Injections, Surjections and More

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Let $I_{m,n}$ denote the set of all injections $\{1,\ldots,m\}\to\{1,\ldots,n\}$ where $m\leq n$. An element of $I_{m,n}$ can be thought of as a permutation on n symbols taken m at a time. We define $I_{0,n}$ to possess one element (the empty permutation) for convenience; therefore [1, 2, 3]

$$\#\,I_{m,n}=\frac{n!}{(n-m)!}$$

and

$$\#\bigcup_{0\leq m\leq n}I_{m,n}=\sum_{k=0}^{n}\frac{n!}{k!}=\begin{cases}\lfloor n!e\rfloor & \text{if } n>0,\\ 1 & \text{if } n=0\end{cases}$$

where e is the natural logarithmic base [4]. In counting all injections, we treat extensions as distinct; for example, the function $f:\{1,2\}\to\{1,2\}$ with $f(x)=x$ is not the same as the function $g:\{1,2\}\to\{1,2,3\}$ with $g(x)=x$, nor is it the same as the function $h:\{1,2,3\}\to\{1,2,3\}$ with $h(x)=x$.

Let $J_{n,m}$ denote the set of all surjections $\{1,\ldots,n\}\to\{1,\ldots,m\}$ where $n\geq m$. An element of $J_{n,m}$ can be thought of as an ordered m-tuple consisting of preimage blocks (m disjoint nonempty sets that cover n symbols). We define $J_{0,0}$ to possess one element (the empty tuple) for convenience; therefore [5, 6, 7]

$$\#\,J_{n,m}=\sum_{j=0}^{m}(-1)^{j}\binom{m}{j}(m-j)^{n}=m!S_{n,m}$$

and

$$\#\bigcup_{0\leq m\leq n}J_{n,m}=\frac{1}{2}\sum_{k=0}^{\infty}\frac{k^{n}}{2^{k}}\sim\frac{n!}{2}\left(\frac{1}{\ln(2)}\right)^{n+1}\sim\frac{n!}{2\ln(2)}\,(1.4426950408...)^{n}$$

as $n\to\infty$, where $S_{n,m}$ is a Stirling number of the second kind [8]. In counting all surjections, we treat extensions as distinct; for example, the preceding function f is not the same as the function $g:\{1,2,3\}\to\{1,2\}$ with $g(x)=x$ mod 2, nor is it the same as the preceding function h.

Various refinements of surjections are available. An **ℓ-surjection** has the property that every value in the range $\{1,\ldots,m\}$ is taken with multiplicity at least ℓ. (The phrase "double surjection" was used in [6], while "2-surjection" meant something different.) Asymptotic counting results for 2-surjections, 3-surjections and 4-surjections are

$$\frac{n!}{(1+r)r}(0.8724532496...)^n \qquad \text{where} \qquad r=1.1461932206... \text{ solves } e^r=2+r,$$

$$\frac{n!}{\left(1+\frac{1}{2}r^2\right)r}(0.6377063010...)^n \qquad \text{where} \qquad r=1.5681199923... \text{ solves } 2e^r=4+2r+r^2,$$

$$\frac{n!}{\left(1+\frac{1}{6}r^3\right)r}(0.5060319662...)^n \qquad \text{where} \qquad r=1.9761597421... \text{ solves } 6e^r=12+6r+3r^2+r^3$$

respectively (the numerical value within parentheses is $1/r$). The formulas for $\ell=3$ and 4 are due to Kotesovec [5].

Another way of imagining a surjection is as a **labeled clique**, that is, a hierarchy on $\{1,\ldots,n\}$ in which vertical ordering is important but horizontal ordering is not. We illustrate $\#\,J_{3,1}=1$, $\#\,J_{3,2}=6$, $\#\,J_{3,3}=6$ here:

$$\begin{array}{|c|}\hline 1,2,3\\\hline\end{array},$$

$$\begin{array}{|c|}\hline 1\\2,3\\\hline\end{array},\quad \begin{array}{|c|}\hline 2\\1,3\\\hline\end{array},\quad \begin{array}{|c|}\hline 3\\1,2\\\hline\end{array},\quad \begin{array}{|c|}\hline 1,2\\3\\\hline\end{array},\quad \begin{array}{|c|}\hline 1,3\\2\\\hline\end{array},\quad \begin{array}{|c|}\hline 2,3\\1\\\hline\end{array},$$

$$\begin{array}{|c|}\hline 1\\2\\3\\\hline\end{array},\quad \begin{array}{|c|}\hline 1\\3\\2\\\hline\end{array},\quad \begin{array}{|c|}\hline 2\\3\\1\\\hline\end{array},\quad \begin{array}{|c|}\hline 2\\1\\3\\\hline\end{array},\quad \begin{array}{|c|}\hline 3\\1\\2\\\hline\end{array},\quad \begin{array}{|c|}\hline 3\\2\\1\\\hline\end{array}.$$

If we remove labels, then just 4 hierarchies emerge:

$$\begin{array}{|c|}\hline *,*,*\\\hline\end{array},\quad \begin{array}{|c|}\hline *\\*,*\\\hline\end{array},\quad \begin{array}{|c|}\hline *,*\\*\\\hline\end{array},\quad \begin{array}{|c|}\hline *\\*\\*\\\hline\end{array}.$$

More generally [9], the number of **unlabeled cliques** on n integers is 2^n.

