Continued Fraction Transformation. IV

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Let $\lfloor x+iy\rfloor = \lfloor x\rfloor + i\lfloor y\rfloor$, where i is the imaginary unit. Extending the regular continued fraction algorithm [1] from the real interval $[0,1]$ to the complex square $[0,1]+i[0,1]$ is problematic: the transformation

$$T(z) = \begin{cases} \dfrac{1}{z} - \left\lfloor \dfrac{1}{z} \right\rfloor & \text{if } z \neq 0, \\ 0 & \text{if } z = 0 \end{cases}$$

gives divergent continued fractions of the form

$$\frac{1|}{|-i} + \frac{1|}{|-i} + \frac{1|}{|-i} + \cdots$$

whenever

$$z = \frac{\sqrt{p}}{p-1} + i\frac{1}{2}$$

for any odd prime number p. This observation appears to be new. Nakada [2] noted divergence given any z satisfying both $|z|>1$ and $|z-i|>1$, for which $p=3$ is a limiting case.

Extending the nearest integer continued fraction algorithm [3] to the complex square $[-1/2,1/2]+i[-1/2,1/2]$ at least makes sense! The transformation here is

$$T(z) = \begin{cases} \dfrac{1}{z} - \left\lfloor \dfrac{1}{z} + \dfrac{1}{2} \right\rfloor & \text{if } z \neq 0, \\ 0 & \text{if } z = 0 \end{cases}$$

and is called **Hurwitz's algorithm** [4, 5]. Consider the eight regions into which the four circular arcs $|z\pm 1|=1$, $|z\pm i|=1$ partition the square. The additional four circular arcs $|z\pm 1\pm i|=1$ subdivide four of the regions, making a total of twelve. Hensley [6, 7, 8] proved that the invariant density function for T is smooth on the interiors of the twelve regions and continuous everywhere except perhaps along the eight circular arcs. No closed-form expression for the density is known. For a complex random variable Z following this distribution, Monte Carlo simulation suggests that

$$\mathrm{E}(\ln(|Z|)) = 1.092766....$$

We shall not pursue this topic further, opting instead to discuss the most natural extension from $\mathbb{R}$ to $\mathbb{C}$ yet found of continued fraction theory.

0.1. Schmidt's Complex Continued Fractions. Define matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Regular continued fractions can be thought of as infinite products of matrices; for example,

$$\pi = 3 + \frac{1|}{|7} + \frac{1|}{|15} + \frac{1|}{|1} + \frac{1|}{|292} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|2} + \frac{1|}{|1} + \frac{1|}{|3} + \cdots$$

is identified with

$$A^3BA^7BA^{15}BA^1BA^{292}BA^1BA^1BA^1BA^2BA^1BA^3B\cdots.$$

If the above product is multiplied on the right by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

yielding

$$\begin{pmatrix} p^{(1)} & p^{(2)} & p^{(1)} + p^{(2)} \\ q^{(1)} & q^{(2)} & q^{(1)} + q^{(2)} \end{pmatrix},$$

then the ratios $p^{(1)}/q^{(1)}$, $p^{(2)}/q^{(2)}$ and $\left(p^{(1)} + p^{(2)}\right) / \left(q^{(1)} + q^{(2)}\right)$ each approach π as more terms are included in the product. For later convenience, let $p^{(3)} = p^{(1)} + p^{(2)}$ and $q^{(3)} = q^{(1)} + q^{(2)}$.

Define instead matrices [9, 10, 11, 12]

$$V_1 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \qquad V_3 = \begin{pmatrix} 1-i & i \\ -i & 1+i \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 & -1+i \\ 0 & i \end{pmatrix}, \qquad E_3 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}.$$

With this enhanced "alphabet", the real number π can be represented by

$$E_2V_1^2V_3^7V_1^{15}V_3^1V_1^{292}V_3^1V_1^1V_3^1V_1^2V_3^1V_1^3\cdots$$

and the interpretation of convergence (ratios of first-row elements to second-row elements) is identical to before.

