Distance-Avoiding Sets in the Plane

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Fix a real number $d > 0$. Let $D = \{1, d\}$ if $d \neq 1$; otherwise $D = \{1\}$ may simply be written as 1. A subset $S \subseteq \mathbb{R}^n$ is said to **avoid** D if $\|x - y\| \notin D$ for all $x, y \in S$. For example, the union of open balls of radius $1/2$ with centers in $(2\mathbb{Z})^n$ avoids the distance 1. If instead the balls have centers in $(3\mathbb{Z})^n$, then their union avoids $\{1, 2\}$.

It is natural to ask about the "largest possible" S that avoids D. Let B_R denote the ball of radius R with center 0. Assuming S is Lebesgue measurable, its **density**

$$\delta(S) = \limsup_{R \to \infty} \frac{\mu(B_R \cap S)}{\mu(B_R)}$$

quantifies the asymptotic proportion of $\mathbb{R}^n$ occupied by S. We wish to know

$$m_D(\mathbb{R}^n) = \sup \{\delta(S) : S \text{ is measurable and avoids } D\}.$$

The shortage of information regarding $m_D(\mathbb{R}^n)$ is surprising. Until further notice, let $n = 2$ and $d = 1$ for simplicity [1, 2, 3].

On the one hand, the number of $\mathbb{Z}^2$ points within B_R is $\sim \pi R^2$ [4], hence the number of $(2\mathbb{Z})^2$ points within B_R is $\sim (\pi/4) R^2$. Each open disk in our example has area $\pi/4$ and B_R has area πR^2, thus $m_1(\mathbb{R}^2) \geq \pi/16 \approx 0.196$. It turns out we can do better by arranging the disks with centers according to an equilateral triangle lattice, giving $m_1(\mathbb{R}^2) \geq \pi/\left(8\sqrt{3}\right) \approx 0.227$. An additional improvement (replacing six portions of each circular circumference by linear segments) gives $m_1(\mathbb{R}^2) \geq 0.229365$. This is the best lower bound currently known [5, 6].

On the other hand, a configuration called the Moser spindle implies that $m_1(\mathbb{R}^2) \leq 2/7 \approx 0.286$ [7, 8]. Székely [9, 10] improved the upper bound to $12/43 \approx 0.279$. The best result currently known is $m_1(\mathbb{R}^2) \leq 0.258795$ via linear programming techniques [11, 12]. Erdős' conjecture that $m_1(\mathbb{R}^2) < 1/4$ seems out of reach.

Sets avoiding 1 have been studied by combinatorialists because of their association with the *measurable chromatic number* of the plane. What is the minimum number of colors $\chi_m(\mathbb{R}^2)$ required to color all points of $\mathbb{R}^2$ so that any two points at distance 1 receive distinct colors and so that points receiving the same color form Lebesgue measurable sets? It is known only that $5 \leq \chi_m(\mathbb{R}^2) \leq 7$ [13].

Let us now consider the case $n = 2$ and $d = 2$. The number of $(3\mathbb{Z})^2$ points within B_R is $\sim (\pi/9)\, R^2$. Each open disk in our example has area $\pi/4$ and B_R has area $\pi\, R^2$, thus $m_{1,2}(\mathbb{R}^2) \geq \pi/36 \approx 0.087$. Better lower bounds can surely be found, akin to before. We also know that $m_{1,2}(\mathbb{R}^2) \leq 2/9 \approx 0.222$ [9]. No one appears to have pursued this case further.

A more interesting problem is to allow d to vary, in an effort to determine

$$\inf_{d>0} m_{1,d}(\mathbb{R}^2).$$

One line of research gave $m_{1,\sqrt{3}}(\mathbb{R}^2) \leq 2/11 \approx 0.182$ [9], now improved to $m_{1,\sqrt{3}}(\mathbb{R}^2) \leq 0.170213$ [11]. Another direction gives $m_{1,c}(\mathbb{R}^2) \leq 0.141577$, where

$$c = \frac{j_{1,2}}{j_{1,1}} = 1.8309303282...$$

is a ratio of the first two positive zeroes of the Bessel function J_1 [14, 15]. There is no indication [11] that c is necessarily an optimal choice for d.

For $n = 3$ and $d = 1$, a configuration called the Moser-Raiskii spindle implies that $m_1(\mathbb{R}^3) \leq 3/14 \approx 0.214$ [8]. Székely [16] improved the upper bound to $7/37 \approx 0.189$; this was further diminished to $3/16 = 0.1875$ in [13]. The best result currently known is $m_1(\mathbb{R}^3) \leq 0.165609$ [11].

For $n = 4$ and $d = 1$, an early result $m_1(\mathbb{R}^4) \leq 16/125 = 0.128$ [13] was superseded later by 0.112937 [11] and more recently improved to 0.100062 [17]. Upper bounds on $m_1(\mathbb{R}^n)$ are now known up to $n = 24$; lower bounds seem to be relatively neglected.

Let us return finally to a lower bound, mentioned in [13]:

$$\inf_{d>0} m_{1,d}(\mathbb{R}^2) \geq \left(\frac{1}{\chi_m(\mathbb{R}^2)}\right)^2 \geq \left(\frac{1}{7}\right)^2 = \frac{1}{49}$$

and proved in [9]. The gap between $1/49 \approx 0.02$ and ≈ 0.14 deserves to be bridged! We are hopeful that someone will accept this challenge.

0.1. Addendum. Let I be a Lebesgue surface measurable subset of the unit sphere in $\mathbb{R}^3$ with the property that no two vectors in I are orthogonal. Let α denote the largest possible area of such sets I, normalized by 4π. It is known [18] that $0.2928 < \alpha < 0.313$ and the upper bound is (again) the outcome of linear programming techniques.

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