Molteni's Composition Constant

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This essay continues where we left off in [1]: the number of (unordered) partitions of 2^{k-1} as a sum of k powers of 2 is well-understood [2, 3, 4, 5, 6]. What can be said about the number $w(k)$ of (ordered) compositions of 2^{k-1} as a sum of k powers of 2? Clearly $w(1) = w(2) = 1$; $w(3) = 3$ since there are three ways to sort $\{1, 1, 2\}$ and $w(4) = 13$ since there are twelve ways to sort $\{1, 1, 2, 4\}$ plus $8 = 2 + 2 + 2 + 2$. A few more terms of $\{w(k)\}$ appear in [7, 8] but a pattern is far from clear.

The following doubly-indexed recursive formula [9]

$$m_{k,\ell} = \begin{cases} 0 & \text{if } \ell \geq k, \\ 1 & \text{if } k > 1 \text{ and } \ell = k-1, \\ \displaystyle\sum_{j=1}^{2\ell} \binom{k+\ell-1}{2\ell-j} m_{k-\ell,j} & \text{if } 1 \leq \ell < k-1, \end{cases}$$

coupled with $w_k = m_{k,1}$, $k > 1$, makes efficient calculation of many more terms possible. It further allowed Molteni [10] to deduce the asymptotic behavior of $\{w(k)\}$:

$$\lim_{k\to\infty} \left(\frac{w(k)}{k!}\right)^{1/k} = 1.1926743412...$$

– a remarkable achievement! – but an exact formula for this constant seems to be unavailable. The same constant appears in a more general setting when 2^{k-1} is replaced by, for instance, a sum of two distinct powers of 2. As an example, $w'(3) = 6$ since $10 = 2+8$, there are three ways to sort $\{1, 1, 8\}$ plus three ways to sort $\{2, 4, 4\}$, and such a portfolio is maximal. Replacing w by w' in the limiting expression does not change the constant.

0.1. Euler Binary Partitions. Given $d \geq 2$ and $n \geq 0$, let $b_d(n)$ denote the number of integer sequences $x_1, x_2, x_3, \ldots$ satisfying $0 \leq x_i \leq d-1$ for all i for which $n = \sum_{i=0}^{\infty} x_i 2^i$. Clearly $b_2(n) = 1$ for all n, $\{b_3(n)\}$ is related to Stern's sequence [11], and $b_4(n) = \lfloor n/2 \rfloor + 1$ for all n. Define

$$\kappa_d = \liminf_{n\to\infty} \frac{\ln(b_d(n))}{\ln(n)}, \qquad \lambda_d = \limsup_{n\to\infty} \frac{\ln(b_d(n))}{\ln(n)}.$$

The most interesting asymptotics occur for odd d and we list several results here [12, 13, 14, 15, 16]:

$$2^{\kappa_3} = 1, \qquad 2^{\lambda_3} = \varphi = \left(1+\sqrt{5}\right)/2 = 1.6180339887...;$$

$$2^{\kappa_5} = 1+\sqrt{2} = 2.4142135623..., \qquad 2^{\lambda_5} = 2.5386157635...$$

has minimal polynomial $z^4 - 2z^3 - 2z^2 + 2z - 1$;

$$2^{\kappa_7} = 3.4918910516..., \qquad 2^{\lambda_7} = 3.5115471416...$$

have minimal polynomials $z^5 - z^4 - 7z^3 - 5z^2 - 3z - 1$ and $z^3 - 4z^2 + 2z - 1$, respectively; and

$$2^{\kappa_9} = 4.4944928370..., \qquad 2^{\lambda_9} = 4.5030994219...$$

have minimal polynomials $z^3 - 4z^2 - 2z - 1$ and $z^8 - 3z^7 - 9z^6 + 9z^5 + 5z^4 - z^3 - z^2 - z + 1$, respectively.

0.2. Joint Spectral Radius. The joint spectral radius [17] of two real 2×2 matrices A, B is the maximum possible exponential rate of growth of long products of As and Bs. The set $\{A, B\}$ is said to have the finiteness property if there exists a periodic product that attains this maximal rate of growth. At one point, it was believed that every set $\{A, B\}$ satisfies the finiteness property. This was eventually disproved; the first explicit counterexample was given in [18]. It takes the form

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = c \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

where the constant c requires elaboration. Define

$$e_{n+1} = e_n e_{n-1} - e_{n-2}, \qquad e_0 = 1, \qquad e_1 = 2, \qquad e_2 = 2$$

and

$$f_{n+1} = f_n + f_{n-1}, \qquad f_0 = 0, \qquad f_1 = 1$$

(the latter is the Fibonacci sequence). It follows that

$$\begin{aligned} c &= \lim_{n\to\infty} \left(\frac{e_n^{f_{n+1}}}{e_{n+1}^{f_n}} \right)^{(-1)^n} = \prod_{n=1}^{\infty} \left(1 - \frac{e_{n-1}}{e_{n+1}e_n}\right)^{(-1)^n f_{n+1}} \\ &= 0.7493265463... \end{aligned}$$

converges unconditionally. No uniqueness claims have been made about c; we are simply attracted by its intricate construction. The authors of [18] wondered whether c is irrational, tying it to the Fibonacci substitution $0 \to 01$, $1 \to 0$ [19] and to the quantity $1/\varphi^2 = \left(3-\sqrt{5}\right)/2$. They conjectured that $\tilde{c}$ is irrational, where $\tilde{c}$ (unspecified but distinct from c) is tied to the substitution $0 \to 001$, $1 \to 0$ and to the quantity $1 - 1/\sqrt{2}$. We hope to report more about $\tilde{c}$ later.

References

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