For the complex number e^i, the matrix representation can be proved to be [9, 11]

$$CCV_3^1CCV_3^3CCV_3^5CCV_3^7CCV_3^9CCV_3^{11}CCV_3^{13}CCV_3^{15}CCV_3^{17}CCV_3^{19}\cdots$$

and for the number $\pi e^i/4$, it can be calculated to be

$$\begin{aligned}
&V_2^1E_2V_3^1CV_1^2E_3V_2^1CE_3CE_1CE_2CE_1V_3^6CV_2^1V_3^4E_2CE_2V_1^2CV_2^1V_3^1V_1^1E_3C\\
&E_1V_2^2CV_3^1V_1^1V_3^4E_2V_3^2CE_1V_2^3CV_3^1E_2V_3^1CV_2^1E_1V_2^2CV_1^1E_2V_1^1CV_2^6E_3V_2^{12}C\\
&V_3^1V_1^1V_2^1V_1^1E_3CV_1^{80}E_3V_1^{32}CV_2^1V_1^1E_3CE_2V_1^1CE_1V_2^3CV_1^1V_2^2E_3V_2^3V_1^1CV_1^1V_2^1\\
&E_3V_2^1V_1^2V_2^2CV_3^8E_1V_3^{19}V_2^5V_3^1CE_2V_1^1CV_3^6E_1V_3^6CV_1^1E_2V_2^1CV_3^2E_2V_3^3CV_2^5E_1V_2^4\\
&CV_2^1E_1V_2^2CV_1^2E_3V_1^4CV_2^1V_1^3E_3CV_3^1E_1V_3^2CE_2V_2^1V_1^1V_2^1V_3^1CV_1^1V_2^2E_3V_2^2CV_3^1\\
&E_2CE_3V_1^2CV_2^2E_3CE_1V_3^1V_1^1V_3^1CV_2^8E_3CV_3^1E_1V_3^2CV_2^3V_1^3E_3V_1^3V_3^1CE_1V_3^1C\\
&V_1^1E_2CE_3V_3^1CE_1V_3^2V_2^3V_3^1CV_2^1V_1^1V_3^1V_1^5V_3^1V_1^2E_2CE_1V_2^2CE_2V_1^1CV_3^2V_1^1E_3V_1^1\\
&CV_2^1V_1^1V_3^1E_1V_3^3CE_3CV_2^1E_1CE_3CV_3^3V_1^2V_3^1V_2^1V_1^1E_3CV_2^1E_1V_1^1V_2^1V_3^1CE_2CE_1\\
&V_3^1CE_3V_1^2CE_2CE_3CE_1CV_1^1V_2^1E_3CE_1V_2^1CV_2^4E_3V_1^3CV_3^2E_2V_1^4CE_1CV_1^2V_3^2\\
&V_1^1V_3^1E_2V_3^5CV_1^1V_2^1V_3^1V_2^2E_2CV_3^1E_1V_3^1CV_3^1V_2^1V_1^1V_2^1V_1^8E_2V_1^4CV_2^2E_3CE_2CV_1^1\\
&E_2V_1^2CV_3^1E_3V_2^1CV_1^1V_3^1V_1^2V_3^4E_2V_3^3CV_1^3E_2V_1^3CV_1^1E_2V_3^1V_2^1V_1^1CV_1^1E_3V_1^1V_2^1\\
&CV_1^2V_2^1V_1^1E_3V_1^3CV_1^1E_2CV_3^1E_2V_1^2CV_1^6E_2V_1^4CCCV_1^2E_3V_1^2CE_1V_3^1V_2^1V_3^1V_2^1\cdots.
\end{aligned}$$

Note that powers of V_j are collected together, but not powers of C or E_j. The **terms** of the matrix representation are hence

$$T_1=C,\quad T_2=C,\quad T_3=V_3^1,\quad T_4=C,\quad T_5=C,\quad T_6=V_3^3,\quad\ldots$$

for e^i and

$$T_1=V_2^1,\quad T_2=E_2,\quad T_3=V_3^1,\quad T_4=C,\quad T_5=V_1^2,\quad T_6=E_3,\quad\ldots$$

for $\pi e^i/4$. This convention will be crucial later: the phrase "full terms" will sometimes be used for emphasis. We now give **Schmidt's algorithm** for generating such **chains** of matrices.

