Fraenkel Asymmetry

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For simplicity, we restrict attention to subregions of the plane. Let $\Omega \subseteq \mathbb{R}^2$ be the closure of a bounded, open, connected set of area $|\Omega|$ with piecewise continuously differentiable boundary and perimeter p. The classical isoperimetric inequality:

$$p(\Omega) \geq (4\pi\,|\Omega|)^{1/2} \qquad \text{with equality iff } \Omega \text{ is a disk}$$

can be expressed as

$$\delta(\Omega) \geq 0 \qquad \text{with equality iff } \Omega \text{ is a disk}$$

where the **isoperimetric deficit** is

$$\delta(\Omega) = \frac{p(\Omega)}{(4\pi\,|\Omega|)^{1/2}} - 1.$$

We wish to refine $\delta(\Omega) \geq 0$ so that the right-hand side vanishes only on disks and measures to what degree Ω deviates from a disk. Out of many possible choices, we examine **Fraenkel asymmetry** [1, 2, 3]

$$\alpha(\Omega) = \inf\left\{ \frac{|(\Omega \setminus D) \cup (D \setminus \Omega)|}{|\Omega|} : D \text{ a disk with } |D| = |\Omega| \right\}.$$

Note the symmetric difference of sets in the numerator (some authors employ $|\Omega \setminus D|$ instead, hence their results are off by a factor of 2). Before understanding best constants for the inequality $\delta(\Omega) \geq c\,\alpha(\Omega)^2$, that is, extreme values of the ratio $\delta(\Omega)/\alpha(\Omega)^2$, let us first examine $\alpha(\Omega)$ for several polygonal regions.

The Fraenkel asymmetry of a regular hexagon (side length 1) is

$$\frac{1}{3\sqrt{3}/2} \cdot 12 \int\limits_{\sqrt{3}/2}^{\sqrt{3\sqrt{3}/(2\pi)}} \sqrt{\frac{3\sqrt{3}}{2\pi} - x^2}\, dx$$

$$= \frac{-9\sqrt{\left(2\sqrt{3}-\pi\right)\pi} + 18\sqrt{3}\arccos\left(\sqrt{\pi/\left(2\sqrt{3}\right)}\right)}{\left(3\sqrt{3}/2\right)\pi} = 0.0744657545...$$

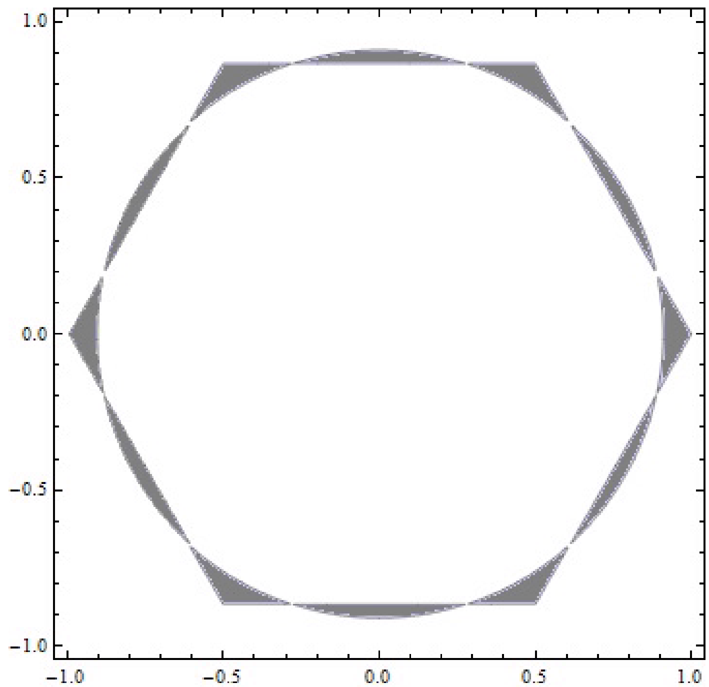

Figure 1: Symmetric difference between regular hexagon and Fraenkel disk.

which is quite close to zero (Figure 1). The square has greater asymmetry

$$\begin{aligned} & 16\int_{1/2}^{1/\sqrt{\pi}}\sqrt{\frac{1}{\pi}-x^2}\,dx \\ = \; & 4-\frac{2\sqrt{(4-\pi)\,\pi}+8\arcsin\left(\sqrt{\pi}/2\right)}{\pi} = 0.1810919376... \end{aligned}$$

and the equilateral triangle has still greater asymmetry

$$\frac{1}{\sqrt{3}/4}\cdot 12\int_{0}^{1/4-\sqrt{3\pi(3\sqrt{3}-\pi)}/(12\pi)}\left(\left(\frac{1}{\sqrt{3}}-\sqrt{3}x\right)-\sqrt{\frac{\sqrt{3}}{4\pi}-x^2}\right)dx = 0.3649426110...$$

(omitting the exact expression, which is complicated).

