Hankel and Toeplitz Determinants

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The most famous Hankel matrix is the Hilbert matrix

$$H_n = \left(\frac{1}{i+j-1}\right)_{1\leq i,j\leq n}$$

which has determinant equal to a ratio of Barnes G-function values:

$$\det(H_n) = \frac{\prod_{k=1}^{n-1}(k!)^4}{\prod_{\ell=1}^{2n-1}\ell!} = \frac{G(n+1)^4}{G(2n+1)} \to 0$$

as $n\to\infty$. More precisely [1],

$$\frac{\det(H_n)}{4^{-n^2}(2\pi)^n n^{-1/4}} \to 2^{1/12}e^{1/4}A^{-3} = 0.6450024485...$$

where A denotes the Glaisher-Kinkelin constant [2]. Such Hankel determinants are important in random matrix theory and applications [3], but we shall forsake all this, giving instead only a few examples [4, 5, 6]. Another interesting fact is that $\det(H_n)$ is always the reciprocal of a positive integer [7].

The Hankel determinant of Euler numbers [8] is, in absolute value,

$$\begin{aligned} \left|E_{i+j}\right|_{0\leq i,j\leq n-1} &= \prod_{k=1}^{n-1}(k!)^2 = G(n+1)^2 \\ &\sim \frac{e^{\frac{1}{6}}}{A^2}e^{-\frac{3}{2}n^2}(2\pi)^n n^{n^2-\frac{1}{6}} \end{aligned}$$

as $n\to\infty$. The simplicity of this result contrasts with the following. The Hankel determinant of Bernoulli numbers [9] is, in absolute value,

$$\begin{aligned} \left|B_{i+j}\right|_{0\leq i,j\leq n-1} &= \prod_{k=1}^{n-1}\frac{(k!)^6}{(2k)!(2k+1)!} \\ &= \frac{2^{\frac{1}{12}}e^{\frac{1}{4}}}{A^3}4^{-n^2}(2\pi)^n\frac{G(n+1)^4}{G(n+1/2)G(n+3/2)} \\ &\sim \frac{2^{\frac{1}{12}}e^{\frac{5}{12}}}{A^5}4^{-n^2}e^{-\frac{3}{2}n^2}(2\pi)^{2n}n^{n^2-\frac{5}{12}} \end{aligned}$$

as $n \to \infty$. We mention three formulas of Krattenthaler [10]:

$$\left| \frac{B_{2i+2j+2}}{(2i+2j+2)!} \right|_{0 \le i,j \le n-1} = 4^{-n^2} \prod_{k=1}^{2n-1} (2k+1)^{-2n+k},$$

$$\left| \frac{B_{2i+2j+4}}{(2i+2j+4)!} \right|_{0 \le i,j \le n-1} = 4^{-n^2-n} 9^{-n} \prod_{k=1}^{2n-1} (2k+3)^{-2n+k},$$

$$\left| \frac{B_{2i+2j+6}}{(2i+2j+6)!} \right|_{0 \le i,j \le n-1} = (n+1)(2n+3) 4^{-n^2-2n} \prod_{k=1}^{2n+1} (2k+1)^{-2n-2+k}$$

which are always reciprocals of integers (unlike $|E_{i+j}|$ and $|B_{i+j}|$). The asymptotics of these three sequences remain open.

More difficult are determinants of Riemann zeta function values:

$$a_n^{(0)} = |\zeta(i+j)|_{1 \le i,j \le n}, \qquad a_n^{(1)} = |\zeta(i+j+1)|_{1 \le i,j \le n}$$

which evidently satisfy

$$a_n^{(0)} \sim C \cdot \left(\frac{2n+1}{e^{3/2}} \right)^{-(n+1/2)^2}, \qquad a_n^{(1)} \sim \frac{e^{9/8}}{\sqrt{6}} C \cdot \left(\frac{2n}{e^{3/2}} \right)^{-n^2+3/4}$$

thanks to numerical experiments by Zagier [11]. No closed-form expression for the constant $C = 0.351466738331...$ is known.

