Constant of Interpolation

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A bounded entire function is necessarily constant (by Liouville's theorem). For our purposes, let us therefore restrict attention to function f analytic on the upper half plane $\mathrm{Im}(z) > 0$. Define the H^∞-norm of f to be

$$||f||_\infty = \sup_{y>0} |f(x + i\,y)|.$$

Also, given a finite or infinite sequences $W = \{w_j\}$ of complex numbers, define its l^∞-norm by

$$||W||_\infty = \sup_{j\geq 1} |w_j|.$$

We say that a sequence $Z = \{z_j\}$ of distinct complex numbers in the upper half plane is an **interpolating sequence** if there exists an analytic function f for which $||f||_\infty < \infty$ and

$$f(z_j) = w_j, \qquad j = 1, 2, 3, ...$$

for each sequence W with $||W||_\infty < \infty$. In words, Z has the property that, for any bounded W, there must be a bounded analytic interpolant f taking z_j to w_j for all j. There may be many such f. We wish to be as efficient as possible and define $M(Z)$ to be the smallest constant C such that

$$||f||_\infty \leq C \cdot ||W||_\infty$$

always; if Z is not an interpolating sequence, define instead $M(Z) = \infty$. Carleson [1, 2, 3, 4] proved that $M(Z) < \infty$ if and only if a uniform separation criterion

$$\delta = \inf_{k\geq 1} \prod_{j\neq k} \left| \frac{z_j - z_k}{z_j - \bar{z}_k} \right| > 0$$

is met.

Define the **Blaschke product** corresponding to Z by [4]

$$B(z) = \prod_{n\geq 1} \frac{|z_n^2 + 1|}{z_n^2 + 1} \frac{z - z_n}{z - \bar{z}_n}$$

with the understanding that, if $z = i$ (the imaginary unit), then the left hand factor is to be interpreted as 1. If Z is an interpolating sequence, then B is uniformly convergent on compact subsets of the upper half plane and hence represents an analytic function. Further, $||B||_\infty = 1$ and B vanishes only at the points z_n. Let

$$B_k(z) = \frac{z - \bar{z}_k}{z - z_k} B(z)$$

so that we may write $\delta = \inf_{k \geq 1} |B_k(z_k)|$. Also let $z_j = x_j + i\, y_j$.

Beurling [5], Jones [6] and Havin [7] examined the problem of exhibiting an explicit formula for f. Nicolau, Ortega-Cerdà & Seip [8] used this work as a basis for estimating $M(Z)$. Define

$$\Phi(Z) = \sup_{k \geq 1} \sum_{y_j \leq y_k} \frac{4y_j(y_j + y_k)}{|z_j - \bar{z}_k|^2} \frac{1}{|B_j(z_j)|},$$

$$\Psi(Z) = \sup_{k \geq 1} \sum_{n \geq 1} \frac{4y_k y_n}{|z_k - \bar{z}_n|^2} \frac{1}{|B_n(z_n)|}.$$

Then, for every interpolating sequence Z in the upper half plane, we have

$$\frac{1}{2} \leq \frac{M(Z)}{\Phi(Z)} \leq \kappa, \qquad 1 \leq \frac{M(Z)}{\Psi(Z)} \leq \lambda$$

for constants κ and λ satisfying

$$2.2661... = \frac{\pi}{2\ln(2)} \leq \kappa \leq e = 2.7182...,$$

$$1.5707... = \frac{\pi}{2} \leq \lambda \leq 2e = 5.4365....$$

Can these bounds be improved? Also, can simpler expressions than Φ or Ψ for the denominators be found?

An alternative definition of $M(Z)$ is related to Nevanlinna-Pick theory [4, 9, 10]. Let $M_n(Z)$ be the smallest constant C_n such that the matrix $A = (a_{j,k})$ with

$$a_{j,k} = \frac{1 - \bar{w}_j w_k}{z_j - \bar{z}_k}, \qquad j = 1, 2, ..., n, \qquad k = 1, 2, ..., n,$$

is nonnegative definite whenever $||W||_\infty < 1/C_n$. The constant of interpolation $M(Z)$ is thus $M_n(Z)$ if Z consists of exactly n points and $\lim_{n \to \infty} M_n(Z)$ if Z is infinite [8].

We could alternatively restrict attention to functions f analytic on the unit disk $|z| < 1$. Some relevant formulas in this new setting are

$$\delta = \inf_{k \geq 1} \prod_{j \neq k} \left| \frac{z_j - z_k}{\bar{z}_j z_k - 1} \right|,$$

$$B(z) = \prod_{n\geq 1} \frac{|z_n|}{z_n} \frac{z - z_n}{\bar{z}_n z - 1},$$

$$a_{j,k} = \frac{1 - \bar{w}_j w_k}{1 - \bar{z}_j z_k}, \qquad j = 1, 2, ..., n, \qquad k = 1, 2, ..., n.$$

Similar interpolation questions can be asked for the H^p-norm on the unit disk (for example):

$$||f||_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(r\, e^{i\theta}) \right|^p d\theta \right)^{1/p}$$

where $1 < p < \infty$ [4, 11]. It would be good to see results paralleling those in [8] for $p = 2$ and $p = 1$.

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