Let $\ell \geq 2/\sqrt{\pi}$. If Ω is the rectangle with vertices $(\pm\ell/2, \pm 1/(2\ell))$, clearly $|\Omega| = 1$ and

$$\alpha(\Omega) = -\frac{1}{\ell^2}\sqrt{\frac{4\ell^2-\pi}{\pi}}+\frac{4}{\pi}\arcsin\left(\sqrt{\frac{4\ell^2-\pi}{4\ell^2}}\right) \to 2$$

as $\ell \to \infty$. Fraenkel asymmetry can never exceed 2; from

$$p(\Omega) = 2\left(\ell + \frac{1}{\ell}\right) \sim 2\ell$$

we deduce

$$\alpha(\Omega) \sim 2 - \frac{8}{\sqrt{\pi}}\frac{1}{p} + \frac{4\sqrt{\pi}}{3}\frac{1}{p^3}.$$

This example is inefficient (in terms of perimeter) by comparison with the following.

Let $0 < \theta \leq \arctan(\pi/4)$ and

$$f(\theta) = \frac{\sqrt{\pi}}{4}\frac{\cos(\theta)^2}{\sin(\theta)}, \qquad g(\theta) = \frac{1}{\sqrt{\pi}}\sin(\theta).$$

Consider the rectangle with vertices $(\pm f(\theta), \pm g(\theta))$, capped on the right and left by semicircles. The equation of the boundary in the first quadrant only is

$$y = \begin{cases} g(\theta) & \text{if } 0 \leq x \leq f(\theta), \\ \sqrt{g(\theta)^2 - (x - f(\theta))^2} & \text{if } f(\theta) < x \leq f(\theta) + g(\theta). \end{cases}$$

The region Ω' in Figure 2, called a **biscuit**, satisfies $|\Omega'| = 1$ and [4, 5]

$$\alpha(\Omega') = \frac{2}{\pi}\left(\pi - 2\theta - 2\sin(\theta)\cos(\theta)\right) \to 2$$

as $\theta \to 0^+$. From

$$p(\Omega') = \sqrt{\pi}\,\frac{1 + \sin(\theta)^2}{\sin(\theta)} \sim \frac{\sqrt{\pi}}{\theta}$$

we deduce

$$\alpha(\Omega') \sim 2 - \frac{8}{\sqrt{\pi}}\frac{1}{p} + \frac{8\sqrt{\pi}}{3}\frac{1}{p^3}.$$

The third term when expanding $\alpha(\Omega')$ is greater than that for $\alpha(\Omega)$. These asymptotics are consistent with a theorem that, among all *convex* sets Ω of unit area and fixed perimeter

$$p \geq p_0 = \frac{2}{\sqrt{\pi}}\frac{\pi^2 + 8}{\sqrt{\pi^2 + 16}} = 3.9643784229...,$$

the biscuit maximizes α. Write $E_p = \Omega'$ for convenience. Since $\delta(\Omega) = p(4\pi)^{-1/2} - 1$ is fixed, E_p coincides with the solution of a restricted version of the earlier optimization problem.

