inequality provides that [33, 34, 35, 36]

$$\mu_2(\Omega) \leq \left(\frac{\omega_n}{|\Omega|}\right)^{2/n} p^2_{\frac{n}{2},1}$$

with equality if and only if Ω is a ball.

The **clamped plate** problem involves the bi-Laplacian with the following boundary conditions:

$$\triangle^2 w = \Lambda w \qquad \text{in } \Omega$$

$$w = \frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial\Omega.$$

We seek the smallest eigenvalue $\Lambda_1(\Omega)$. When is $\Lambda_1(\Omega)$ minimal? The **Nadirashvili-Ashbaugh-Benguria** inequality provides that [37, 38, 39]

$$\Lambda_1(\Omega) \geq \left(\frac{\omega_n}{|\Omega|}\right)^{4/n} q^4_{\frac{n}{2}-1,1}$$

with equality if and only if Ω is a ball. This has been rigorously proved only for $2 \leq n \leq 3$, but it is known to be true for $n \geq 4$ up to a constant factor $\to 1$ as $n \to \infty$. Only the case $n = 2$ was mentioned in [32].

The **buckling load** problem involves both the Laplacian and bi-Laplacian with the following boundary conditions:

$$\triangle^2 z = -M \triangle z \qquad \text{in } \Omega$$

$$z = \frac{\partial z}{\partial n} = 0 \qquad \text{on } \partial\Omega.$$

We seek the smallest eigenvalue $M_1(\Omega)$. When is $M_1(\Omega)$ minimal? Pólya & Szegö [39, 40] conjectured that

$$M_1(\Omega) \geq \left(\frac{\omega_n}{|\Omega|}\right)^{2/n} j^2_{\frac{n}{2},1}$$

with equality if and only if Ω is a ball, but this is only known to be true up to a constant factor $\to 1$ as $n \to \infty$.

We return to the original Dirichlet problem to state one more idea. The **Payne-Pólya-Weinberger** conjecture**,** proved by Ashbaugh & Benguria [41, 42, 43]**,** involves the maximal ratio of the two smallest eigenvalues $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$:

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{j^2_{\frac{n}{2},1}}{j^2_{\frac{n}{2}-1,1}}$$

with equality if and only if Ω is a ball. For example, when $n = 2$, the right hand side is 2.5387339670... What can be said about the maximal ratios of two arbitrary eigenvalues? [44]

0.2. Other Best Constants. Bessel function zeroes occur in best constants associated with Nash's inequality [45], uncertainty inequalities [46], and with an improved version of Hardy's inequality [47, 48, 49, 50, 51]. We hope to include more examples here later.

We close with remarks about the multiplicities of the zeroes. It appears that, for fixed $\nu > 0$, the positive zeroes $j''_{\nu,s}$ of the second derivative $J''_\nu(x)$ are all simple, like those of $J_\nu(x)$ and $J'_\nu(x)$. This is no longer true when considering positive zeroes $j'''_{\nu,s}$ of the third derivative $J'''_\nu(x)$: there exists a value $\nu_0 = 0.755378...$ for which J'''_{ν_0} has a double zero $x_0 = 0.959621...$ [52, 53]. Related papers include [54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64], the latter of which are more concerned with the strictly increasing behavior of $j''_{\nu,s}$ as a function of ν for fixed s (rather than of s for fixed ν).

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