Continued Fraction Transformation. III

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July 1, 2007

We continue the discussion from our earlier essays [1, 2], turning attention first to two variations on regular continued fractions (RCFs). For reasons of space, only first-order results (means) will be presented. After this, we exhibit formulas connected with Lüroth representations and with ordinary decimal representations.

0.1. Nearest Integer Continued Fractions. Define $T : [-1/2, 1/2] \to [-1/2, 1/2]$ by

$$T(x) = \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} + \dfrac{1}{2} \right\rfloor & \text{if } -1/2 \le x \le 1/2 \text{ and } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For example,

$$\begin{array}{rl} \pi - 3 = 0.141592..., & \left\lfloor \frac{1}{\pi-3} + \frac{1}{2} \right\rfloor = 7, \\ T(\pi - 3) = 0.062513..., & \left\lfloor \frac{1}{T(\pi-3)} + \frac{1}{2} \right\rfloor = 16, \\ T^2(\pi - 3) = -0.003405..., & \left\lfloor \frac{1}{T^2(\pi-3)} + \frac{1}{2} \right\rfloor = -294, \\ T^3(\pi - 3) = 0.365409..., & \left\lfloor \frac{1}{T^3(\pi-3)} + \frac{1}{2} \right\rfloor = 3, \\ T^4(\pi - 3) = -0.263340..., & \left\lfloor \frac{1}{T^4(\pi-3)} + \frac{1}{2} \right\rfloor = -4 \end{array}$$

and

$$\begin{aligned} \pi &= 3 + \frac{1|}{|7} + \frac{1|}{|16} + \frac{1|}{|-294} + \frac{1|}{|3} + \frac{1|}{|-4} + \frac{1|}{|5} + \frac{1|}{|-15} + \frac{1|}{|-3} + \frac{1|}{|2} + \cdots \\ &= 3 + \frac{1|}{|7} + \frac{1|}{|16} - \frac{1|}{|294} - \frac{1|}{|3} - \frac{1|}{|4} - \frac{1|}{|5} - \frac{1|}{|15} + \frac{1|}{|3} - \frac{1|}{|2} + \cdots \end{aligned}$$

is the **nearest integer continued fraction** (NICF) expansion for π. This is also called a **centered continued fraction**. Let X be a random variable in $[-1/2, 1/2]$ with density

$$\frac{d}{dx} \mathrm{P}(X \le x) = \begin{cases} \dfrac{1}{\ln(\varphi)} \dfrac{1}{\varphi + 1 + x} & \text{if } -1/2 \le x < 0, \\ \dfrac{1}{\ln(\varphi)} \dfrac{1}{\varphi + x} & \text{if } 0 \le x \le 1/2 \end{cases}$$

where $\varphi = (1+\sqrt{5})/2$ denotes the Golden mean [3]. What is the mean of $\ln(|X|)$? This is equal to the asymptotic mean of $(1/n)\ln q_n$, corresponding to denominators q_n in the partial convergents to x :

$$\frac{p_1}{q_1}=\frac{3}{1},\quad \frac{p_2}{q_2}=\frac{22}{7},\quad \frac{p_3}{q_3}=\frac{355}{113},\quad \frac{p_4}{q_4}=\frac{104348}{33215},\quad \frac{p_5}{q_5}=\frac{312689}{99532},\quad ...,$$

as $n \to \infty$. It follows that [4, 5]

$$\begin{aligned} \mathrm{E}(\ln(|X|)) &= \frac{1}{\ln(\varphi)}\int_{-1/2}^{0}\frac{\ln(-x)}{\varphi+1+x}dx+\frac{1}{\ln(\varphi)}\int_{0}^{1/2}\frac{\ln(x)}{\varphi+x}dx \\ &= -\frac{\pi^2}{12\ln(\varphi)} = -1.7091579853... = \mathrm{E}(\ln(|TX|)). \end{aligned}$$