A famous Toeplitz matrix, called the alternating Hilbert matrix in [12], is

$$\tilde{H}_n = \left(\frac{1}{i-j} \right)_{1 \le i,j \le n}$$

where we understand the diagonal elements to be 0. Schur [13] proved long ago that the maximum eigenvalue (in modulus) of both H_n and $\tilde{H}_n$ is less than π and approaches π as $n \to \infty$. The determinant is, of course, the product of all eigenvalues. When n is odd, $\det(\hat{H}_n) = 0$. When n is even, a closed-form expression for $\det(\hat{H}_n)$ seems to be unavailable, despite the existence of a combinatorial approach [14]. Note that the "symbol" associated with $\hat{H}_n$ is

$$\sum_{r=1}^{\infty} \frac{e^{ir\theta}}{-r} + \sum_{r=1}^{\infty} \frac{e^{-ir\theta}}{r} = i(\theta - \pi)$$

for $0 < \theta < 2\pi$, hence a theorem due to Grenander & Szegő [15] gives

$$\lim_{\substack{n \to \infty \\ n \text{ even}}} \frac{1}{n} \ln \left(\det(\hat{H}_n) \right) = \frac{1}{2\pi} \int_0^{2\pi} \ln \left[i(\theta - \pi) \right] d\theta = -1 + \ln(\pi) = 0.1447298858....$$

A refined estimate shown subsequently in [15], potentially governing the value of

$$\lim_{n\to\infty} \det(\hat{H}_n)\cdot\left(\frac{\pi}{e}\right)^n,$$

has conditions that must be verified.

Consider finally another Toeplitz matrix

$$K_n = \left(\frac{1}{1+|i-j|}\right)_{1\le i,j\le n}$$

for which little is known. The "symbol" here is

$$\sum_{r=0}^{\infty}\frac{e^{ir\theta}}{1+r}+\sum_{r=1}^{\infty}\frac{e^{-ir\theta}}{1+r} = -1-e^{i\theta}\ln\left(1-e^{-i\theta}\right)-e^{-i\theta}\ln\left(1-e^{i\theta}\right)$$

for $0<\theta<2\pi$, hence

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\ln\left(\det(K_n)\right) &= \frac{1}{2\pi}\int_0^{2\pi}\ln\left[-1-e^{i\theta}\ln\left(1-e^{-i\theta}\right)-e^{-i\theta}\ln\left(1-e^{i\theta}\right)\right]d\theta \\ &= -0.3100863233....\end{aligned}$$

An exact formula for this constant is desired; might, at least, the integral be simplified in some way?

0.1. Combinatorial Approach. Assume that n is even. Let S denote the set of all $(n/2)$-tuples of ordered pairs:

$$(p_k,q_k)_{k=1}^{n/2}$$

of positive integers $p_k<q_k$ satisfying

$$\bigcup_{k=1}^{n/2}\{p_k,q_k\}=\{1,2,\ldots,n\}$$

and $p_1<p_2<\ldots<p_{n/2}$. Note that the qs need not be in ascending order. Let us verify a formula in [14]:

$$\det(\hat{H}_n)=\sum_{(p_k,q_k)_{k=1}^{n/2}\in S}\prod_{k=1}^{n/2}\frac{1}{(q_k-p_k)^2}$$

for $n = 4$. Three such 2-tuples exist:

$$p_1 = 1 < p_2 = 2 < q_1 = 3 < q_2 = 4,$$

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yielding

$$\frac{1}{(3-1)^2(4-2)^2} + \frac{1}{(4-1)^2(3-2)^2} + \frac{1}{(2-1)^2(4-3)^2} = \frac{169}{144} = \det(\hat{H}_4).$$

The case $\det(\hat{H}_2) = 1$ is trivial; the case $\det(\hat{H}_6) = 6723649/4665600$ will require some effort. We wonder if a simple method for computing the size of S, as a function of n, can be found. An analogous approach for $\det(K_n)$ would also be good to see.

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