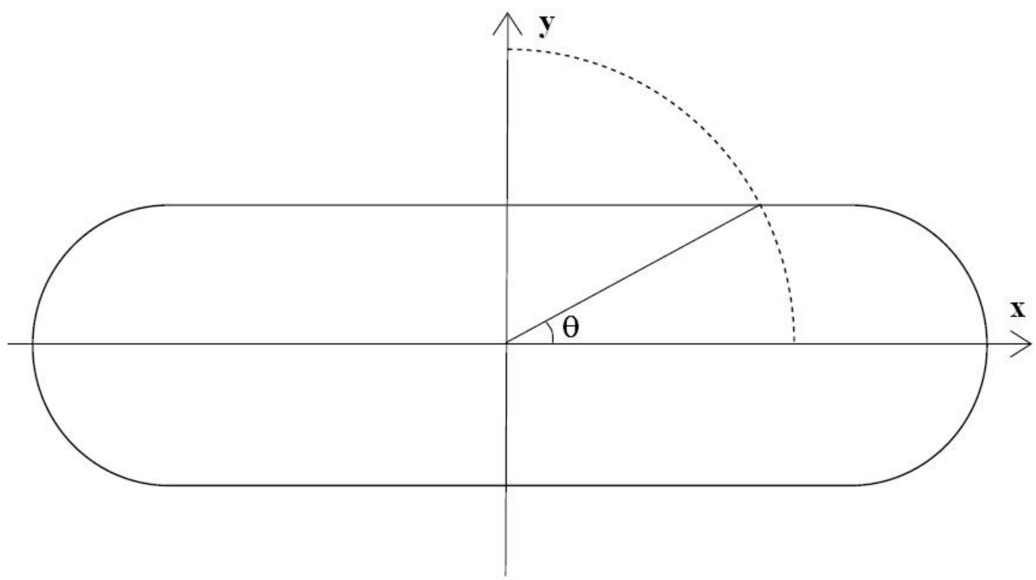

Figure 2: For a biscuit (or stadium or racetrack) of unit area, θ is the angle determined by the intersection between its boundary and the circle with common center, radius $1/\sqrt{\pi}$.

If $2\sqrt{\pi} < p < p_0$, then the maximizing convex set E_p is called an **oval** whose boundary consists of four symmetrically placed circular arcs. We omit all details except to remark that $\arctan(\pi/4) < \theta < \pi/4$ for these. Also of interest is [5, 6, 7]

$$\min_{p>2\sqrt{\pi}} \frac{\delta(E_p)}{\alpha(E_p)^2} = 0.4055851970... = \frac{1}{4}(1.6223407880...)$$

which is achieved for a specific biscuit. Allowing non-convex sets to enter the discussion,

$$\frac{\delta(E_{\mathrm{nc}})}{\alpha(E_{\mathrm{nc}})^2} \approx 0.39314$$

is achieved by a certain set, called a **mask**, whose boundary involves eight circular arcs. Proof of this latter new assertion has not yet appeared.

Finally, we turn to an older topic: the calculation of maximal coefficients c_k in the asymptotic estimate

$$\delta(\Omega) \geq \sum_{k=1}^{m} c_k \alpha(\Omega)^k + o\left(\alpha(\Omega)^m\right)$$

for arbitrary Ω. The fact that $c_k = 0$ for odd k and [8, 9, 10]

$$c_2 = \frac{\pi}{8(4-\pi)} = 0.4574740457... = \frac{1}{4}(1.8298961831...)$$

has been known since the 1990s; the fact that [6]

$$c_4 = -\frac{\pi^3(3\pi-14)(5\pi-16)}{96(4-\pi)^4(\pi-2)} = -0.6962146734...,$$

$$\begin{aligned} c_6 &= \frac{\pi^5(-759808+1619648\pi-1386576\pi^2+612992\pi^3-148024\pi^4+18552\pi^5-945\pi^5)}{2880(4-\pi)^7(\pi-2)^4} \\ &= -1.7607874382... \end{aligned}$$

was found only in 2013. Verification makes use of a sequence of ovals converging to the disk ($\theta \to (\pi/4)^-$).

We witnessed two measures of asymmetry (in a different context) in [11]; Reuleaux polygons are mentioned in [12]. Yet another measure – *Hausdorff asymmetry* – is found in [13].

0.1. Geometric Uncertainty Principle. For the following, an assumption of finite perimeter is not needed, thus hypotheses may be weakened. Let $\Omega \subseteq \mathbb{R}^2$ be an open bounded region with a given decomposition

$$\Omega = \bigcup_{j=1}^{N} \Omega_j$$

into disjoint Lebesgue measurable sets Ω_j. Define the j^{th} **area deviation**

$$\sigma(\Omega_j) = \frac{|\Omega_j| - \min_{1\leq i\leq N} |\Omega_i|}{|\Omega_j|}$$

which satisfies $0 \leq \sigma(\Omega_j) \leq 1$ and, like $\alpha(\Omega_j)$, is scale-invariant. Steinerberger [14] proved the remarkable existence of a universal constant $\kappa > 0$ such that, for sufficiently large N depending only on Ω, the sum

$$\left(\sum_{j=1}^{N} \frac{|\Omega_j|}{|\Omega|}\alpha(\Omega_j)\right) + \left(\sum_{j=1}^{N} \frac{|\Omega_j|}{|\Omega|}\sigma(\Omega_j)\right) \geq \kappa.$$