Also, what is the mean of $\ln(|a_1|)$, where $a_1, a_2, a_3, \ldots$ denote the partial denominators (digits) of X ? Using the substitution $y = \pm 1/x$, it follows that [6, 7]

$$\begin{aligned} \mathrm{E}\left(\ln\left|\left\lfloor\frac{1}{X}+\frac{1}{2}\right\rfloor\right|\right) &= \frac{1}{\ln(\varphi)}\int_{-1/2}^{0}\frac{\ln\left\lfloor-\frac{1}{x}+\frac{1}{2}\right\rfloor}{\varphi+1+x}dx+\frac{1}{\ln(\varphi)}\int_{0}^{1/2}\frac{\ln\left\lfloor\frac{1}{x}+\frac{1}{2}\right\rfloor}{\varphi+x}dx \\ &= \frac{1}{\ln(\varphi)}\int_{2}^{\infty}\left(\frac{\ln\left\lfloor y+\frac{1}{2}\right\rfloor}{y((\varphi+1)y-1)}+\frac{\ln\left\lfloor y+\frac{1}{2}\right\rfloor}{y(\varphi y+1)}\right)dy \\ &= \frac{1}{\ln(\varphi)}\int_{2}^{5/2}\left(\frac{\ln(2)}{y((\varphi+1)y-1)}+\frac{\ln(2)}{y(\varphi y+1)}\right)dy \\ &\quad +\frac{1}{\ln(\varphi)}\sum_{n=3}^{\infty}\int_{n-1/2}^{n+1/2}\left(\frac{\ln(n)}{y((\varphi+1)y-1)}+\frac{\ln(n)}{y(\varphi y+1)}\right)dy \\ &= \frac{\ln(2)}{\ln(\varphi)}\ln\left(\frac{5\varphi+3}{5\varphi+2}\right) \\ &\quad +\frac{1}{\ln(\varphi)}\sum_{n=3}^{\infty}\ln(n)\ln\left(\frac{(\varphi+1)(n+\frac{1}{2})-1}{(\varphi+1)(n-\frac{1}{2})-1}\frac{\varphi(n-\frac{1}{2})+1}{\varphi(n+\frac{1}{2})+1}\right) \\ &= 1.6964441175... = \mathrm{E}\left(\ln\left|\left\lfloor\frac{1}{TX}+\frac{1}{2}\right\rfloor\right|\right). \end{aligned}$$

These two constants are the NICF analogs of Lévy's constant and Khintchine's constant, respectively. A Central Limit Theorem exists in both cases [8], but the associated variances have not yet been numerically evaluated.

0.2. Odd Digit Continued Fractions. Define $T:[0,1]\to[0,1]$ by

$$T(x)=\begin{cases}\dfrac{1}{x}-\left\lfloor\dfrac{1}{x}\right\rfloor & \text{if } \left\lfloor\dfrac{1}{x}\right\rfloor\equiv 1 \bmod 2 \text{ and } x\neq 0,\\ \left\lceil\dfrac{1}{x}\right\rceil-\dfrac{1}{x} & \text{if } \left\lceil\dfrac{1}{x}\right\rceil\equiv 1 \bmod 2 \text{ and } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

For example,

$$\begin{array}{ll} \pi-3=0.141592..., & \left\lfloor\frac{1}{\pi-3}\right\rfloor=7,\\ T(\pi-3)=0.062513..., & \left\lfloor\frac{1}{T(\pi-3)}\right\rfloor=15,\\ T^2(\pi-3)=0.996594..., & \left\lfloor\frac{1}{T^2(\pi-3)}\right\rfloor=1,\\ T^3(\pi-3)=0.003417..., & \left\lceil\frac{1}{T^3(\pi-3)}\right\rceil=293,\\ T^4(\pi-3)=0.365409..., & \left\lceil\frac{1}{T^4(\pi-3)}\right\rceil=3,\\ T^5(\pi-3)=0.263340..., & \left\lfloor\frac{1}{T^5(\pi-3)}\right\rfloor=3,\\ T^6(\pi-3)=0.797366..., & \left\lfloor\frac{1}{T^6(\pi-3)}\right\rfloor=1 \end{array}$$

and

$$\begin{aligned}\pi &= 3+\frac{1|}{|7}+\frac{1|}{|15}+\frac{1|}{|1}+\frac{1|}{|293}-\frac{1|}{|3}-\frac{1|}{|3}+\frac{1|}{|1}+\frac{1|}{|3}+\frac{1|}{|1}+\frac{1|}{|15}+\cdots\\ &= 3+\frac{1|}{|7}+\frac{1|}{|15}+\frac{1|}{|1}+\frac{1|}{|293}+\frac{1|}{|-3}+\frac{1|}{|3}+\frac{1|}{|1}+\frac{1|}{|3}+\frac{1|}{|1}+\frac{1|}{|15}+\cdots\end{aligned}$$

