Two-Colorings of Positive Integers

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Let $f : \{1, 2, 3, \ldots\} \to \{-1, 1\}$ be an arbitrary function. Given a threshold $M > 0$, we ask two questions:

- Do there exist integers $a > 0$, $b \geq 0$, $\ell > 0$ such that

$$|f(a+b) + f(2a+b) + f(3a+b) + \cdots + f(\ell\, a+b)| > M?$$

- Do there exist integers $a > 0$, $\ell > 0$ such that

$$|f(a) + f(2a) + f(3a) + \cdots + f(\ell\, a)| > M?$$

The answer to the first question is yes. In words, every two-coloring of the positive integers has unbounded discrepancy, taken over the family of arithmetic progressions. Restricting attention to the subset $\{1, 2, 3, \ldots, n\}$, we have [1, 2, 3, 4]

$$c\, n^{1/4} \leq P(n) = \min_{f} \max_{\substack{a,b,\ell \\ \ell\, a+b \leq n}} \left| \sum_{k=1}^{\ell} f(k\, a+b) \right| \leq C\, n^{1/4}$$

for all n, with constants $c \geq 1/20$ and $C < \infty$. The lower bound on c is improved to $1/7$ in [5]; no finite upper bound on C is apparently known. It is natural to wonder about the numerical values of

$$\liminf_{n\to\infty} n^{-1/4} P(n), \qquad \limsup_{n\to\infty} n^{-1/4} P(n).$$

The second question, due to Erdös [6, 7, 8] and Chudakov [9, 10], remains open. It is remarkable that, upon mere constraint to homogeneity ($b = 0$), the problem becomes unsolved! If we expand the family under consideration, more can be said. For almost all real numbers $\alpha \geq 1$, there exists $\ell > 0$ such that [11, 12, 13]

$$|f(\lfloor \alpha \rfloor) + f(\lfloor 2\alpha \rfloor) + f(\lfloor 3\alpha \rfloor) + \cdots + f(\lfloor \ell\, \alpha \rfloor)| > M.$$

Such **quasi-arithmetic progressions** collapse to homogeneous arithmetic progressions when α is an integer. Even though the set S of counterexamples α has measure zero, it is not known whether S avoids all integers.

We also examine the expression [14, 15]

$$Q(n)=\min_{f}\max_{\substack{a,b,\ell\\ a<b\\ \ell+b\leq n}}\left|\sum_{k=1}^{\ell}f(k+a)f(k+b)\right|$$

and wonder about the numerical values of

$$\liminf_{n\to\infty}n^{-1/2}Q(n),\qquad\limsup_{n\to\infty}n^{-1/2}Q(n).$$

0.1. Addendum. If f is random (independently taking the values ±1 with probability $1/2$ at each integer $1\leq k\leq n$), then an asymptotic statement can be made about the mean: [16]

$$\mathrm{E}\left(\left|f(1)+f(2)+f(3)+\cdots+f(n)\right|\right)\sim\sqrt{\frac{2\,n}{\pi}}$$

as $n\to\infty$, and likewise

$$\mathrm{E}\left(\left|f(1)f(1+b)+f(2)f(2+b)+f(3)f(3+b)+\cdots+f(n-b)f(n)\right|\right)\sim\sqrt{\frac{2\,n}{\pi}}$$

for fixed $b\geq1$. A proof of the latter can be based on [17]; the order of $Q(n)$ is $n^{1/2}$ (in agreement) whereas the order of $P(n)$ is only $n^{1/4}$ (in disagreement).

From [18, 19], we learned of the Polymath wiki – which documents massively collaborative online mathematical projects – and which includes work on problems given here [20].

Nikolov & Talwar [21], building on Alon & Kalai [22], have shown that the following statement is true for infinitely many positive integers n. There is a set $W\subseteq\{1,\ldots,n\}$ of square-free integers such that, for any $f:W\to\{-1,1\}$, there exists a positive integer a so that

$$\left|\sum_{w\in W,\,a|w}f(w)\right|=n^{1/O(\ln(\ln(n)))}$$

as $n\to\infty$. (If we were permitted to define $f=0$ outside of W, then the Erdös-Chudakov problem would be solved. The values of f, however, are restricted to ±1, disallowing such a construction.)

Konev & Lisitsa [23, 24], assisted by computer, exhibited a length 1160 sequence whose discrepancy is bounded by $M = 2$, but proved that such cannot be true for any sequence of length ≥ 1161. Hence the Erdös-Chudakov conjecture (for infinite sequences) is true for $M = 2$. They also exhibited a length 13000 sequence whose discrepancy is bounded by $M = 3$. Gowers' survey [25] contains additional discussion.

Tao [26] evidently has the final word on this subject: every two-coloring of the positive integers has unbounded discrepancy, taken over the family of homogeneous arithmetic progressions. The existence of near-counterexamples (four are given in [26]) serve to isolate the key difficulty of the problem. The argument also applies to functions f taking values in the unit sphere of a real or complex Hilbert space.

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