Continued Fraction Transformation

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We are interested in iterates of the **continued fraction transformation** $T : [0,1] \to [0,1]$ defined by [1]

$$T(x) = \begin{cases} \left\{ \dfrac{1}{x} \right\} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0 \end{cases}$$

where $\{\xi\} = \xi - \lfloor \xi \rfloor$ denotes the fractional part of ξ. For example,

$$\begin{array}{ll} \pi - 3 = 0.141592..., & \left\lfloor \frac{1}{\pi-3} \right\rfloor = 7, \\ T(\pi - 3) = 0.062513..., & \left\lfloor \frac{1}{T(\pi-3)} \right\rfloor = 15, \\ T^2(\pi - 3) = 0.996594..., & \left\lfloor \frac{1}{T^2(\pi-3)} \right\rfloor = 1, \\ T^3(\pi - 3) = 0.003417..., & \left\lfloor \frac{1}{T^3(\pi-3)} \right\rfloor = 292, \\ T^4(\pi - 3) = 0.634591..., & \left\lfloor \frac{1}{T^4(\pi-3)} \right\rfloor = 1 \end{array}$$

and

$$\pi = 3 + \frac{1|}{|7} + \frac{1|}{|15} + \frac{1|}{|1} + \frac{1|}{|292} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|2} + \frac{1|}{|1} + \frac{1|}{|3} + \cdots$$

is the regular continued fraction expansion for π. In words, T discards the first "digit" in any expansion, that is,

$$T\left(\frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots \right) = \frac{1|}{|a_2} + \frac{1|}{|a_3} + \frac{1|}{|a_4} + \cdots .$$

What can be said about the moments of $T^j X$ and of $\ln(T^j X)$, where X is a random variable in $[0,1]$? There are two cases: the first when X follows the uniform distribution, and the second when X follows the **Gauss-Kuzmin distribution**:

$$\mathrm{P}(X \leq x) = \frac{\ln(x+1)}{\ln(2)}.$$

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We will later study the partial convergents to x, for example,

$$\frac{p_1}{q_1} = \frac{3}{1}, \quad \frac{p_2}{q_2} = \frac{22}{7}, \quad \frac{p_3}{q_3} = \frac{333}{106}, \quad \frac{p_4}{q_4} = \frac{355}{113}, \quad \frac{p_5}{q_5} = \frac{103993}{33102}, \quad ...$$

when $x = \pi$. The asymptotic distribution of denominators Q_n, corresponding to uniformly distributed X as $n \to \infty$, turns out to be related to our earlier work on $\ln(T^j X)$ statistics.

0.1. Uniform Distribution. Let γ denote the Euler-Mascheroni constant [2], ζ denote the Riemann zeta function and Li_k denote the k^{th} polylogarithm function [3]. If X is a random variable following the uniform distribution on $[0, 1]$, then

$$\mathrm{E}(X) = \int_0^1 x\,dx = \frac{1}{2}, \qquad \mathrm{E}(X^2) = \int_0^1 x^2 dx = \frac{1}{3},$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \frac{1}{12}$$

and, via the substitution $y = 1/x$,

$$\begin{aligned} \mathrm{E}(TX) &= \int_0^1 \left\{\frac{1}{x}\right\} dx = \int_1^\infty \frac{\{y\}}{y^2} dy = \sum_{n=1}^\infty \int_n^{n+1} \frac{y-n}{y^2} dy \\ &= \sum_{n=1}^\infty \left(\ln\left(\frac{n+1}{n}\right) - \frac{1}{n+1} \right) = 1 - \gamma = 0.4227843351... \end{aligned}$$

(which is related to de la Vallée Poussin's theorem [2, 4]),

$$\mathrm{E}((TX)^2) = \ln(2\pi) - \gamma - 1,$$

$$\mathrm{Var}(TX) = \ln(2\pi) - \gamma^2 + \gamma - 2 = 0.0819148075... = (0.2862076300...)^2,$$

$$\mathrm{E}(X \cdot TX) = 1 - \frac{\pi^2}{12},$$

$$\mathrm{Cov}(X, TX) = \mathrm{E}(X \cdot TX) - \mathrm{E}(X)\,\mathrm{E}(TX) = \frac{1}{12}\left(6 - \pi^2 + 6\gamma\right),$$

