Continued Fraction Transformation. II

Steven Finch

June 15, 2007

As in our earlier essay [1], define $T:[0,1]\to[0,1]$ by

$$T(x)=\begin{cases}\left\{\dfrac{1}{x}\right\} & \text{if } 0<x\leq 1,\\ 0 & \text{if } x=0\end{cases}$$

where $\{\xi\}=\xi-\lfloor\xi\rfloor$ denotes the fractional part of ξ. Previously, we examined the moments of T^jX and of $\ln(T^jX)$, where X is a random variable in $[0,1]$. The distribution of X was assumed to be either uniform or Gauss-Kuzmin.

What can be said about the moments of $\lfloor 1/T^jX\rfloor$ and of $\ln\lfloor 1/T^jX\rfloor$? An answer to this question helps in determining the asymptotic distribution of the first n continued fraction "digits", corresponding to uniformly distributed X as $n\to\infty$.

0.1. Uniform Distribution. Let γ denote the Euler-Mascheroni constant, ψ denote the digamma function, and ζ denote the Riemann zeta function. If X is a random variable following the uniform distribution on $[0,1]$, then

$$\mathrm{E}\left\lfloor\frac{1}{X}\right\rfloor=\int_1^\infty\frac{\lfloor y\rfloor}{y^2}dy\sim\sum_{n\leq N}\int_n^{n+1}\frac{n}{y^2}dy\sim\sum_{n\leq N}n\left(\frac{1}{n}-\frac{1}{n+1}\right)\sim\sum_{n\leq N}\frac{1}{n+1}\sim\ln(N),$$

$$\begin{aligned}\mathrm{E}\left\lfloor\frac{1}{TX}\right\rfloor &= \int_1^\infty\left\lfloor\frac{1}{\{y\}}\right\rfloor\frac{dy}{y^2}\sim\sum_{n\leq N}\int_n^{n+1}\left\lfloor\frac{1}{y-n}\right\rfloor\frac{dy}{y^2}\\ &\sim \sum_{n\leq N}\int_0^1\left\lfloor\frac{1}{z}\right\rfloor\frac{dy}{(z+n)^2}\sim\sum_{n\leq N}\int_1^\infty\frac{\lfloor w\rfloor}{(1+n\,w)^2}dw\\ &\sim \sum_{n\leq N}\sum_{m\leq N}\int_m^{m+1}\frac{m}{(1+n\,w)^2}dw\sim\sum_{n\leq N}\sum_{m\leq N}\frac{m}{n}\left(\frac{1}{1+n\,m}-\frac{1}{1+n(m+1)}\right)\\ &\sim \sum_{n\leq N}\sum_{m\leq N}\frac{1}{n(1+n\,m)}\sim\sum_{m\leq N}\left(\psi\left(1+\frac{1}{m}\right)+\gamma\right)\sim\frac{\pi^2}{6}\ln(N)\end{aligned}$$

as $N\to\infty$, via the substitutions $y=1/x$, $z=y-n$ and $w=1/z$. Hence both expected values are infinite. By contrast,

$$\begin{aligned}\mathrm{E}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor\right) &= \int_1^\infty \frac{\ln\lfloor y\rfloor}{y^2}dy=\sum_{n=1}^\infty\int_n^{n+1}\frac{\ln(n)}{y^2}dy=\sum_{n=1}^\infty\frac{\ln(n)}{n(n+1)}\\ &= -\sum_{k=2}^\infty(-1)^k\zeta'(k)=0.7885305659...\end{aligned}$$

(Lüroth analog of Khintchine's constant [2]),

$$\mathrm{E}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor^2\right)=\sum_{n=1}^\infty\frac{\ln(n)^2}{n(n+1)}=\sum_{k=2}^\infty(-1)^k\zeta''(k),$$

$$\mathrm{Var}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor\right)=1.1759638742...=(1.0844186803...)^2,$$