is the **odd digit continued fraction** (ODCF) expansion for π. The phrase "partial denominator" or "partial quotient" often replaces the word "digit". Let X be a random variable in $[0,1]$ with density

$$\frac{d}{dx}\,\mathrm{P}(X\leq x)=\frac{1}{3\ln(\varphi)}\left(\frac{1}{\varphi-1+x}+\frac{1}{\varphi+1-x}\right)$$

where φ is as before. What is the mean of $\ln(X)$? This is equal to the asymptotic mean of $(1/n)\ln q_n$, corresponding to denominators q_n in the partial convergents to x :

$$\frac{p_1}{q_1}=\frac{3}{1},\quad \frac{p_2}{q_2}=\frac{22}{7},\quad \frac{p_3}{q_3}=\frac{333}{106},\quad \frac{p_4}{q_4}=\frac{355}{113},\quad \frac{p_5}{q_5}=\frac{104348}{33215},\quad ...,$$

as $n \to \infty$. It follows that [9, 10]

$$\begin{aligned} \mathrm{E}(\ln(X)) &= \frac{1}{3\ln(\varphi)} \int_0^1 \left(\frac{\ln(x)}{\varphi - 1 + x} + \frac{\ln(x)}{\varphi + 1 - x} \right) dx \\ &= -\frac{\pi^2}{18\ln(\varphi)} = -1.1394386568... = \mathrm{E}(\ln(TX)). \end{aligned}$$

Also, what is the mean of $\ln(|a_1|)$, where a_1, a_2, a_3, ... denote the digits of X ? Let $\lfloor z \rceil = \lfloor z \rfloor$ if $\lfloor z \rfloor$ is odd and $\lfloor z \rceil = \lceil z \rceil$ otherwise. Using the substitution $y = 1/x$, it follows that [11]

$$\begin{aligned} \mathrm{E}\left(\ln \left\lfloor \frac{1}{X} \right\rceil\right) &= \frac{1}{3\ln(\varphi)} \int_0^1 \ln \left\lfloor \frac{1}{x} \right\rceil \left(\frac{1}{\varphi - 1 + x} + \frac{1}{\varphi + 1 - x} \right) dx \\ &= \frac{1}{3\ln(\varphi)} \int_1^\infty \ln \lfloor y \rceil \left(\frac{1}{y((\varphi - 1)y + 1)} + \frac{1}{y((\varphi + 1)y - 1)} \right) dy \\ &= \frac{1}{3\ln(\varphi)} \sum_{n=1}^{\infty} \int_{2n}^{2n+2} \ln(2n+1) \left(\frac{1}{y((\varphi - 1)y + 1)} + \frac{1}{y((\varphi + 1)y - 1)} \right) dy \\ &= \frac{1}{3\ln(\varphi)} \sum_{n=1}^{\infty} \ln(2n+1) \ln \left(\frac{2(\varphi+1)(n+1) - 1}{2(\varphi+1)n - 1} \frac{2(\varphi-1)n + 1}{2(\varphi-1)(n+1) + 1} \right) \\ &= 1.0283554474... = \mathrm{E}\left(\ln \left\lfloor \frac{1}{TX} \right\rceil\right). \end{aligned}$$

These two constants are the ODCF analogs of Lévy's constant and Khintchine's constant, respectively. A Central Limit Theorem exists in both cases [8], but again the associated variances have not yet been numerically evaluated.

0.3. Lüroth Representations. Define $A : [0, 1] \to [0, 1]$ by

$$A(x) = \begin{cases} \left\lfloor \frac{1}{x} \right\rfloor \left(x \left\lceil \frac{1}{x} \right\rceil - 1 \right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

and $B : [0, 1] \to [0, 1]$ by

$$B(x) = \begin{cases} \left\lceil \frac{1}{x} \right\rceil \left(1 - x \left\lfloor \frac{1}{x} \right\rfloor \right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For example,