Let $\mathbb{C}$ and $\mathbb{C}^*$ denote two distinct complex planes. Define sets

$$F(I)=\{z\in\mathbb{C}:\mathrm{Im}(z)\geq 0\},$$

$$F^*(I)=\left\{z\in\mathbb{C}^*:0\leq\mathrm{Re}(z)\leq 1,\ \mathrm{Im}(z)\geq 0,\ \left|z-\frac{1}{2}\right|\geq\frac{1}{2}\right\}$$

and subsets

$$F(V_1)=\{z\in F(I):\mathrm{Im}(z)\geq 1\},$$

$$F(V_2) = \left\{ z \in F(I) : \left| z - \frac{i}{2} \right| \leq \frac{1}{2} \right\},$$

$$F(V_3) = \left\{ z \in F(I) : \left| z - \left(1 + \frac{i}{2}\right) \right| \leq \frac{1}{2} \right\},$$

$$F(C) = \left\{ \begin{array}{c} z \in F(I) : 0 < \mathrm{Re}(z) < 1,\ \frac{1}{2} < \mathrm{Im}(z) < 1, \\ \left| z - \frac{i}{2} \right| > \frac{1}{2},\ \left| z - \left(1 + \frac{i}{2}\right) \right| > \frac{1}{2} \end{array} \right\},$$

$$F(E_1) = \left\{ \begin{array}{c} z \in F(I) : 0 \leq \mathrm{Re}(z) < 1,\ 0 \leq \mathrm{Im}(z) < \frac{1}{2}, \\ \left| z - \frac{i}{2} \right| > \frac{1}{2},\ \left| z - \left(1 + \frac{i}{2}\right) \right| > \frac{1}{2} \end{array} \right\},$$

$$F(E_2) = \left\{ z \in F(I) : \mathrm{Re}(z) > 1,\ 0 \leq \mathrm{Im}(z) < 1,\ \left| z - \left(1 + \frac{i}{2}\right) \right| > \frac{1}{2} \right\},$$

$$F(E_3) = \left\{ z \in F(I) : \mathrm{Re}(z) < 0,\ 0 \leq \mathrm{Im}(z) < 1,\ \left| z - \frac{i}{2} \right| > \frac{1}{2} \right\},$$

$$F^*(V_1) = \left\{ z \in F^*(I) : 0 \leq \mathrm{Re}(z) \leq 1,\ \mathrm{Im}(z) > 1,\ \left| z - \left(\frac{1}{2} + i\right) \right| > \frac{1}{2} \right\},$$

$$F^*(V_2) = \left\{ \begin{array}{c} z \in F^*(I) : 0 \leq \mathrm{Re}(z) < \frac{1}{2},\ 0 \leq \mathrm{Im}(z) \leq 1, \\ \left| z - \frac{1}{2} \right| \geq \frac{1}{2},\ \left| z - \left(\frac{1}{2} + i\right) \right| > \frac{1}{2} \end{array} \right\},$$

$$F^*(V_3) = \left\{ \begin{array}{c} z \in F^*(I) : \frac{1}{2} < \mathrm{Re}(z) \leq 1,\ 0 \leq \mathrm{Im}(z) \leq 1, \\ \left| z - \frac{1}{2} \right| \geq \frac{1}{2},\ \left| z - \left(\frac{1}{2} + i\right) \right| > \frac{1}{2} \end{array} \right\},$$

$$F^*(C) = \left\{ z \in F^*(I) : \left| z - \left(\frac{1}{2} + i\right) \right| \leq \frac{1}{2} \right\}.$$

The letter F suggests "Farey set" and $F(C)$, for instance, is the image of the interior of $F^*(I)$ under the action of C, where Cz is the value of the linear fractional function

$$Cz = \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix} z = \frac{z + (-1+i)}{(1-i)z + i}, \qquad z \in F^*(I).$$

Note that each of the seven matrices is invertible and, for instance,

$$C^{-1}z = \begin{pmatrix} -1 & 1+i \\ -1-i & i \end{pmatrix} z = \frac{-z + (1+i)}{(-1-i)z + i}.$$

Schmidt's transformation T maps the disjoint union $F(I) \cup F^*(I)$ into $F(I) \cup F^*(I)$ via the following formula:

$$
\begin{aligned}
T(z,\varepsilon) &= \begin{cases} \left(V_j^{-1} z, \varepsilon\right) & \text{if } \left(z \in F(V_j) \wedge \varepsilon = 1\right) \vee \left(z \in F^*(V_j) \wedge \varepsilon = 0\right), \\ \left(E_j^{-1} z, 1-\varepsilon\right) & \text{if } z \in F(E_j) \wedge \varepsilon = 1, \\ \left(C^{-1} z, 1-\varepsilon\right) & \text{if } \left(z \in F(C) \wedge \varepsilon = 1\right) \vee \left(z \in F^*(C) \wedge \varepsilon = 0\right) \end{cases} \\
&= \begin{cases} (z-i, \varepsilon) & \text{if } \left(z \in F(V_1) \wedge \varepsilon = 1\right) \vee \left(z \in F^*(V_1) \wedge \varepsilon = 0\right), \\ \left(\dfrac{z}{iz+1}, \varepsilon\right) & \text{if } \left(z \in F(V_2) \wedge \varepsilon = 1\right) \vee \left(z \in F^*(V_2) \wedge \varepsilon = 0\right), \\ \left(\dfrac{(1+i)z-i}{iz+(1-i)}, \varepsilon\right) & \text{if } \left(z \in F(V_3) \wedge \varepsilon = 1\right) \vee \left(z \in F^*(V_3) \wedge \varepsilon = 0\right), \\ \left(\dfrac{z}{(1+i)z-i}, 1-\varepsilon\right) & \text{if } z \in F(E_1) \wedge \varepsilon = 1, \\ \left(\dfrac{z-(1+i)}{-i}, 1-\varepsilon\right) & \text{if } z \in F(E_2) \wedge \varepsilon = 1, \\ (-iz, 1-\varepsilon) & \text{if } z \in F(E_3) \wedge \varepsilon = 1, \\ \left(\dfrac{-z+(1+i)}{(-1-i)z+i}, 1-\varepsilon\right) & \text{if } \left(z \in F(C) \wedge \varepsilon = 1\right) \vee \left(z \in F^*(C) \wedge \varepsilon = 0\right) \end{cases}
\end{aligned}
$$

where $j = 1,2,3$ and $\varepsilon = 0,1$. The chains for π, e^i and $\pi e^i/4$ were obtained by iterating T with starting value $\varepsilon = 1$, meaning that π, e^i and $\pi e^i/4$ are thought of as residing in $F(I)$. Clearly $\pi \notin F^*(I)$ and $e^i \notin F^*(I)$, but $\pi e^i/4$ can thought of as residing in $F^*(I)$ as well. Starting with $\varepsilon = 0$ instead, the **dual chain** for $\pi e^i/4$ is

$$
\begin{gathered}
CV_1^2 E_2 V_3^1 V_1^1 C E_1 C V_3^1 E_1 C V_3^1 E_2 V_3^1 C V_2^5 E_1 V_1^1 C E_2 V_1^4 C V_3^2 E_2 V_2^2 C V_2^1 E_3 \\
V_2^1 C V_3^2 E_1 V_2^1 V_3^1 C V_1^1 E_2 V_1^3 C V_3^4 E_1 V_3^1 C E_2 C V_3^3 E_1 V_3^1 C E_3 V_1^1 C V_2^{12} E_1 V_2^7 \\
V_1^1 C E_1 V_3^1 V_2^1 V_3^1 C V_3^{32} E_1 V_3^{80} C E_1 V_2^1 C E_3 C V_3^1 V_1^3 E_2 C E_1 V_3^1 C V_1^2 E_3 V_1^1 C E_1 \ldots.
\end{gathered}
$$

Dual chains will not be mentioned again, since the ergodic results for chains we seek are the same as ergodic results for dual chains. The associated geometry of Schmidt's algorithm is well-illustrated in [6, 13].

0.2. Invariant Density. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$
h(x,y) = \frac{1}{x\,y} - \frac{1}{x^2} \arctan\left(\frac{x}{y}\right)
$$

and $\tilde{f} : F(I) \cup F^*(I) \to F(I) \cup F^*(I)$ be given by

$$
\tilde{f}(z) = \begin{cases} \dfrac{1}{2\pi^2}\left(h(x,y) + h(1-x,y) + h(x^2-x+y^2,y)\right) & \text{if } z = x+iy \in F(I), \\ \dfrac{1}{2\pi}\dfrac{1}{y^2} & \text{if } z = x+iy \in F^*(I). \end{cases}
$$

The probability density function $\tilde{f}$ is continuous everywhere except at the points $0, 1 \in F(I)$ and $0, 1 \in F^*(I)$.

Define a constant

$$\kappa = \frac{24}{\sqrt{15}} \arccos\left(\frac{1}{4}\right) - 2\pi$$

and the Jacobian determinant

$$\|V_j z\| = \left|\frac{d}{dz}(V_j z)\right|^2$$

for each $j = 1, 2, 3$. For example,

$$V_2 z = \frac{z}{-i\,z + 1} = \frac{x}{x^2 + (y+1)^2} + i\frac{x^2 + y(y+1)}{x^2 + (y+1)^2},$$

$$V_3 z = \frac{(1-i)z + i}{-i\,z + (1+i)} = \frac{x(x-1) + (y+1)^2}{(x-1)^2 + (y+1)^2} + i\frac{(x-1)^2 + y(y+1)}{(x-1)^2 + (y+1)^2}$$

and

$$\|V_2 z\| = \frac{1}{|-i\,z + 1|^4} = \frac{1}{\left(x^2 + (y+1)^2\right)^2},$$

$$\|V_3 z\| = \frac{1}{|-i\,z + (1+i)|^4} = \frac{1}{\left((x-1)^2 + (y+1)^2\right)^2}.$$