$$\begin{aligned}\mathrm{E}\left(\ln\left\lfloor\frac{1}{TX}\right\rfloor\right) &= \sum_{n=1}^\infty\sum_{m=1}^\infty\frac{\ln(m)}{n}\left(\frac{1}{1+n\,m}-\frac{1}{1+n(m+1)}\right)\\ &= \sum_{n=1}^\infty\sum_{m=2}^\infty\frac{\ln(m)-\ln(m-1)}{n(1+n\,m)}\\ &= \sum_{m=2}^\infty(\ln(m)-\ln(m-1))\left(\psi\left(1+\frac{1}{m}\right)+\gamma\right)\\ &= \sum_{k=2}^\infty(-1)^k\zeta(k)\sum_{j=1}^\infty\binom{1-k}{j}\zeta'(j+k-1)\\ &= 1.06479...,\end{aligned}$$

$$\begin{aligned}\mathrm{E}\left(\ln\left\lfloor\frac{1}{TX}\right\rfloor^2\right) &= \sum_{n=1}^\infty\sum_{m=1}^\infty\frac{\ln(m)^2}{n}\left(\frac{1}{1+n\,m}-\frac{1}{1+n(m+1)}\right)\\ &= \sum_{m=2}^\infty\left(\ln(m)^2-\ln(m-1)^2\right)\left(\psi\left(1+\frac{1}{m}\right)+\gamma\right)\\ &= -\sum_{k=2}^\infty(-1)^k\zeta(k)\sum_{j=1}^\infty\binom{1-k}{j}\zeta''(j+k-1),\end{aligned}$$

$$\mathrm{Var}\left(\ln\left\lfloor\frac{1}{TX}\right\rfloor\right)=1.49522...=(1.22279...)^2.$$

We shall not attempt to compute the cross-moments

$$\mathrm{E}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor\cdot\ln\left\lfloor\frac{1}{TX}\right\rfloor\right)\qquad\text{or}\qquad\rho\left(\ln\left\lfloor\frac{1}{X}\right\rfloor,\ln\left\lfloor\frac{1}{TX}\right\rfloor\right)$$

and leave these as open problems.

0.2. Gauss-Kuzmin Distribution. If X is a random variable following the Gauss-Kuzmin distribution on $[0,1]$, then

$$\begin{aligned}\mathrm{E}\left\lfloor\frac{1}{X}\right\rfloor &= \frac{1}{\ln(2)}\int_1^\infty\frac{\lfloor y\rfloor}{y(y+1)}dy\sim\frac{1}{\ln(2)}\sum_{n\le N}\int_n^{n+1}\frac{n}{y(y+1)}dy\\ &\sim \frac{1}{\ln(2)}\sum_{n\le N}n\ln\left(1+\frac{1}{n(n+2)}\right)\sim\frac{1}{\ln(2)}\ln(N)\sim\mathrm{E}\left\lfloor\frac{1}{TX}\right\rfloor\end{aligned}$$

as $N\to\infty$. Hence both expected values are infinite. By contrast,

$$\begin{aligned}\mathrm{E}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor\right) &= \frac{1}{\ln(2)}\sum_{n=1}^\infty\ln(n)\ln\left(1+\frac{1}{n(n+2)}\right)\\ &= \frac{1}{\ln(2)}\sum_{j=2}^\infty(-1)^j\frac{2\zeta'(j)-2^j\left(\zeta'(j)+\frac{\ln(2)}{2^j}+\frac{\ln(3)}{3^j}\right)}{j}\\ &\quad+(1-\ln(2))+\tfrac{\ln(3)}{\ln(2)}\left(\tfrac{2}{3}-\ln(\tfrac{5}{3})\right)\\ &= 0.9878490568...=\ln(K)=\mathrm{E}\left(\ln\left\lfloor\frac{1}{TX}\right\rfloor\right)\end{aligned}$$

(Khintchine's constant [2]),

$$\begin{aligned}\mathrm{E}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor^2\right) &= \frac{1}{\ln(2)}\sum_{n=1}^\infty\ln(n)^2\ln\left(1+\frac{1}{n(n+2)}\right)\\ &= -\frac{1}{\ln(2)}\sum_{j=2}^\infty(-1)^j\frac{2\zeta''(j)-2^j\left(\zeta''(j)-\frac{\ln(2)^2}{2^j}-\frac{\ln(3)^2}{3^j}\right)}{j}\\ &\quad+\ln(2)\left(1-\ln(2)\right)+\tfrac{\ln(3)^2}{\ln(2)}\left(\tfrac{2}{3}-\ln(\tfrac{5}{3})\right)\\ &= \mathrm{E}\left(\ln\left\lfloor\frac{1}{TX}\right\rfloor^2\right),\end{aligned}$$

$$\mathrm{Var}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor\right)=1.4094310970...=(1.1871946331...)^2=\mathrm{Var}\left(\ln\left\lfloor\frac{1}{TX}\right\rfloor\right).$$