$$\rho(X, TX) = \frac{\mathrm{Cov}(X, TX)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(TX)}} = \frac{6 - \pi^2 + 6\gamma}{\sqrt{12}\sqrt{\ln(2\pi) - \gamma^2 + \gamma - 2}} = -0.4098133678...$$

where ρ denotes cross-correlation. Likewise,

$$\mathrm{E}(\ln(X)) = -1, \qquad \mathrm{E}(\ln(X)^2) = 2, \qquad \mathrm{Var}(\ln(X)) = 1,$$

and, via the substitutions $y = 1/x$ and $z = y - n$,

$$\begin{aligned}
\mathrm{E}(\ln(TX)) &= \int_0^1 \ln\left\{\frac{1}{x}\right\} dx = \int_1^\infty \frac{\ln\{y\}}{y^2} dy = \sum_{n=1}^\infty \int_n^{n+1} \frac{\ln(y-n)}{y^2} dy \\
&= \sum_{n=1}^\infty \int_0^1 \frac{\ln(z)}{(z+n)^2} dz = -\sum_{n=1}^\infty \frac{1}{n} \ln\left(\frac{n+1}{n}\right) \\
&= -\left(\ln(2) + \sum_{k=2}^\infty (-1)^k \frac{\zeta(k)-1}{k-1}\right) = -1.2577468869...
\end{aligned}$$

(this constant appears elsewhere [5, 6]),

$$\mathrm{E}(\ln(TX)^2) = -2\sum_{n=1}^\infty \frac{1}{n} \mathrm{Li}_2\left(-\frac{1}{n}\right) = \zeta(2) - 2\sum_{k=1}^\infty (-1)^k \frac{\zeta(k+1)-1}{k^2},$$

$$\mathrm{Var}(\ln(TX)) = 1.2665694005... = (1.1254196552...)^2,$$

$$\begin{aligned}
\mathrm{E}(\ln(X)\cdot\ln(TX)) &= \sum_{n=1}^\infty \frac{1}{n}\left[\ln\left(\frac{n+1}{n}\right)(1+\ln(n)) - \mathrm{Li}_2\left(\frac{1}{n+1}\right)\right] \\
&= -\zeta(2) + \sum_{k=2}^\infty \left[\left(\zeta(2) - \sum_{\ell=1}^{k-1} \frac{1}{\ell^2}\right)(\zeta(k)-1) - \left(1 + \frac{(-1)^k}{k-1}\right)\zeta'(k)\right],
\end{aligned}$$

$$\rho(\ln(X), \ln(TX)) = -0.2275522084....$$

The cumulative distribution for TX can be expressed in terms of the digamma function:

$$F(x) = \mathrm{P}(TX \le x) = \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+x}\right) = \gamma + \psi(x+1),$$

and its density in terms of the trigamma function:

$$f(x) = \sum_{n=1}^\infty \frac{1}{(n+x)^2} = \psi'(x+1).$$

For example, the median of TX is $F^{-1}(1/2) = 0.3846747346....$ The cumulative distribution for T^2X is

$$\begin{aligned}
G(x) &= \mathrm{P}(T^2X \le x) = \sum_{n=1}^\infty \left(F\left(\frac{1}{n}\right) - F\left(\frac{1}{n+x}\right)\right) \\
&= \sum_{n=1}^\infty \left(\psi\left(\frac{1}{n}+1\right) - \psi\left(\frac{1}{n+x}+1\right)\right),
\end{aligned}$$

its density is

$$g(x)=\sum_{n=1}^{\infty}\psi'\left(\frac{1}{n+x}+1\right)\frac{1}{(n+x)^2},$$

and its median is $G^{-1}(1/2)=0.42278....$ It is certainly inconvenient that $F\neq G$!

0.2. Gauss-Kuzmin Distribution. If X is a random variable following the Gauss-Kuzmin distribution on $[0,1]$, then

$$\mathrm{E}(X)=\frac{1}{\ln(2)}-1=0.4426950408...=\mathrm{E}(TX),$$

$$\mathrm{E}(X^2)=1-\frac{1}{2\ln(2)}=\mathrm{E}((TX)^2),$$

$$\mathrm{Var}(X)=\frac{(3/2)\ln(2)-1}{\ln(2)^2}=0.0826735803...=(0.2875301381...)^2=\mathrm{Var}(TX)$$

by invariance under T, and

$$\mathrm{E}(X\cdot TX)=1-\frac{\gamma}{\ln(2)},\qquad \mathrm{Cov}(X,TX)=\frac{(2-\gamma)\ln(2)-1}{\ln(2)^2},$$

