Condition Numbers of Matrices

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Let A be a real $n\times n$ matrix and

$$\|A\| = \sup_{\|x\|=1} \|A\,x\|$$

denote its Euclidean operator norm (often called the 2-norm). If A is nonsingular, then its **condition number** $\kappa(A)$ is defined by

$$\kappa(A) = \|A\|\,\left\|A^{-1}\right\| = \frac{\sigma_1(A)}{\sigma_n(A)}$$

where $\sigma_1 \geq \sigma_1 \geq \ldots \geq \sigma_n \geq 0$ are the singular values of A. The σs constitute lengths of the semi-axes of the hyperellipsoid $E = \{A\,x : \|x\| = 1\}$ in n-dimensional space; thus κ measures elongation of E at its extreme [1]. The role that κ plays in numerical analysis cannot be overstated: real matrices with large κ are called **ill-conditioned** whereas matrices with small κ are called **well-conditioned**. In a nutshell, κ quantifies the sensitivity of x to pertubations in A and b when solving the linear system $A\,x = b$.

It remains to understand meaning of "large" versus "small" in this context. Let the entries of A be independent normally distributed random variables with mean 0 and variance 1. Edelman [2] proved that the condition number κ_n satisfies

$$\mathrm{E}\left(\ln(\kappa_n)\right) = \ln(n) + c + o(1)$$

as $n\to\infty$, where

$$c = -\frac{1}{2}\tilde{c} + \ln(2) = 1.5370894353... = \ln(4.6510334182...),$$

$$\begin{aligned}\tilde{c} &= \int\limits_0^\infty \ln(x)\frac{1+\sqrt{x}}{2\sqrt{x}}\exp\left(-\frac{x}{2}-\sqrt{x}\right)dx \\ &= -2\gamma - 2e^{1/2}\int\limits_1^\infty \frac{1}{y+1}\exp\left(-\frac{1}{2}y^2\right)dy \\ &= -1.6878845096...\end{aligned}$$

and γ is the Euler-Mascheroni constant [3]. Therefore random dense matrices are well-conditioned, in the sense that κ_n grows only linearly with n.

Let A be the same as before except all superdiagonal entries are zero and all diagonal elements are one. That is, A is a unit lower triangular matrix, all of whose subdiagonal entries are independent $N(0,1)$. Viswanath & Trefethen [4] proved that

$$\begin{aligned}\sqrt[n]{\kappa_n} &\to \exp\left[\frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\ln\left(1+x^2\right)\exp\left(-\frac{1}{2}x^2\right)dx\right] \\ &= 1.3056834105...\end{aligned}$$

almost surely as $n \to \infty$. Therefore random unit lower triangular matrices are ill-conditioned, in the sense that κ_n grows exponentially with n. Such behavior is in striking contrast to the linear growth for random dense matrices.

Similar conclusions follow if we replace the normal distribution by, say, the Cauchy distribution with density function

$$\frac{1}{\pi}\frac{1}{1+x^2}$$

for $-\infty < x < \infty$. An exact limiting expression for $\mathrm{E}\left(\ln(\kappa_n/n)\right)$ analogous to that in [2] is unknown, although Monte Carlo simulation suggests that a constant c indeed exists and is close to 7.0. For random unit lower triangular matrices, we have [4]

$$\begin{aligned}\sqrt[n]{\kappa_n} &\to \exp\left[\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{\ln\left(1+|x|\right)}{1+x^2}dx\right] \\ &= \exp\left(\frac{\ln(2)}{2}+\frac{2G}{\pi}\right) = 2.5337372794...\end{aligned}$$

almost surely as $n \to \infty$, where G is Catalan's constant [5]. An interesting variation arises if we allow the diagonal entries of the latter to be independent Cauchy as well (rather than fixed at unity):

$$\begin{aligned}\sqrt[n]{\kappa_n} &\to \exp\left[\frac{2}{\pi^2}\int\limits_{-\infty}^{\infty}\frac{\ln\left(1+|x|\right)\ln\left(|x|\right)}{x^2-1}dx\right] \\ &= \exp\left(\ln(2)+\frac{7\zeta(3)}{2\pi^2}\right) = 3.0630941933...\end{aligned}$$

almost surely as $n \to \infty$, where $\zeta(3)$ is Apéry's constant [6].

We can extend our discussion to complex matrices. Let the real and imaginary parts of entries of A be independent normally distributed random variables with mean 0 and variance 1. From [2], we have

$$\mathrm{E}\left(\ln(\kappa_n)\right) = \ln(n) + d + o(1)$$

as $n \to \infty$, where

$$d = -\frac{1}{2}\tilde{d} + \frac{3}{2}\ln(2) = 0.9817550130... = \ln(2.6691365030...),$$

$$\tilde{d} = \int_0^\infty \ln(x)\frac{1}{2}\exp\left(-\frac{x}{2}\right)dx = \ln(2) - \gamma = 0.1159315156....$$

If we replace the normal distribution by the Cauchy distribution, then simulation suggests that d indeed exists and is close to 6.4.

Finally, let real/imaginary parts of entries of unit lower triangular A be independent normal with mean 0 and variance 1/2 (different scaling than previously). From [4], we have

$$\begin{aligned} \sqrt[n]{\kappa_n} &\to \exp\left[\frac{1}{4}\int_0^\infty \ln\left(1+\frac{x}{2}\right)\exp\left(-\frac{x}{2}\right)dx\right] \\ &= \exp\left(-\frac{e}{2}\,\mathrm{Ei}(-1)\right) = 1.3473957848... \end{aligned}$$

almost surely as $n \to \infty$, where Ei is the exponential integral [7]. Numerical values when replacing the normal distribution here by the Cauchy distribution (for some choice of scaling) remain open. Other choices of densities are possible (symmetric strictly stable distributions, for example) and corresponding constants would be good to see someday.

References

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