The joint expectation

$$\mathrm{E}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor\cdot\ln\left\lfloor\frac{1}{TX}\right\rfloor\right)$$

simplifies to

$$\frac{1}{\ln(2)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\ln(n)\ln(m)\ln\left(1+\frac{1}{(1+(n+1)m)(1+n(m+1))}\right)$$

and can be numerically evaluated via suitable generalization of Kummer's method [3]. It follows that

$$\rho\left(\ln\left\lfloor\frac{1}{X}\right\rfloor,\ln\left\lfloor\frac{1}{TX}\right\rfloor\right)=-0.0876526887...=r_1.$$

0.3. Variance of Sample Mean. The sample mean

$$\hat{\mu}_n(X)=\frac{1}{n}\sum_{0\leq j<n}\ln\left\lfloor\frac{1}{T^jX}\right\rfloor$$

satisfies

$$\lim_{n\to\infty}\mathrm{E}\left(\hat{\mu}_n(X)\right)=\ln(K)=0.9878490568...=\mu,$$

$$\begin{aligned}\lim_{n\to\infty}n\,\mathrm{Var}\left(\hat{\mu}_n(X)\right) &= \lim_{n\to\infty}\frac{1}{n}\sum_{\substack{0\leq j<n,\\0\leq k<n}}\mathrm{Cov}(\ln(T^jX),\ln(T^kX))=\sigma^2\\ &\approx \mathrm{Var}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor\right)\left(1+\frac{2r_1}{1-r_1}\right)\approx 1.2\end{aligned}$$

for a wide variety of initial distributions for X on $[0,1]$. (No negative sign is introduced this time in the definition of $\hat{\mu}_n(X)$, unlike before.)

0.4. Continued Fraction Digits. If a_1, a_2, a_3, $\ldots$ denote the partial denominators (digits) of X, then it is clear that

$$\ln\left((a_1a_2a_3\cdots a_n)^{\frac{1}{n}}\right)=\frac{1}{n}\sum_{0\leq j<n}\ln\left\lfloor\frac{1}{T^jX}\right\rfloor$$

(no nonzero error ε_n is present here). Baladi & Vallée [4] proved that the following Central Limit Theorem is true:

$$\lim_{n\to\infty}\mathrm{P}\left(\frac{\frac{1}{n}\left(\ln a_1+\ln a_2+\cdots+\ln a_n\right)-\mu}{\frac{\sigma}{\sqrt{n}}}\leq t\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}\exp\left(-\frac{u^2}{2}\right)du.$$

and Lhote [5] computed that

$$\sigma^2 = 1.2297301427... = (1.1089319829...)^2.$$

What happens if we omit the logarithms on the left-hand side? Since a_k has infinite expectation, it is not surprising that asymptotic normality fails. Lévy [6], Philipp [7], Heinrich [8] and Hensley [9] proved that

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{\ln(2)}{n}\sum_{k=1}^{n} a_k - (\ln(n) - \gamma - \ln(\ln(2))) \le t\right) = \int_{-\infty}^{t} f(u)\,du$$

where the density f of the limiting stable distribution $S(1, 1, \pi/2, 0; 1)$ is given by

$$f(u) = \frac{1}{\pi}\int_{0}^{\infty} \sin(\pi\, v) \exp(-v\ln(v) - u\, v)\, dv.$$

See Figure 1. The median of f is 1.35578... and the mode of f is $-0.22278...$. Extreme asymmetry is the most noticeable feature here!