$$\rho(X,TX)=\frac{(2-\gamma)\ln(2)-1}{(3/2)\ln(2)-1}=-0.3474517057....$$

Likewise,

$$\mathrm{E}(\ln(X))=-\frac{\pi^2}{12\ln(2)}=-1.1865691104...=\mathrm{E}(\ln(TX)),$$

$$\mathrm{E}(\ln(X)^2)=\frac{3\zeta(3)}{2\ln(2)}=\mathrm{E}(\ln(TX)^2),$$

$$\begin{aligned}\mathrm{Var}(\ln(X)) &= \frac{216\ln(2)\zeta(3)-\pi^4}{144\ln(2)^2}=1.1933560457...\\ &= (1.0924083695...)^2=\mathrm{Var}(\ln(TX)),\end{aligned}$$

$$\begin{aligned}\mathrm{E}(\ln(X)\cdot\ln(TX)) &= \frac{1}{\ln(2)}\sum_{n=1}^{\infty}\left[\frac{1}{2}\ln\left(\frac{n+1}{n}\right)^2\ln((n+1)n)+\ln(n)\,\mathrm{Li}_2\left(-\frac{1}{n}\right)\right.\\ &\quad -\ln(n+1)\,\mathrm{Li}_2\left(-\frac{1}{n+1}\right)+\ln(n+1)\,\mathrm{Li}_2\left(\frac{1}{(n+1)^2}\right)\\ &\quad \left.+2\,\mathrm{Li}_3\left(-\frac{1}{n}\right)-2\,\mathrm{Li}_3\left(-\frac{1}{n+1}\right)+\mathrm{Li}_3\left(\frac{1}{(n+1)^2}\right)\right]\\ &= \frac{1}{\ln(2)}\left[-\frac{3\zeta(3)}{2}+\sum_{k=1}^{\infty}\left(\frac{\zeta(2k)-1}{k^3}-\frac{\zeta'(2k)}{k^2}+\frac{\zeta''(2k)}{2k}\right)\right],\end{aligned}$$

$$\rho(\ln(X), \ln(TX)) = -0.1858801270... = r_1.$$

The median of $T^j X$ is $\sqrt{2} - 1 = 0.4142135623...$ for every j. We wish to understand the decay rate of $\rho(X, T^j X)$ and $\rho(\ln(X), \ln(T^j X))$ as j increases, but this appears to be a difficult problem.

0.3. Variance of Sample Mean. Let us consider the sample mean

$$\hat{\mu}_n(X) = -\frac{1}{n} \sum_{0 \le j < n} \ln(T^j X),$$

that is, the average of the time series $\ln(X)$, $\ln(TX)$, ..., $\ln(T^{n-1}X)$ built from iterates of T evaluated at X. (The negative sign will simplify subsequent formulation.) It can be proved that

$$\lim_{n\to\infty} \mathrm{E}\left(\hat{\mu}_n(X)\right) = \frac{\pi^2}{12\ln(2)} = 1.1865691104... = \mu,$$

$$\begin{aligned} \lim_{n\to\infty} n \operatorname{Var}\left(\hat{\mu}_n(X)\right) &= \lim_{n\to\infty} \frac{1}{n} \sum_{\substack{0 \le j < n, \\ 0 \le k < n}} \operatorname{Cov}(\ln(T^j X), \ln(T^k X)) = \sigma^2 \\ &\approx \frac{216\ln(2)\zeta(3) - \pi^4}{144\ln(2)^2} \left(1 + \frac{2r_1}{1 - r_1}\right) \approx 0.8 \end{aligned}$$

for a wide variety of initial distributions for X on $[0, 1]$. The latter is a poor numerical estimate (since it presumes that the lag-ℓ correlation r_ℓ is approximately r_1^ℓ, which is not true). It is inspired, in part, by Salamin [7]. A more precise estimate will be given shortly.

