Hardy-Littlewood Maximal Inequalities

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The operators M and N defined here were first introduced by Hardy & Littlewood [1]. These tools are useful in several areas, e.g., harmonic analysis [2], but we disregard the applications entirely and focus rather on properties of M and N in themselves.

0.1. One Dimension, Uncentered. For a locally integrable function $f : \mathbb{R} \to \mathbb{R}$, define

$$(Mf)(x) = \sup_{\substack{a<x\\ b>x}} \frac{1}{b-a} \int_a^b |f(t)|\, dt \ .$$

In the Banach space $L_p(\mathbb{R})$, $1 \leq p < \infty$, with norm

$$||f||_p = \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{\frac{1}{p}},$$

we examine the inequality

$$||Mf||_p \leq c_p \cdot ||f||_p$$

and ask for the best constant c_p. (By "best", we mean that c_p is the smallest positive constant for which the inequality holds for all f.) It is known, for $1 < p < \infty$, that c_p is the unique positive solution of [3]

$$(p-1)x^p - px^{p-1} - 1 = 0 \ ;$$

hence, for example, we have $c_2 = 1 + \sqrt{2}$ and $\lim_{p\to\infty} c_p = 1$.

For $p = 1$, we examine instead the weak type $(1, 1)$ inequality

$$|\{x : (Mf)(x) > \lambda\}| \leq C \cdot \frac{1}{\lambda} \cdot ||f||_1$$

where $|S|$ denotes the Lebesgue measure of a measurable set $S \subseteq \mathbb{R}$ and $\lambda > 0$. In this case, it is comparatively simple to prove that $C = 2$ is the best constant [4], valid for all f and all λ.

0.2. One Dimension, Centered. For a locally integrable function $f : \mathbb{R} \to \mathbb{R}$, define

$$(Nf)(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|\, dt \ .$$

Soria & Carbery [5, 6, 7] conjectured that $C = 3/2$ is the best constant for the weak type $(1,1)$ inequality

$$|\{x : (Nf)(x) > \lambda\}| \leq C \cdot \frac{1}{\lambda} \cdot ||f||_1 \ .$$

Aldaz [8] refuted this conjecture and showed that $37/24 \leq C \leq (9+\sqrt{41})/8$. Further progress was made in [9, 10] before Melas [4] established that

$$C = \frac{11+\sqrt{61}}{12} = 1.5675208063...$$

The impressive proof underlying this formula is far more complicated than the corresponding uncentered result [0.1].

For the strong type (p,p) inequality with $p > 1$, Dror, Ganguli & Strichartz [7] conjectured that the best constant c_p is given by

$$c_p = \frac{(y+1)^{\frac{p-1}{p}} + (y-1)^{\frac{p-1}{p}}}{2y^{\frac{p-1}{p}}}$$

where $y > 1$ uniquely satisfies

$$\left(1 - \frac{y}{p}\right)^p (y+1) - \left(1 + \frac{y}{p}\right)^p (y-1) = 0 \ ;$$

hence, for example, $c_2 = \sqrt[4]{27}/\sqrt{2}$ and $\lim_{p\to\infty} c_p = 1$. Grafakos, Montgomery-Smith & Motrunich confirmed the truth of this formula for a special class of "bell-shaped" functions, but expressed doubt that it holds for all $f \in L_p(\mathbb{R})$. The problem remains unsolved.

0.3. n Dimensions, Uncentered. Let $n \geq 2$. For a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, define

$$(M_n f)(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(t)|\, dt \ ,$$

where the supremum is taken over all compact cubes Q with sides parallel to the coordinate axes, subject only to $x \in Q$. For fixed $1 < p < \infty$, the best constant $c_{p,n}$ must grow at least exponentially as $n \to \infty$ [3]. This result is also true if we replace cubes by balls.

0.4. n Dimensions, Centered. Let $n \geq 2$. Define similarly

$$(N_n f)(x) = \sup_Q \frac{1}{|Q|} \int\limits_Q |f(t)|\, dt \ ,$$

where we insist not only that $x \in Q$, but additionally that each cube Q is centered at x. For the weak type $(1,1)$ inequality, we have lower bounds on the best constants C_n, for example [12]

$$C_2 \geq \frac{3+\sqrt{2}\left(2\sqrt{3}-1\right)}{4},$$

$$\liminf_{n\to\infty} C_n \geq \frac{47\sqrt{2}}{36}.$$

It would be good someday to know the exact values of these constants. Moreover, we have $C_1 < C_2$ and $C_n \leq C_{n+1}$ for all n [13]. Stein & Strömberg [14] demonstrated that C_n grows at most like $O(n \ln(n))$ and like $O(n)$ if we replace cubes by balls.

Let us return finally to the strong type (p,p) setting. There exists a constant K for which [14]

$$c_{p,n} \leq K \cdot \frac{p}{p-1} \cdot n$$

for all p and n. If we replace cubes by balls, then n can be further replaced by $\sqrt{n}$. Also, it is possible to write

$$c_{p,n} \leq F(p)$$

for all n, in the case of balls (but the expression $F(p)$ may have to grow more rapidly than $p/(p-1)$ as $p \to 1^+$). Thus, for fixed $1 < p < \infty$, $c_{p,n}$ is bounded as $n \to \infty$. This result contrasts strikingly with the uncentered case [0.3].

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