Radii in Geometric Function Theory

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First, we talk about geometry. A region $R \subseteq \mathbb{C}$ is **convex** if, for any two points $p, q \in R$, the line segment $pq \subseteq R$. A region $R \subseteq \mathbb{C}$ is **starlike** with respect to the origin if $0 \in R$ and if, for any point $p \in R$, the line segment $0p \subseteq R$.

Next, we talk about functions. A complex analytic function f defined on an open region is **univalent** (or **schlicht**) if f is one-to-one; that is, $f(z) = f(w)$ if and only if $z = w$. Let

$$D = \{z : |z| < 1\} \qquad \text{(the open disk of radius 1)},$$

$$E = \{z : 0 < |z| < 1\} \qquad \text{(the open punctured disk)},$$

$$S = \left\{ \text{univalent } f \text{ on } D \text{ with } f(z) = z + \sum_{n=2}^{\infty} a_n z^n \right\},$$

$$\Sigma = \left\{ \text{univalent } f \text{ on } E \text{ with } f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \right\}.$$

Geometry and functions now come together. The various subclasses of S include

$$\begin{aligned} CV &= \{f \in S : f(D) \text{ is convex}\} \\ &= \left\{ f \in S : \operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0 \text{ for all } z \in D \right\}, \end{aligned}$$

the class of convex functions on D, and

$$\begin{aligned} ST &= \{f \in S : f(D) \text{ is starlike with respect to } 0\} \\ &= \left\{ f \in S : \operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right) > 0 \text{ for all } z \in D \right\}, \end{aligned}$$

the class of starlike functions on D. We will mostly discuss S (the analytic case), but will mention Σ (the meromorphic case) occasionally in the following [1, 2, 3, 4, 5].

0.1. Radius of Convexity. Define $D_r = \{z : |z| < r\}$, the open disk of radius r, for each $r > 0$. For each $f \in S$, let $r(f)$ be the supremum of all numbers r such that $f(D_r)$ is convex. The **radius of convexity** for S is [1]

$$\rho_{cv}(S) = \inf_{f \in S} r(f) = 2 - \sqrt{3} = 0.2679491924...$$

and is achieved by the Koebe function $f(z) = z(1-z)^{-2}$. This fact was first proved by Nevanlinna [6]. Generalization of ρ_{cv} to any subclass of S gives rise to some interesting optimization problems. Trivially we have

$$\rho_{cv}(CV) = 1, \qquad \rho_{cv}(ST) = 2 - \sqrt{3}$$

(the latter follows since the Koebe function is starlike). Define, however, the special class of starlike functions of order α:

$$S^*_\alpha = \left\{ f \in S : \operatorname{Re}\left(z\frac{f'(z)}{f(z)} \right) > \alpha \text{ for all } z \in D \right\}.$$

Zmorovic [7], extending work in [8, 9, 10], proved that

$$\rho_{cv}(S^*_\alpha) = \begin{cases} \dfrac{1}{2 - 3\alpha + \sqrt{(1-\alpha)(3-5\alpha)}} & \text{if } 0 \le \alpha < \alpha_0, \\ \left(\dfrac{5\alpha - 1}{4\alpha^2 - \alpha + 1 + 4\alpha\sqrt{\alpha^2 - 3\alpha + 2}} \right)^{\frac{1}{2}} & \text{if } \alpha_0 \le \alpha < 1, \end{cases}$$

where $\alpha_0 = 0.3349596751...$ is the smallest positive zero of $20\alpha^4 - 52\alpha^3 + 15\alpha^2 + 12\alpha - 4$. Note that $\rho_{cv}(S^*_0) = 2 - \sqrt{3}$, as expected.

We turn attention to the class Σ. Define $E_r = \{z : 0 < |z| < r\}$ and, for $f \in \Sigma$, let $r(f)$ be the supremum of all numbers r such that the complement of $f(E_r)$ in $\mathbb{C}$ is convex. Goluzin [5, 11] proved that

$$\rho_{cv}(\Sigma) = \inf_{f \in \Sigma} r(f) = x = 0.5600798519...$$

where x is the unique positive solution of the equation

$$\frac{E(x)}{K(x)} + \frac{x^2}{8} - \frac{7}{8} = 0$$

and $K(x)$, $E(x)$ are complete elliptic integrals of the first and second kind [12]. Letting

$$\Sigma^*_\beta = \left\{ f \in \Sigma : \operatorname{Re}\left(z\frac{f'(z)}{f(z)} \right) < -\beta \text{ for all } z \in E \right\},$$

we also have [7, 9, 11, 13, 14]