As a footnote, let us return to some very simple ideas. If X_1, X_2, ..., X_n is an independent sample from the uniform distribution and Y_1, Y_2, ..., Y_n is an independent sample from the Gauss-Kuzmin distribution, then

$$\mathrm{P}\left(\frac{\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{2}}{\frac{1}{6}\sqrt{\frac{3}{n}}} \le t\right) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} \exp\left(-\frac{u^2}{2}\right) du \leftarrow \mathrm{P}\left(\frac{\frac{1}{n}\sum_{k=1}^{n} Y_k - \left(\frac{1}{\ln(2)} - 1\right)}{\frac{1}{\ln(2)}\sqrt{\frac{(3/2)\ln(2) - 1}{n}}} \le t\right)$$

as $n \to \infty$. Also, the distributions of reciprocals have densities

$$\mathrm{P}\left(\frac{1}{X} \le t\right) = \begin{cases} \dfrac{1}{t^2} & \text{if } t \ge 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathrm{P}\left(\frac{1}{Y} \le t\right) = \begin{cases} \dfrac{1}{\ln(2)}\dfrac{1}{t\,(t+1)} & \text{if } t \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

The expectations of $1/X$ and of $1/Y$ are infinite. Our ideas hence become vastly more complicated at this point [9]:

$$\mathrm{P}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{X_k} - (\ln(n) + 1 - \gamma) \le t\right) \to \int_{-\infty}^{t} f(u)\,du$$

where f is exactly as before, and

$$\mathrm{P}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{Y_k}-\frac{\ln(n)+1-\ln(2)-\gamma}{\ln(2)}\leq t\right)\to\int\limits_{-\infty}^{t}g(u)\,du$$

where

$$g(u)=\frac{1}{\pi}\int\limits_{0}^{\infty}\sin\left(\frac{\pi\,v}{\ln(2)}\right)\exp\left(-\frac{v}{\ln(2)}\ln(v)-u\,v\right)\,dv$$

is the density of the limiting stable distribution $S(1,1,\pi/(2\ln(2)),0;1)$. The median of g is 2.48474... and the mode of g is 0.20735...; asymmetry again dominates. A wealth of materials on calculating stable distributions is available [10, 11, 12].

0.5. Acknowledgements. I thank for Pascal Sebah for computing r_1, Loïck Lhote for computing σ^2, John Nolan for his STABLE software [12], and Doug Hensley for helpful discussions.

References

[1] S. R. Finch, Continued fraction transformation, unpublished note (2007).

[2] S. R. Finch, Khintchine-Lévy constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 59–65.

[3] P. Sebah, Accelerating convergence of a double series, unpublished note (2007).

[4] V. Baladi and B. Vallée, Euclidean algorithms are Gaussian, *J. Number Theory* 110 (2005) 331–386; MR2122613 (2006e:11192).

[5] L. Lhote, The variance associated with Khintchine's constant, unpublished note (2007).

[6] P. Lévy, Fractions continues aléatoires, *Rend. Circ. Mat. Palermo* 1 (1952) 170–208; also in *Œuvres de Paul Lévy*, v. VI, *Théorie des jeux*, ed. D. Dugué, P. Deheuvels and M. Ibéro, Gauthier-Villars, 1980, pp. 361–399; MR0066583 (16,600e) and MR0586767 (83c:01070).

[7] W. Philipp, Limit theorems for sums of partial quotients of continued fractions, *Monatsh. Math.* 105 (1988) 195–206; MR0939942 (89e:60069).

[8] L. Heinrich, Rates of convergence in stable limit theorems for sums of exponentially ψ-mixing random variables with an application to metric theory of continued fractions, *Math. Nachr.* 131 (1987) 149–165; MR0908807 (89a:60061).

[9] D. Hensley, The statistics of the continued fraction digit sum, *Pacific J. Math.* 192 (2000) 103–120; MR1741027 (2000k:11092); Zbl. 1015.11038.

[10] J. P. Nolan, Numerical calculation of stable densities and distribution functions. Heavy tails and highly volatile phenomena, *Comm. Statist. Stochastic Models* 13 (1997) 759–774; MR1482292 (98f:60024).

[11] I. A. Belov, On the computation of the probability density function of α-stable distributions, *Proc. 10*$^{\text{th}}$ *Internat. Conf. Mathematical Modelling and Analysis – 2*$^{\text{nd}}$ *Internat. Conf. Computational Methods in Applied Mathematics*, Trakai, 2005, ed. R. Čiegis, pp. 333–341; MR2194689; http://www.techmat.vgtu.lt/~art/proc/file/BeloIg.pdf.

[12] J. P. Nolan, *Stable Distributions. Models for Heavy Tailed Data*, Birkhäuser, to appear; http://auapps.american.edu/jpnolan/www/stable/stable.html.