A **labeled society** on $\{1,\ldots,n\}$ is created by distributing the elements into cliques. The ordering of the cliques is not important. Let S_n denote the number of such societies and s_n denote the unlabeled analog. The cliques are visually separated by bars and (as before) hierarchy within a clique is indicated by the vertical arrangement. We illustrate $S_3 = 23$ and $s_3 = 7$, omitting the 13 one-clique cases for the former and the 4 one-clique cases for the latter (which were already given):

$$\begin{array}{|cc|c|}\hline 1,2 & & 3 \\ \hline\end{array}, \quad \begin{array}{|cc|c|}\hline 1,3 & & 2 \\ \hline\end{array}, \quad \begin{array}{|cc|c|}\hline 2,3 & & 1 \\ \hline\end{array},$$

$$\begin{array}{|c|c|}\hline 1 & \\ 2 & 3 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 1 & \\ 3 & 2 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 2 & \\ 3 & 1 \\ \hline\end{array},$$

$$\begin{array}{|c|c|}\hline 2 & \\ 1 & 3 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 3 & \\ 1 & 2 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 3 & \\ 2 & 1 \\ \hline\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array};$$

$$\begin{array}{|c|c|}\hline *,* & * \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline * & \\ * & * \\ \hline\end{array}, \quad \begin{array}{|c|cc|}\hline * & * & * \\ \hline\end{array}.$$

More generally [10, 11, 12],

$$S_n = \frac{d^n}{dx^n} \exp\left(\frac{1}{2-e^x} - 1\right)\bigg|_{x=0}, \qquad s_n = \frac{1}{n!}\frac{d^n}{dx^n}\prod_{k=1}^{\infty}\frac{1}{(1-x^k)^{2^{k-1}}}\bigg|_{x=0}$$

and

$$S_n \sim C\frac{e^{\sqrt{2n/\ln(2)}}}{n^{3/4}\ln(2)^n}n!, \qquad s_n \sim \frac{c}{\sqrt{2\pi}}\frac{e^{\sqrt{2n}-1/4}}{n^{3/4}}2^{n-3/4}$$

as $n \to \infty$, where

$$C = \frac{1}{4\sqrt{\pi}}\left(\frac{2}{e}\right)^{3/4}\left(\frac{e^{1/\ln(2)}}{\ln(2)}\right)^{1/4} = (1038.9726974426...)^{-1/4},$$

$$c = \exp\left(\sum_{j=2}^{\infty}\frac{1}{j\,(2^j-1)}\right) = 1.3976490050....$$

The constant c, overlooked in [10], was subsequently determined in [13].

Let us focus entirely on the labeled scenario henceforth. A clique is **elitist** if, given any two adjacent levels, the number of elements in the higher level never exceeds the number of elements in the lower level. Define R_n to be the number of elitist cliques on $\{1,\ldots,n\}$. Clearly $R_2 = 3$ and $R_3 = 10$. More generally [9, 12, 14],

$$R_n = \frac{d^n}{dx^n}\prod_{k=1}^{\infty}\left(1-\frac{x^k}{k!}\right)^{-1}\bigg|_{x=0}$$

and $R_n \sim B\, n!$ as $n \to \infty$, where

$$B = \prod_{k=2}^{\infty} \left(1 - \frac{1}{k!}\right)^{-1} = 2.5294774720....$$

Another interpretation involves multinomial coefficients [15]: for suitably large m,

$$(x_1 + x_2 + \cdots + x_m)^2 = \sum_i x_i^2 + 2 \sum_{i<j} x_i x_j,$$

$$(x_1 + x_2 + \cdots + x_m)^3 = \sum_i x_i^3 + 3 \sum_{i \neq j} x_i x_j^2 + 6 \sum_{i<j<k} x_i x_j x_k,$$

$$(x_1+x_2+\cdots+x_m)^4 = \sum_i x_i^4 + 4 \sum_{i \neq j} x_i x_j^3 + 6 \sum_{i<j} x_i^2 x_j^2 + 12 \sum_{\substack{i<j, \\ i \neq k, j \neq k}} x_i x_j x_k^2 + 24 \sum_{i<j<k<\ell} x_i x_j x_k x_\ell$$

hence $R_2 = 1 + 2$, $R_3 = 1 + 3 + 6$ and $R_4 = 1 + 4 + 6 + 12 + 24$.