$$\begin{aligned} a_1 &= \left\lfloor \tfrac{1}{\pi-3} \right\rfloor = 7, & b_1 &= \left\lfloor \tfrac{1}{\pi-3} \right\rfloor = 7, \\ a_2 &= \left\lfloor \tfrac{1}{A(\pi-3)} \right\rfloor = 1, & b_2 &= \left\lfloor \tfrac{1}{B(\pi-3)} \right\rfloor = 14, \\ a_3 &= \left\lfloor \tfrac{1}{A^2(\pi-3)} \right\rfloor = 1, & b_3 &= \left\lfloor \tfrac{1}{B^2(\pi-3)} \right\rfloor = 7, \\ a_4 &= \left\lfloor \tfrac{1}{A^3(\pi-3)} \right\rfloor = 1, & b_4 &= \left\lfloor \tfrac{1}{B^3(\pi-3)} \right\rfloor = 1, \\ a_5 &= \left\lfloor \tfrac{1}{A^4(\pi-3)} \right\rfloor = 2, & b_5 &= \left\lfloor \tfrac{1}{B^4(\pi-3)} \right\rfloor = 1, \\ a_6 &= \left\lfloor \tfrac{1}{A^5(\pi-3)} \right\rfloor = 1, & b_6 &= \left\lfloor \tfrac{1}{B^5(\pi-3)} \right\rfloor = 1, \\ a_7 &= \left\lfloor \tfrac{1}{A^6(\pi-3)} \right\rfloor = 4, & b_7 &= \left\lfloor \tfrac{1}{B^6(\pi-3)} \right\rfloor = 15 \\ a_8 &= \left\lfloor \tfrac{1}{A^7(\pi-3)} \right\rfloor = 23, & b_8 &= \left\lfloor \tfrac{1}{B^7(\pi-3)} \right\rfloor = 1 \end{aligned}$$

and

$$\pi = 3 + \frac{1}{a_1+1} + \sum_{n=2}^{\infty} \left(\prod_{k=1}^{n-1} \frac{1}{a_k(a_k+1)} \right) \frac{1}{a_n+1} = 3 + \frac{1}{b_1} + \sum_{n=2}^{\infty} \left(\prod_{k=1}^{n-1} \frac{1}{b_k(b_k+1)} \right) \frac{(-1)^{n-1}}{b_n}$$

are the **positive Lüroth** and **alternating Lüroth representations** for π, respectively. The limiting constants are the same whether we use as or bs. For uniformly distributed X, it follows that

$$\mathrm{E}\left(\ln \left\lfloor \frac{1}{X} \right\rfloor \right) = \sum_{n=1}^{\infty} \frac{\ln(n)}{n(n+1)} = -\sum_{k=2}^{\infty} (-1)^k \zeta'(k) = 0.7885305659...$$

(Lüroth analog of Khintchine's constant [4, 12, 13]),

$$\mathrm{E}\left(\ln \left\lceil \frac{1}{X} \right\rceil \right) = \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n(n+1)} = -\sum_{k=2}^{\infty} \zeta'(k) = 1.2577468869...$$

(which appeared earlier [1]),

$$\begin{aligned} \mathrm{E}\left(\ln \left\lfloor \frac{1}{X} \right\rfloor + \ln \left\lceil \frac{1}{X} \right\rceil \right) &= \sum_{n=1}^{\infty} \frac{\ln(n(n+1))}{n(n+1)} \\ &= -2 \sum_{k=1}^{\infty} \zeta'(2k) = 2.0462774528... \end{aligned}$$

(Lüroth analog of Lévy's constant [14]),

$$\mathrm{E}\left(\ln \left\lfloor \frac{1}{X} \right\rfloor^2 \right) = \sum_{n=1}^{\infty} \frac{\ln(n)^2}{n(n+1)} = \sum_{k=2}^{\infty} (-1)^k \zeta''(k),$$

$$\mathrm{E}\left(\ln\left\lceil\frac{1}{X}\right\rceil^2\right)=\sum_{n=1}^{\infty}\frac{\ln(n+1)^2}{n(n+1)}=\sum_{k=2}^{\infty}\zeta''(k),$$

$$\mathrm{Var}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor\right)=1.1759638742...=(1.0844186803...)^2,$$

$$\mathrm{Var}\left(\ln\left\lceil\frac{1}{X}\right\rceil\right)=0.7543859444...=(0.8685539387...)^2,$$

$$\begin{aligned}\mathrm{E}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor\cdot\ln\left\lceil\frac{1}{X}\right\rceil\right) &= \sum_{n=1}^{\infty}\frac{\ln(n)\ln(n+1)}{n(n+1)}\\ &= \sum_{k=2}^{\infty}(-1)^k\zeta''(k)+\sum_{j=1}^{\infty}\frac{1}{j}\sum_{k=2}^{\infty}(-1)^{j+k}\zeta'(j+k),\end{aligned}$$

$$\mathrm{Var}\left(\ln\left\lfloor\frac{1}{X}\right\rfloor+\ln\left\lceil\frac{1}{X}\right\rceil\right)=3.8012096188...=(1.9496691049...)^2.$$