The invariant probability density function f is given by

$$f(z) = \begin{cases} \dfrac{\pi}{\kappa}\tilde{f}(z) & \text{if } z \in F(E_1) \cup F(E_2) \cup F(E_3) \cup F(C) \cup F^*(C), \\ \dfrac{\pi}{\kappa}\left(\tilde{f}(z) - \tilde{f}(V_j z)\,\|V_j z\|\right) & \text{if } z \in F(V_j) \cup F^*(V_j),\ 1 \le j \le 3 \end{cases}$$

where, as always, a union involving F and F^* is a disjoint one. Consequences of this remarkable explicit formula follow in the next two sections. Note, for example,

$$f(z) = \frac{\pi}{\kappa}\left(\frac{1}{y^2} - \frac{1}{\left(x^2 + y(y+1)\right)^2}\right)$$

for $z \in F^*(V_2)$ and

$$f(z) = \frac{\pi}{\kappa}\left(\frac{1}{y^2} - \frac{1}{\left((x-1)^2 + y(y+1)\right)^2}\right)$$

for $z \in F^*(V_3)$. Over and beyond the singularities at points $0, 1 \in F(I)$ and $0, 1 \in F^*(I)$, there are jump discontinuities at the boundaries of $F(V_j)$ and $F^*(C)$.

0.3. Analog of Khintchine's Constant. For each term T_n in the matrix representation of z, define the corresponding continued fraction "digit"

$$\alpha_n(z) = \begin{cases} m & \text{if } T_n = V_j^m \text{ for some } 1 \le j \le 3, \\ 1 & \text{otherwise.} \end{cases}$$

In the case $z = \pi e^i/4$, we have $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $\alpha_5 = 2$, $\alpha_6 = 1$ and $\alpha_{16} = 6$, $\alpha_{19} = 4$. Define also

$$\Phi(x) = \pi - \frac{2}{\sqrt{1-x^2}} \arccos(x).$$

It can be shown that

$$F(V_1^m) = \left\{ z \in F(I) : \mathrm{Im}(z) \ge m \right\},$$

$$F(V_2^m) = \left\{ z \in F(I) : \left| z - \frac{i}{2m} \right| \le \frac{1}{2m} \right\},$$

$$F(V_3^m) = \left\{ z \in F(I) : \left| z - \left(1 + \frac{i}{2m} \right) \right| \le \frac{1}{2m} \right\},$$

$$F^*(V_1^m) = \left\{ z \in F^*(I) : 0 \le \mathrm{Re}(z) \le 1,\ \mathrm{Im}(z) > m,\ \left| z - \left(\frac{1}{2} + m\, i \right) \right| > \frac{1}{2} \right\},$$

$$F^*(V_2^m) = \left\{ \begin{array}{c} z \in F^*(I) : 0 \le \mathrm{Re}(z) < \dfrac{1}{m^2+1},\ 0 \le \mathrm{Im}(z) \le \dfrac{1}{m}, \\ \left| z - \dfrac{1}{2} \right| \ge \dfrac{1}{2},\ \left| z - \left(\dfrac{1}{2m^2} + \dfrac{i}{m} \right) \right| > \dfrac{1}{2m^2} \end{array} \right\},$$

$$F^*(V_3^m) = \left\{ \begin{array}{c} z \in F^*(I) : \dfrac{m^2}{m^2+1} < \mathrm{Re}(z) \le 1,\ 0 \le \mathrm{Im}(z) \le \dfrac{1}{m}, \\ \left| z - \dfrac{1}{2} \right| \ge \dfrac{1}{2},\ \left| z - \left(\dfrac{2m^2-1}{2m^2} + \dfrac{i}{m} \right) \right| > \dfrac{1}{2m^2} \end{array} \right\}$$

and hence

$$\int\limits_{F(V_j^m)} f(z)\, dz = \frac{1}{2\kappa} \left(\Phi\left(\frac{1}{2m} \right) - \Phi\left(\frac{1}{2(m+1)} \right) \right) = \int\limits_{F^*(V_j^m)} f(z)\, dz$$