$$\rho_{cv}(\Sigma_\beta^*) = \begin{cases} \left(\dfrac{4\beta - 5 + 4\sqrt{\beta^2 - \beta + 1}}{8\beta - 3}\right)^{\frac{1}{2}} & \text{if } 0 \le \beta < \beta_0, \\ \dfrac{1}{\beta + \sqrt{(1-\beta)(3\beta - 1)}} & \text{if } \beta_0 \le \beta < 1, \end{cases}$$

where $\beta_0 = 0.8673407553...$ is the largest positive zero of $12\beta^4 - 28\beta^3 + 33\beta^2 - 20\beta + 4$. Note here that $\rho_{cv}(\Sigma_0^*) = 1/\sqrt{3} = 0.577... > 0.560... = x$. In this case, the extremal function is not starlike, which accounts for the strict inequality.

0.2. Radius of Starlikeness. For each $f \in S$, let $r(f)$ be the supremum of all numbers r such that $f(D_r)$ is starlike with respect to the origin. The **radius of starlikeness** for S is [1]

$$\rho_{st}(S) = \inf_{f \in S} r(f) = \frac{1 - e^{-\pi/2}}{1 + e^{-\pi/2}} = \tanh\left(\frac{\pi}{4}\right) = 0.6557942026...$$

and this fact was first discovered by Grunsky [15].

Goluzin [5, 16] found several interesting generalizations. Define a region $R \subseteq \mathbb{C}$ to be n**-starlike** with respect to the origin if $0 \in R$ and if every point of R can be connected with 0 by a piecewise linear curve that lies entirely in R and that consists of no more than n line segments. Let δ_n be the supremum of all r such that an arbitrary $f \in S$ maps D_r onto an n-starlike region with respect to 0. Then

$$\tanh\left(\frac{\pi}{4}\right) = \delta_1 \le \delta_2 \le \delta_3 \le \cdots, \qquad \delta_n \ge \tanh\left(\frac{n\pi}{4}\right),$$

but values for δ_n, $n \ge 2$, are unknown. See also [17, 18].

Likewise, let ϵ_n be the supremum of all r such that an arbitrary $f \in \Sigma$ maps E_r onto a region, the complement of which is n-starlike with respect to 0. Then

$$0.85 < \epsilon_1, \qquad 1 - 1.11 \exp\left(\frac{-n\pi}{2}\right) < \epsilon_n \qquad \text{for all } n > 1.$$

An exact expression for ϵ_1 would be good to see someday.

0.3. Radius of Close-to-Convexity. A region $R \subseteq \mathbb{C}$ is **close-to-convex** (or **linearly accessible**) if its complement is a union of closed half-lines such that the corresponding open half-lines are pairwise disjoint. Any starlike region is close-to-convex. A half-annulus is also close-to-convex, but this property fails for any larger subsection of an annulus.

An analytic function $f : D \to \mathbb{C}$ is close-to-convex if $f(D)$ is close-to-convex. Equivalently, f is close-to-convex if there is a convex function $g : D \to \mathbb{C}$ such that $\mathrm{Re}(f'(z)/g'(z)) > 0$ for all $z \in D$ [1, 19, 20, 21, 22, 23, 24, 25]. It can be shown that every close-to-convex function is univalent.

Define

$$\begin{aligned} CC &= \{f \in S : f(D) \text{ is close-to-convex}\} \\ &= \left\{ f \in S : \begin{array}{l} \displaystyle\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) d\theta > -\pi, \text{ where } z = re^{i\theta}, \\ \text{for each } 0 < r < 1 \text{ and each pair } 0 < \theta_1 < \theta_2 < 2\pi \end{array} \right\}. \end{aligned}$$

Let $\rho_{cc}(S)$ be the supremum of all r such that an arbitrary $f \in S$ maps D_r onto a close-to-convex region. Krzyz [26] determined that

$$\rho_{cc}(S) = y = 0.8098139153...$$

where y is the unique real solution of the equation

$$2\arctan\left(\frac{\kappa(y)}{\lambda(y)}\right) + \ln\left(1 + \lambda(y)^2\right) - 2\ln\left(\frac{2y}{1 - y^2}\right) = 0$$

in the interval $0 < y < 1$, $\kappa(y) = (1 + y^2)/(1 - y^2)$, and $\lambda = \lambda(y)$ is the unique real solution of the equation

$$\lambda^3 - \kappa(y)\lambda^2 + \kappa(y)^2\lambda - \kappa(y) = 0.$$

Sizuk [27] extended this result to the class of close-to-convex functions of order γ.

0.4. Radius of Convexity in One Direction. A region $R \subseteq \mathbb{C}$ is **convex in the direction of the imaginary axis** if, for every vertical line L, the set $L \cap R$ is either empty or connected. Any region that is convex in one direction can be rotated so that it is convex in the imaginary direction [3, 28, 29].