Finally, a society is elitist if all of its cliques are elitist. Define Q_n to be the number of elitist societies on $\{1, \ldots, n\}$. Clearly $Q_2 = 4$ and $Q_3 = 20$. More generally,

$$Q_n = \frac{d^n}{dx^n} \exp\left(\prod_{k=1}^{\infty} \left(1 - \frac{x^k}{k!}\right)^{-1} - 1 \right) \Bigg|_{x=0}$$

but an asymptotic expression for Q_n appears to be open.

In closing, we give a sequence [9, 16]

$$p_n = \frac{1}{n!} \frac{d^n}{dx^n} \left(2 - \prod_{k=1}^{\infty} \left(1 - x^k\right)^{-1} \right)^{-1} \Bigg|_{x=0} = \frac{1}{n!} \frac{d^n}{dx^n} \frac{1}{f(x)} \Bigg|_{x=0}$$

which arises from unlabeled cliques on *set partitions* rather than integers. It is quite similar to the sequence 2^n mentioned earlier. We illustrate $p_3 = 8$ here:

$$\overline{\underline{\left|\{*,*,*\}\right|}}, \quad \overline{\underline{\left|\{*,*\},\{*\}\right|}}, \quad \overline{\underline{\left|\{*\},\{*\},\{*\}\right|}}, \quad \overline{\underline{\left|\begin{array}{c}\{*,*\} \\ \{*\}\end{array}\right|}},$$

$$\overline{\underline{\left|\begin{array}{c}\{*\} \\ \{*,*\}\end{array}\right|}}, \quad \overline{\underline{\left|\begin{array}{c}\{*\} \\ \{*\},\{*\}\end{array}\right|}}, \quad \overline{\underline{\left|\begin{array}{c}\{*\},\{*\} \\ \{*\}\end{array}\right|}}, \quad \overline{\underline{\left|\begin{array}{c}\{*\} \\ \{*\} \\ \{*\}\end{array}\right|}}.$$

It is easily shown that $p_n \sim a\, b^n$ where $b = 2.6983291064...$ is the unique positive solution of the equation $f(1/y) = 0$ and

$$a = \frac{-b}{f'(1/b)} = 0.4141137931....$$

The fit is excellent. Moreover, the occurrence of the Dedekind eta function [17] is unexpected. Replacing $f(x)$ by $f(x) - 1$ spawns another (alternating in sign) integer sequence [16]; we wonder whether this perturbation possesses a combinatorial interpretation. Societies (labeled or not, elitist or not) can also be imposed in the new partitional framework and more asymptotic results await discovery.

REFERENCES

[1] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A000522 and A007526.

[2] L. Halbeisen and N. Hungerbühler, Number theoretic aspects of a combinatorial function. *Notes Number Theory Discrete Math.,* v. 5 (1999) n. 4, 138–150; http://user.math.uzh.ch/halbeisen/publications/pdf/seq.pdf; MR1764301 (2001c:11029).

[3] M. Hassani, Counting and computing by e, arXiv:math/0606613.

[4] S. R. Finch, Natural logarithmic base, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 12–17.

[5] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A000670, A032032, A102233, and A232475.

[6] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, 2009, pp. 106–110, 244–245, 259–261, 335; http://algo.inria.fr/flajolet/Publications/AnaCombi/; MR2483235 (2010h:05005).

[7] H. S. Wilf, *generatingfunctionology*, 2nd ed., Academic Press, 1994, pp. 175–176; http://www.math.upenn.edu/~wilf/DownldGF.html; MR1277813 (95a:05002).

[8] S. R. Finch, Lengyel's constant: Stirling partition numbers, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 316–317.

[9] T. Wieder, The number of certain rankings and hierarchies formed from labeled or unlabeled elements and sets, *Appl. Math. Sci. (Ruse)* 3 (2009) 2707–2724; http://www.m-hikari.com/ams/; MR2563113 (2010m:05025).

[10] N. J. A. Sloane and T. Wieder, The number of hierarchical orderings, *Order* 21 (2004) 83–89; arXiv:math/0307064; MR2128036 (2005k:06012).

[11] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, 2009, p. 571; http://algo.inria.fr/flajolet/Publications/AnaCombi/; MR2483235 (2010h:05005).

[12] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A005651, A034691, A034899, A075729, and A143463.

[13] V. Kotesovec, Asymptotics of sequence A034691, unpublished note (2014), http://oeis.org/A034691/a034691_1.pdf.

[14] A. Knopfmacher, A. M. Odlyzko, B. Pittel, L. B. Richmond, D. Stark, G. Szekeres and N. C. Wormald, The asymptotic number of set partitions with unequal block sizes, *Elec. J. Combin.* 6 (1999) R2; MR1663723 (2000b:05018).

[15] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Reidel, 1974, p. 126; MR0460128 (57 #124).

[16] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A055887 and A082531.

[17] S. R. Finch, Dedekind eta products, unpublished note (2007).