Dirichlet Integral

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Consider the class of complex analytic functions f on the open unit disk Δ with $f(0) = 0$ and finite Dirichlet integral:

$$D(f) = \frac{1}{\pi} \int\limits_{\Delta} |f'(z)|^2 dx\, dy < \infty.$$

Clearly $\pi\, D(f)$ is the area of the region $f(\Delta)$ in $\mathbb{C}$, counting multiplicities [1].

Chang & Marshall [2, 3, 4] proved that there exists a constant $C > 0$ such that $D(f) \leq 1$ implies

$$\frac{1}{2\pi} \int\limits_{0}^{2\pi} \exp\left(|f(e^{i\theta})|^2\right) d\theta \leq C.$$

Andreev & Matheson [5, 6, 7] conjectured that the best constant C is $e = 2.7182818284...$, corresponding to the identity function $f(z) = z$. The mere existence of an extremal function, however, remains open [8]. Interestingly, extremal functions provably exist for the closely-related Trudinger-Moser inequality [9].

In the following, we distinguish the unit disk Δ in z-space from the unit disk in w-space (where $w = f(z)$) by writing $\tilde{\Delta}$ for the latter. Define, for $s > 0$,

$$\Omega(s) = \{z \in \Delta : |f(z)| < s\}$$

and let

$$A(s) = \int\limits_{\Omega(s)} |f'(z)|^2 dx\, dy.$$

Obviously $\Omega(\infty) = \Delta$ and $A(\infty) = \pi\, D(f)$. Marshall [3] asked whether there exists a constant $r > 0$ such that, for any $s > 0$, $A(s) \leq \pi\, s^2$ implies $f(r\,\Delta) \subseteq s\,\tilde{\Delta}$. In words, the constant r is so small that, for any radius s, if

$$\begin{pmatrix} \text{the area of the portion} \\ \text{of } f(\Delta) \text{ lying within } s\,\tilde{\Delta} \end{pmatrix} \text{ is strictly less than } \left(\text{the area of } s\,\tilde{\Delta}\right),$$

then f must map $r\,\Delta$ into $s\,\tilde{\Delta}$ itself.

Poggi-Corradini [10] demonstrated that r exists. Solynin [11] further proved that the best constant r is at least $r_0 = 0.03949...$. In fact, r_0 is best possible for the larger class of analytic functions f that omit two values of a doubly-sheeted Riemann surface corresponding to $z \mapsto \sqrt{z}$. It is given exactly by

$$r_0 = \frac{L\left(\sqrt{\sqrt{2}-1}\right) - K\left(\sqrt{\sqrt{2}-1}\right)}{L\left(\sqrt{\sqrt{2}-1}\right) + K\left(\sqrt{\sqrt{2}-1}\right)} = 0.0394929227...$$

where

$$K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - x^2 \sin(\theta)^2}}\, d\theta, \qquad L(x) = K\left(\sqrt{1-x^2}\right)$$

denote the complete elliptic integrals of the first kind. Unfortunately r_0 is not sharp for Marshall's original class of analytic functions: identifying r here remains open, as is the problem of describing extremal functions.

Marshall [3] pointed out that, if f is univalent, then the associated best value of r is at least $1/16 = 0.0625$. Solynin [11] indicated that the sharp r here is exactly $3 - 2\sqrt{2} = 0.1715728752...$, corresponding to rotations of the Koebe function $f(z) = z/(1-z)^2$.

References

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