Define

$$CD = \{f \in S : f(D) \text{ is convex in the imaginary direction}\}$$

and let $\rho_{cd}(S)$ be the supremum of all numbers r such that an arbitrary $f \in S$ maps D_r onto a region that is convex in the imaginary direction. Umezawa [30] and Goodman & Saff [31] proved that

$$0.394... = 4 - \sqrt{13} \leq \rho_{cd}(S) \leq \sqrt{2} - 1 = 0.414....$$

The exact value of this constant is unknown.

A subclass of CD was considered by Hengartner & Schober [32]:

$$\left\{f \in S : \mathrm{Re}\left((1 - z^2)f'(z)\right) \geq 0 \text{ for all } z \in D\right\}$$

but we omit details. See also [33, 34].

0.5. Radius of Majorization. Let $f : D \to \mathbb{C}$ be analytic with $f(0) = 0$ and $f'(0) \geq 0$. Let $F \in S$. The function f is **subordinate** to F, written $f \preceq F$, if $f(D_r) \subseteq F(D_r)$ for all $0 < r < 1$ [1, 35].

Shah [36, 37], verifying conjectures of Goluzin [5, 38], proved that if $f \preceq F$, then

$$|f(z)| \leq |F(z)| \qquad \text{for all } |z| \leq \tfrac{1}{2}(3-\sqrt{5}) = 0.3819660112...,$$

$$|f'(z)| \leq |F'(z)| \qquad \text{for all } |z| \leq 3-2\sqrt{2} = 0.1715728752....$$

Both of these radii are best possible. If we further assume that f is univalent and $f'(0) > 0$, then [5, 39]

$$|f(z)| \leq |F(z)| \qquad \text{for all } |z| \leq u = 0.3908507887...,$$

where u is the unique real solution of

$$\ln\left(\frac{1+u}{1-u}\right) + 2\arctan(u) = \frac{\pi}{2}.$$

Again, this radius of majorization is best possible. Problems as such (subordination implies majorization) were first examined by Biernacki [40].

Converse problems (majorization implies subordination) were studied by Lewandowski [41]. Under the same conditions as earlier, if $|f(z)| \leq |F(z)|$ for all $z \in D$ and f is not necessarily univalent, then $f \preceq F$ in the disk D_v, where $0.21 < v < 0.29$. The exact value of v is unknown. If f is assumed to be univalent, then the constant $u = 0.390...$ arises again [42, 43].

0.6. Radius of Zeroness. Let $\rho_N(\Sigma)$ be the supremum of all numbers r such that an arbitrary $f \in \Sigma$ never vanishes on the punctured disk E_r. Goluzin [16] proved that $0.86 < \rho_N(\Sigma) \leq \sqrt{3}/2 < 0.867$, but a subsequent theorem of his [5, 44] implies that $\rho_N(\Sigma) = \xi = 0.8649789576...$, where ξ is the unique positive solution of the equation

$$\frac{E(\xi)}{K(\xi)} + \frac{\xi^2}{4} - \frac{3}{4} = 0.$$

This is quite similar to the equation prescribed earlier for the radius of convexity $\rho_{cv}(\Sigma)$.

Given an analytic function f, we may likewise define $\rho_N(f)$ to be the supremum of all numbers r such that f, when restricted to E_r, is never zero. For example,

$$\rho_N(f) = 2|z_0| \qquad \text{for } f(z) = z - \frac{1}{2z_0}z^2 \ \text{ (a quadratic function)}$$

and

$$\rho_N(f) = 2\pi \qquad \text{for } f(z) = \exp(z) - 1 \ \text{ (the exponential function)}.$$

0.7. Radius of Univalence. Given an analytic function f, define the **radius of univalence** of f to be the supremum of all numbers r such that f, when restricted to the disk D_r, is univalent. Let us first consider the case of polynomials. We clearly have

$$\rho_s(f) = |z_0| \qquad \text{for } f(z) = z - \frac{1}{2z_0}z^2$$

in the quadratic case. Kakeya's theorem [45, 46, 47] provides that

$$\sin\left(\frac{\pi}{n}\right) \leq \frac{\rho_s(f)}{|z_0|} \leq 1 \qquad \text{for } f(z) = z + \sum_{k=2}^{n} a_k z^k$$

in the general case, where $n \geq 2$ and $z_0 \neq 0$ is the zero of $f'(z)$ of smallest modulus. These bounds are sharp.