It can be proved whenever $i\neq j$ that digits a_i and a_j are independent random variables (unlike any of the continued fraction expansions we have examined), hence $\rho(\ln a_i,\ln a_j)=0$. As a consequence, two relevant Central Limit Theorems are easy to state: as $n\to\infty$, both of the distributions

$$\mathrm{P}\left(\frac{\left(\frac{1}{n}\sum\limits_{i=1}^{n}\ln(a_i)\right)-0.7885305659...}{\frac{1.0844186803...}{\sqrt{n}}}\leq t\right),\qquad \mathrm{P}\left(\frac{\left(\frac{1}{n}\sum\limits_{i=1}^{n}\ln(a_i(a_i+1))\right)-2.0462774528...}{\frac{1.9496691049...}{\sqrt{n}}}\leq t\right)$$

tend to the standard normal. (For earlier expansions, the computation of σ was complicated by the existence of nonzero correlations.)

Here is an unexplained coincidence. Consider a random ordered (strongly) binary tree with N vertices, where N is odd. Janson [15, 16] recently proved that

$$\mathrm{E}\left(\frac{H}{\sqrt{N}}\cdot\frac{W}{\sqrt{N}}\right)\to 1+\sum_{n=1}^{\infty}\frac{\ln[n(n+1)]}{n(n+1)}=3.0462774528...$$

as $N\to\infty$ (which implies that the cross-correlation between height H and width W is asymptotically $-0.6428251027...$). The appearance of the same infinite series in two seemingly distant settings is fascinating! Why should the joint distribution of height and width of trees be at all related to the ergodic theory of numbers?

Since

$$\mathrm{P}(a_j=k)=\frac{1}{k(k+1)}=\int\limits_{1/(k+1)}^{1/k}dx=\mathrm{P}\left(k<\frac{1}{X}<k+1\right)$$

where X is uniformly distributed, it follows that [2, 17, 18]

$$\mathrm{P}\left(\frac{1}{n}\sum_{j=1}^{n}a_j-(\ln(n)+1-\gamma)\leq t\right)\to\int\limits_{-\infty}^{t}f(u)\,du$$

and f is the density function

$$f(u)=\frac{1}{\pi}\int\limits_{0}^{\infty}\sin(\pi\,v)\exp(-v\ln(v)-u\,v)\,dv$$

of the limiting stable distribution $S(1,1,\pi/2,0;1)$. Similarly precise characterizations of digit sums for NICF and ODCF remain open.

0.4. Ordinary Decimal Representations. At the risk of being anticlimatic, we define $T:[0,1]\to[0,1]$ by

$$T(x)=\{10\,x\}=10\,x-\lfloor 10\,x\rfloor$$

and digits $a_1=\lfloor 10\,x\rfloor$, $a_2=\lfloor 10\,Tx\rfloor$, $a_3=\lfloor 10\,T^2x\rfloor$, ... For uniformly distributed X, it follows that

$$\mathrm{E}(\lfloor 10\,X\rfloor)=\tfrac{9}{2},\qquad \mathrm{Var}(\lfloor 10\,X\rfloor)=\tfrac{33}{4}$$

and, because a_i and a_j are independent random variables whenever $i\neq j$,

$$\mathrm{P}\left(\frac{\frac{1}{n}\sum\limits_{j=1}^{n}a_j-\frac{9}{2}}{\frac{1}{2}\sqrt{\frac{33}{n}}}\leq t\right)\to\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{t}\exp\left(-\frac{u^2}{2}\right)du.$$

We merely mention the Newcomb-Benford law [19, 20, 21], which is a different topic altogether (leading nonzero digit phenomenology) and yet seemingly related.

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