Fermat Numbers and Elite Primes

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The Fermat numbers $F_n = 2^{2^n} + 1$ satisfy a quadratic recurrence [1]

$$F_{n+1} = (F_n - 1)^2 + 1, \qquad n \geq 0$$

and are pairwise coprime. It is conjectured that F_n are always square-free and that, beyond F_4, they are never prime. The latter would imply that there are exactly 31 regular polygons with an odd number G_m of sides that can be constructed by straightedge and compass [2]. The values G_1, G_2, ..., G_{31} encompass all divisors of $2^{32}-1$ except unity [3]. Let $G_0 = 1$. If we scan each row of Pascal's triangle modulo 2 as a binary integer, then the numbers G_m (listed in ascending order) are naturally extended without bound. The reciprocal sum [4]

$$\sum_{m=0}^{\infty} \frac{1}{G_m} = \prod_{n=0}^{\infty} \left(1 + \frac{1}{F_n}\right) = 1.7007354952...$$

is irrational [2]; in contrast,

$$\sum_{m=0}^{\infty} \frac{(-1)^{t_m}}{G_m} = \frac{1}{2}$$

is rational, where $\{t_m\}$ is the Thue-Morse sequence $\{0, 1, 1, 0, 1, 0, 0, 1, 1, 0, ...\}$ [5]. Golomb [6] proved that

$$\sum_{n=0}^{\infty} \frac{1}{F_n} = 0.5960631721...$$

is irrational and Duverney [7] proved that it is transcendental; there is evidence that Mahler possessed these results far earlier [8].

Let P denote the set of all primes p for which there exists n such that p divides F_n. Křížek, Luca & Somer [9] proved that

$$\sum_{p \in P} \frac{1}{p} = 0.5976404758...$$

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is convergent, answering a question raised in [10]. The series $\sum_{d\in D} 1/d$ likewise converges, where D is the set of all divisors $d > 1$ (prime or composite) for which there exists n such that d divides F_n. The smallest element of D not in P is F_5 itself [11, 12].

A prime p is called **elite** [13] if there exists m for which all F_n with $n > m$ are quadratic non-residues of p, that is, the equation

$$x^2 \equiv F_n \bmod p$$

has no solutions x for $n > m$. Let E denote the (infinite?) set of all elite primes. The series [14, 15, 16, 17]

$$\sum_{p\in E} \frac{1}{p} = 0.7007640115...$$

is convergent [9]. This numerical evaluation, as well as that for the series over $p \in P$, is non-rigorous. For our calculation over $p \in E$ to be valid, we would need

$$\#\left\{p \in E : p \leq q\right\} = O(\ln(q))$$

as $q \to \infty$; the best current bound is $O\left(q/\ln(q)^2\right)$, hence improvement in our knowledge of E will be required. Generalization to the numbers $F_{b,n} = b^{2^n} + 1$, for fixed integer $b \geq 2$, is found in [18].

We conclude with the fact that

$$\sum_{n=0}^{\infty} \frac{1}{2^{2^n}} = 0.8164215090...$$

is transcendental, proved by Kempner [19] and revisited in [20].

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