0.4. Partial Convergents. The denominator $Q_n(X)$ of the n^{th} partial convergent to X is connected to our exposition via the formula

$$\underbrace{\ln(Q_n(X))}_{A_n} = \underbrace{-\sum_{0 \le j < n} \ln(T^j X)}_{B_n} + \varepsilon_n$$

where $|\varepsilon_n| < c$ for all n, for some constant c. It is clear that

$$\lim_{n\to\infty} \frac{\mathrm{E}(A_n)}{n} = \lim_{n\to\infty} \frac{\mathrm{E}(B_n)}{n} = \mu$$

and further known [8] that

$$0 < \lim_{n\to\infty} \frac{\operatorname{Var}(A_n)}{n} < \infty.$$

We wish to prove that

$$\lim_{n\to\infty}\frac{\operatorname{Var}(A_n)}{n}=\lim_{n\to\infty}\frac{\operatorname{Var}(B_n)}{n}.$$

From $B_n = A_n - \varepsilon_n$, deduce that

$$\operatorname{Var}(B_n)=\operatorname{Var}(A_n)-2\operatorname{Cov}(A_n,\varepsilon_n)+\operatorname{Var}(\varepsilon_n);$$

hence

$$\begin{aligned}
\left|\operatorname{Var}(A_n)-\operatorname{Var}(B_n)\right| &\leq 2\left|\operatorname{Cov}(A_n,\varepsilon_n)\right|+\operatorname{Var}(\varepsilon_n)\\
&\leq 2\sqrt{\operatorname{Var}(A_n)\operatorname{Var}(\varepsilon_n)}+\operatorname{Var}(\varepsilon_n)\\
&\leq 2\sqrt{\operatorname{Var}(A_n)\operatorname{E}(\varepsilon_n^2)}+\operatorname{E}(\varepsilon_n^2)\\
&\leq 2c\sqrt{\operatorname{Var}(A_n)}+c^2;
\end{aligned}$$

hence

$$\left|\frac{\operatorname{Var}(A_n)}{n}-\frac{\operatorname{Var}(B_n)}{n}\right|\leq 2c\sqrt{\frac{\operatorname{Var}(A_n)}{n^2}}+\frac{c^2}{n}\to 0$$

as $n\to\infty$. In particular,

$$\operatorname{Var}(\ln(Q_n(X)))\sim\sigma^2 n$$

and the importance of computing σ^2 (as attempted using iterates of T) becomes evident.

In fact, the existence of σ^2 (in connection with the denominators Q_n) has been known for a long time. Ibragimov [9], Philipp [10, 11, 12] and others [13, 14, 15, 16, 17, 18, 19] proved the following Central Limit Theorem:

$$\lim_{n\to\infty}\operatorname{P}\left(\frac{\frac{1}{n}\ln(Q_n(X))-\mu}{\frac{\sigma}{\sqrt{n}}}\leq t\right)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{t}\exp\left(-\frac{u^2}{2}\right)du.$$

No numerical estimate of σ^2 appeared until Flajolet & Vallée [8, 20, 21] computed that

$$\begin{aligned}
\sigma^2 &= \lambda_1''(2)-\lambda_1'(2)^2=0.8621470373...=(0.9285187329...)^2\\
&= \frac{1}{4}(9.0803731646...)-\mu^2=(0.5160624088...)\cdot\mu^3,
\end{aligned}$$

where $\lambda_1(s)$ is the dominant eigenvalue of a family of linear operators (indexed by s) on a certain infinite-dimensional function space. Lhote [22, 23] proved that σ^2 is polynomial-time computable and obtained higher accuracy. An elementary expression for σ^2 seems to be impossible. The quantities $4\lambda_1''(2)$ or σ^2/μ^3 are often called **Hensley's constant**.

We close with Loch's theorem [1, 24, 25]:

$$\lim_{n\to\infty}\frac{m(n,x)}{n}=\frac{6\ln(2)\ln(10)}{\pi^2}=0.9702701143...=(1.0306408341...)^{-1}=\alpha$$

for almost all real x, where $m(n,x)$ is the number of partial denominators of x correctly predicted by the first n decimal digits of x. A corresponding Central Limit Theorem was proved by Faivre [26, 27]:

$$\lim_{n\to\infty}\mathrm{P}\left(\frac{\frac{m(n,X)}{n}-\alpha}{\frac{\theta}{\sqrt{n}}}\leq t\right)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{t}\exp\left(-\frac{u^2}{2}\right)du$$

where

$$\begin{aligned}\theta^2 &= \frac{\alpha\,\sigma^2}{\mu^2}=\frac{864\ln(2)^3\ln(10)}{\pi^6}\sigma^2\\ &= 0.5941388048...=(0.7708039990...)^2.\end{aligned}$$

For example, the first 10000 decimal digits of π give 9757 partial denominators, consistent with the value of α. A similar empirical confirmation of the value of θ would be good to see.

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