It is known that κ is at least $1/60000$ and conjectured that $\kappa = 0.0744657545...$, which corresponds to the regular hexagonal tiling of the plane. Another candidate tiling of the plane – Kepler's circle packing with exactly one adjacent **hourglass** per disk (Figure 3) – gives a considerably larger sum.

0.2. Bisecting Chords. As an aside, given a planar measurable convex set Ω, a **bisecting chord** is a line segment whose endpoints lie on the boundary of Ω and which partitions Ω into two subsets of equal area. For example, a disk D of radius $1/2$ possesses infinitely many bisecting chords, all of length 1. The area of such a disk is $\pi/4 = 0.7853981633....$ For most sets Ω, we expect bisecting chord lengths to vary. Suppose Ω has the property that its maximum bisecting chord length is 1. How small can the area of such a set Ω be? Is D the area-minimizing set Ω?

The answer to the second question is no. Define the **Auerbach triangle** Δ (or rounded triangle) to consist of six parts, three linear and three nonlinear, with the topmost part (the dashed curve in Figure 4) given parametrically by [15, 16, 17]

$$x(t) = \frac{e^{4t}-1}{e^{4t}+1} - t, \qquad y(t) = 2\frac{e^{2t}}{e^{4t}+1}, \qquad -\frac{\ln(3)}{4} \leq t \leq \frac{\ln(3)}{4}.$$

Then Δ satisfies the required property, but its area is

$$\frac{\sqrt{3}}{8}\left(8\ln(3) - \ln(3)^2 - 4\right) = 0.7755147827... = \frac{1}{4}(3.1020591308...) < \frac{\pi}{4}.$$

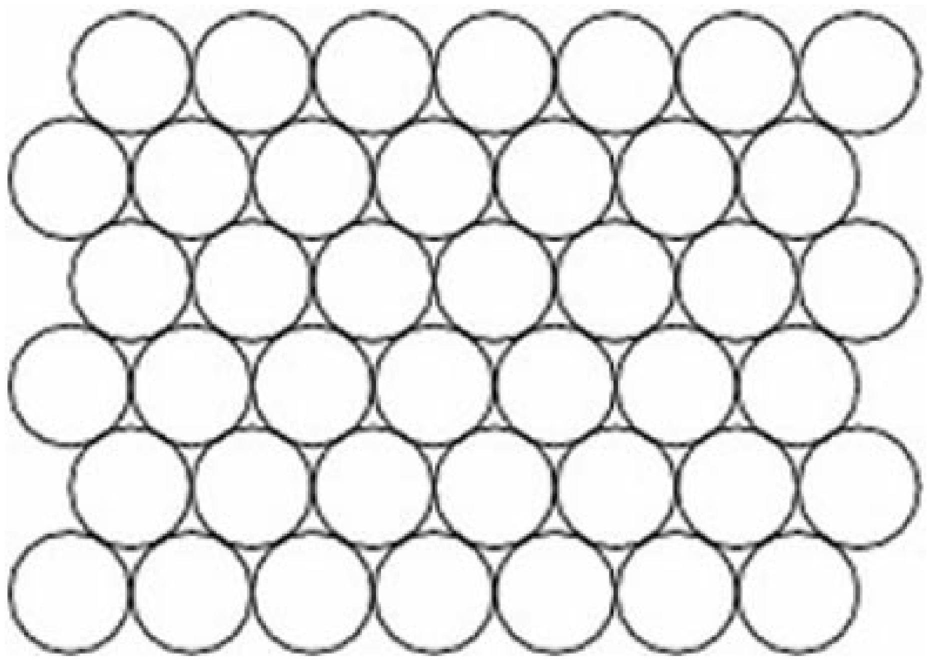

Figure 3: Tiling of the plane using disks and hourglasses in equal proportion.