Now, let us consider the case of transcendental functions. We have

$$\rho_s(f) = \pi \qquad \text{for } f(z) = \exp(z) - 1,$$

as is well-known (although $f'(z)$ never vanishes); [48]

$$\rho_s(f) = 1.5748375891... \qquad \text{for } f(z) = \operatorname{erf}(z),$$

corresponding to the smallest modulus, of points z not on the x-axis, for which $\operatorname{erf}(z)$ is real (see [49] for definition); [50, 51]

$$\rho_s(f) = 0.9241388730... \qquad \text{for } f(z) = \exp(z^2)\operatorname{erf}(z),$$

corresponding to the unique positive solution of $\sqrt{\pi}y\operatorname{Im}(f(iy)) = 1$; [52, 53, 54]

$$\rho_s(f) = p_{\nu,1} \qquad \text{for } f(z) = z^{1-\nu}J_\nu(z),\ \nu > -1,$$

corresponding to the smallest positive zero of $f'(z)$ (see [55] for numerical values); [56]

$$\rho_s(f) = 0.5040830082... \qquad \text{for } f(z) = 1/\Gamma(z),$$

corresponding to the smallest positive zero of $\Gamma'(-z)$; and [57]

$$\rho_s(f) = 0.4616321449... \qquad \text{for } f(z) = \Gamma(z+1),$$

corresponding to the smallest positive zero of $\Gamma'(z+1)$. See also [58].

We digress briefly to other radii. For $f(z) = \exp(z) - 1$, it is known that [59, 60]

$$\rho_{cv}(f) = 1, \qquad \rho_{st}(f) = 2.8329700604...$$

and the latter corresponds to $\sqrt{1+\eta^2}$, where η is the smallest positive solution of the equation

$$\eta\sin(\eta) + \cos(\eta) = \frac{1}{e}.$$

See also [61, 62].

0.8. Sums and Products. Here are two procedures for combining univalent functions:

$$S+S=\{h:h(z)=tf(z)+(1-t)g(z) \text{ for some } f,g\in S \text{ and } 0\le t\le 1\},$$

$$S\cdot S=\left\{h:h(z)=f(z)^t g(z)^{1-t} \text{ for some } f,g\in S \text{ and } 0\le t\le 1\right\}.$$

On the one hand, MacGregor [63] demonstrated that

$$\rho_s(S+S)=\sin\left(\frac{\pi}{8}\right)=\frac{1}{2}\sqrt{2-\sqrt{2}}=0.3826834323...$$

$$\rho_s(CV+CV)=\frac{\sqrt{2}}{2}=0.7071067811...$$

and Robertson [64] showed that

$$\rho_s(ST+ST)=\chi=0.4035150049...$$

where χ is the unique positive zero of $\chi^6+5\chi^4+79\chi^2-13$. Further results appear in [65, 66, 67]. On the other hand, we have [3]

$$CV\cdot CV\subseteq ST\cdot ST\subseteq ST, \qquad CV\cdot CV\nsubseteq CV$$

but virtually nothing is known about the class $S\cdot S$.

0.9. Derivatives and Integrals. Define the following classes of functions:

$$T=\left\{f:f(z)=\frac{1}{2}\frac{d}{dz}\left(zg(z)\right) \text{ for some } g\in S\right\},$$

$$U_\alpha=\left\{f:f(z)=\int_0^z\left(\frac{g(w)}{w}\right)^\alpha dw \text{ for some } g\in S\right\},$$

$$V_\beta=\left\{f:f(z)=\int_0^z g'(w)^\beta dw \text{ for some } g\in S\right\},$$

where α, β are complex numbers and hence the logarithmic branch is selected so that $f'(0)=1$. Barnard [68, 69] and Pearce [70], building on Robinson [71], proved that

$$0.49<\rho_s(T)\le\tfrac{1}{2}, \qquad 0.435<\rho_{st}(T)<0.445.$$

In particular, these two constants must be distinct.

Biernacki [72] claimed that $\rho_s(U_1) = 1$, but this was disproved by Krzyz & Lewandowski [73]. It was later shown [74] that $0.91 < \rho_s(U_1) \leq \tanh(\pi) < 0.9963$. Let A denote the set of all complex numbers α for which $U_\alpha \subseteq S$. Kim & Merkes [75] proved that $D_{1/4} \subseteq A \subseteq D_{1/2}$; we wonder whether $D_r \subseteq A$ for some $r > 1/4$.

Trivially $\rho_s(V_1) = 1$. Let B denote the set of all complex numbers β for which $V_\beta \subseteq S$. Royster [76] and Pfaltzgraff [77] proved that $D_{1/4} \subseteq B \subseteq D_{1/3} \cup \{1\}$; we again wonder whether $D_r \subseteq B$ for some $r > 1/4$. See also [78, 79].

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