Colliding Dice Probabilities

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Let K, L be congruent regular polyhedra in $\mathbb{R}^3$. Let g denote a rigid motion of $\mathbb{R}^3$, that is, $g(x) = \Phi x + \tau$ where Φ is a 3×3 rotation matrix and τ is a translation 3-vector. The polyhedra K, $g(L)$ are said to **touch** if $K \cap g(L) \neq \varnothing$ but $\mathrm{int}(K) \cap \mathrm{int}(g(L)) = \varnothing$. Alternatively, we may think of ΦL moving toward K in the direction τ, stopping precisely when the two polyhedra collide.

Let us sample the space SO_3 of matrices Φ according to the uniform distribution (Haar measure, normalized to 1). The space of vectors τ is slightly harder to describe. Let

$$K - \Phi L = \{y - \Phi x : y \in K \text{ and } x \in L\}$$

be the Minkowski sum of K and the reflected image $-\Phi L$ of ΦL. Another way to characterize $K - \Phi L$ is as the convex hull of all pairwise sums of vertices of K and $-\Phi L$. Clearly

$$\tau \in \mathrm{bd}(K - \Phi L) \qquad \text{if and only if} \qquad \text{the polyhedra } K, g(L) \text{ touch.}$$

Thus we sample the space $\mathrm{bd}(K - \Phi L)$ uniformly (area measure), which is complicated only by the intricate variety of possible faces of $K - \Phi L$.

With independent Φ and τ as described, it is clear that

$$\mathrm{P}\{\text{collision is edge-to-edge}\} > 0,$$

$$\mathrm{P}\{\text{collision is vertex-to-face or face-to-vertex}\} > 0$$

and that no other types of collisions occur with positive likelihood. What is unclear is the relative magnitude of these two probabilities.

Answering a question asked by Firey, McMullen [1, 2] proved that the edge-to-edge collisions are strictly more likely than vertex-to-face collisions. In the case of two cubes (cubical dice), the exact values of the probabilities are

$$\frac{3\pi}{3\pi+8} = 0.5408836762... > 0.4591163237... = \frac{8}{3\pi+8}.$$

More generally, we have [3]

$$\frac{\pi V_1^2}{8V_0V_2+\pi V_1^2} > \frac{8V_0V_2}{8V_0V_2+\pi V_1^2}$$

where $V_0 = 1$ is the Euler characteristic of K, $\frac{1}{2}V_1$ is the mean width b (to be defined shortly), $2V_2$ is the surface area a and V_3 is the volume. For the unit cube, it follows that $b = 3/2$ and $a = 6$.

In the case of two regular tetrahedra (tetrahedral dice), we have

$$b = \tfrac{3}{2\pi}\arccos\left(-\tfrac{1}{3}\right), \qquad a = \sqrt{3}$$

and hence

$$\frac{9\arccos\left(-\frac{1}{3}\right)^2}{4\sqrt{3}\pi+9\arccos\left(-\frac{1}{3}\right)^2} = 0.6015106899... > 0.3984893100... = \frac{4\sqrt{3}\pi}{4\sqrt{3}\pi+9\arccos\left(-\frac{1}{3}\right)^2}.$$

In the case of two regular octahedra (octahedral dice), we have

$$b = \tfrac{3}{\pi}\arccos\left(\tfrac{1}{3}\right), \qquad a = 2\sqrt{3}$$

and hence

$$\frac{9\arccos\left(\frac{1}{3}\right)^2}{2\sqrt{3}\pi+9\arccos\left(\frac{1}{3}\right)^2} = 0.5561691925... > 0.4438308074... = \frac{2\sqrt{3}\pi}{2\sqrt{3}\pi+9\arccos\left(\frac{1}{3}\right)^2}.$$

These specific numerical results are apparently new. For tetrahedra, verification by simulation appears in [4], using [5, 6]. The touching is vertex-to-face or face-to-vertex if and only if τ lies in a triangular face of $K - \Phi L$. (All other faces of $K - \Phi L$ are parallelograms.) Hence it suffices to assess the ratio of surface area of triangles only to surface area of the whole. The cases of two cubes or of two octahedra are more difficult.

0.1. Mean Width. Let C be a convex body in $\mathbb{R}^3$. In earlier essays [7, 8, 9], the words "width" or "breadth" were used to denote the *minimum* distance between all pairs of parallel C-supporting planes. Here, we instead take the *mean* of all such distances, calling this b. The phrase **mean width** [10, 11] is used, as well as **mean breadth** [12] and **mean caliper diameter** [13, 14].

Closed-form expressions for b exist when C is a convex polyhedron. Numerical confirmation of such formulas is possible via quadratic programming (since the optimization constraints are linear) [4].

0.2. Intrinsic Volumes. Let P be a rectangular parallelepiped in $\mathbb{R}^3$ of dimensions z_1, z_2, z_3. It is well-known that

$$V_3(P) = z_1 z_2 z_3,$$

$$V_2(P) = z_1 z_2 + z_1 z_3 + z_2 z_3 = \tfrac{1}{2}a,$$

$$V_1(P) = z_1 + z_2 + z_3 = 2b$$

are the elementary symmetric polynomials in three variables. In $\mathbb{R}^n$, there are n such intrinsic volumes, corresponding to the n elementary symmetric polynomials [11]. Little is known about higher-dimensional intrinsic volumes and the isoperimetric inequalities among them. Limiting approximation arguments enable us to compute $V_j(C)$ for arbitrary convex C. Additionally, let $V_0(C) = 1$. Hadwiger's famous theorem [3] gives that $V_0, V_1, \ldots, V_n$ are a basis of the space of all additive continuous measures that are invariant under rigid motions.

0.3. Acknowledgement. Rolf Schneider generously proposed the method underlying the tetrahedral simulation. More about mean width computations for convex polyhedra is found in [15, 16, 17, 18, 19, 20], for certain other convex bodies in [21, 22, 23], and a specific non-convex body in [24].

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