This numerical value is the answer to the first question. A third question is: How large can the perimeter of such a set Ω be? Note that the perimeter of Δ is $3\ln(3) = 3.2958368660... > \pi$ and Δ evidently is the perimeter-maximizing set Ω as well. Related materials include [18, 19, 20, 22, 21, 23].

0.3. Addendum. Let Ω be the ellipse $x^2/\ell^2 + \ell^2 y^2 \leq 1/\pi$ and Ω' be the rhombus with vertices $(\pm\ell, 0)$, $(0, \pm 1/(2\ell))$. Clearly $|\Omega| = |\Omega'| = 1$ and

$$\alpha(\Omega) = \frac{4}{\pi}\left[\arcsin\left(\frac{\ell}{\sqrt{1+\ell^2}}\right) - \arcsin\left(\frac{1}{\sqrt{1+\ell^2}}\right)\right],$$

$$\alpha(\Omega') = 8\int_0^{\xi}\left[\sqrt{\frac{1}{\pi} - x^2} - \frac{1}{2\ell^2}(\ell - x)\right] dx$$

where

$$\xi = \frac{\ell}{1+4\ell^4} + \frac{2\ell^2\sqrt{1+(4\ell^2-\pi)\ell^2}}{(1+4\ell^4)\sqrt{\pi}}$$

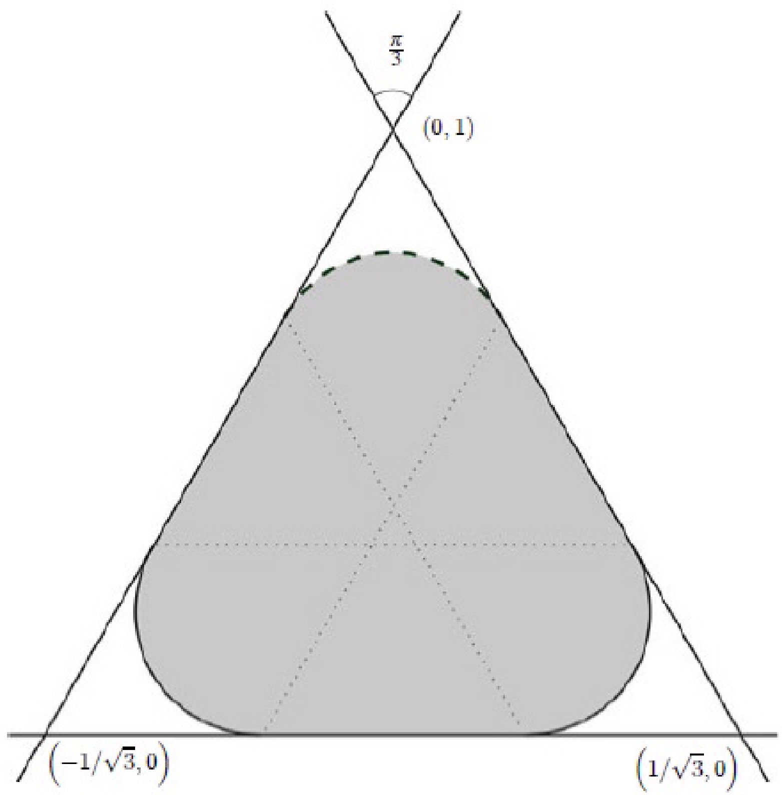

Figure 4: Auerbach triangle with unit bisecting (halving) chords.

(the exact expression for $\alpha(\Omega')$ is complicated). From

$$p(\Omega) = \frac{4\ell}{\sqrt{\pi}} \int_0^{\pi/2} \sqrt{1 - \left(1 - \frac{1}{\ell^4}\right) \cos(\theta)^2}\, d\theta \sim \frac{4\ell}{\sqrt{\pi}}$$

(an elliptic integral of the second kind) and

$$p(\Omega') = 4\sqrt{\ell^2 + \frac{1}{4\ell^2}} \sim 4\ell$$

we deduce that, as $\ell \to \infty$,

$$\alpha(\Omega) \sim 2 - \frac{32}{\pi^{3/2}} \frac{1}{p}, \qquad \alpha(\Omega') \sim 2 - \frac{16}{\sqrt{\pi}} \frac{1}{p}$$

which again are inefficient by comparison with a biscuit. More computations of Fraenkel asymmetry are found in [24], related to the study of various *triangle centers* [25, 26].

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