for each $1 \leq j \leq 3$ and all $m \geq 1$. By ergodicity, the sum $(1/N)\sum_{n \leq N} \ln(\alpha_n(z))$ tends almost certainly as $N \to \infty$ to

$$\begin{aligned} \int\limits_{F(I)\cup F^*(I)} \ln(\alpha_1(z))f(z)\,dz &= 2\sum_{j=1}^{3}\sum_{m=1}^{\infty} \int\limits_{F(V_j^m)-F(V_j^{m+1})} \ln(m)f(z)\,dz \\ &= \frac{3}{\kappa}\sum_{m=2}^{\infty}\ln(m)\left(\Phi\left(\frac{1}{2m}\right)-2\Phi\left(\frac{1}{2(m+1)}\right)+\Phi\left(\frac{1}{2(m+2)}\right)\right) \\ &= \frac{3}{\kappa}\left(\ln(2)\Phi\left(\frac{1}{4}\right)+\sum_{m=3}^{\infty}\ln\left(1-\frac{1}{(m-1)^2}\right)\Phi\left(\frac{1}{2m}\right)\right) \\ &= \ln(1.2617651749...) = 0.2325116730.... \end{aligned}$$

which is the Schmidt analog of Khintchine's constant [10].

In the real case, the almost-certain divergence of $(1/N)\sum_{n \leq N} \alpha_n(z)$ is well-known. It is interesting that in the complex case, the mean converges to

$$\begin{aligned} &2\sum_{j=1}^{3}\left(\sum_{m=1}^{\infty}\int\limits_{F(V_j^m)-F(V_j^{m+1})} m\,f(z)\,dz\right)+\sum_{j=1}^{3}\int\limits_{F(E_j)} f(z)\,dz+\int\limits_{F(C)} f(z)\,dz+\int\limits_{F^*(C)} f(z)\,dz \\ =\;& \frac{3}{\kappa}\Phi\left(\frac{1}{2}\right)+\left(\sum_{j=1}^{3}\int\limits_{F(E_j)} f(z)\,dz+\int\limits_{F(C)} f(z)\,dz\right)+\frac{\pi}{\kappa}\left(\frac{2}{\sqrt{3}}-1\right) \\ =\;& \frac{3\pi}{\kappa}\left(1-\frac{4}{3\sqrt{3}}\right)+\frac{\pi}{\kappa}\left(\frac{2}{\sqrt{3}}-1\right)+\frac{\pi}{\kappa}\left(\frac{2}{\sqrt{3}}-1\right)=\frac{\pi}{\kappa}=1.6667324083.... \end{aligned}$$

The variance, however, is divergent.

0.4. Analog of Lévy's Constant. We wish to compute the almost-certain limit of $(1/n)\ln\left|q_n^{(\ell)}\right|$ as $n \to \infty$, corresponding to denominators $q_n^{(\ell)}$ in the partial convergents to z. The limit turns out to be independent of $1 \leq \ell \leq 3$. There are two variations:

- the **powerless** scenario, in which $q_n^{(\ell)}$ is evaluated at each iteration of Schmidt's algorithm (powers of V_j are irrelevant)
- the **powerful** scenario, in which $q_n^{(\ell)}$ is evaluated only at iterations that "close" a term T_k (only those powers of V_j constituting full terms are relevant, as well as any terms E_j and C).

The first gives a simpler result, but the second is more consistent with the real case. As an example, look at

$$V_2^1 E_2 V_3^1 C V_1 = \begin{pmatrix} 1+3i & -6+i \\ 4+i & -3+7i \end{pmatrix}, \qquad V_2^1 E_2 V_3^1 C V_1^2 = \begin{pmatrix} 1+3i & -9+2i \\ 4+i & -4+11i \end{pmatrix}$$

from the matrix representation of $\pi e^i/4$. In the powerless way of counting, the ratio $p_5^{(2)}/q_5^{(2)}$ is $(-6+i)/(-3+7i)$ and $p_6^{(2)}/q_6^{(2)}$ is $(-9+2i)/(-4+11i)$. In the powerful way of counting, the ratio $p_5^{(2)}/q_5^{(2)}$ is $(-9+2i)/(-4+11i)$. Both variations are interesting to us.

For the powerless scenario, let

$$\tilde{T}_1(z) = \begin{pmatrix} \tilde{a}_1(z) & \tilde{b}_1(z) \\ \tilde{c}_1(z) & \tilde{d}_1(z) \end{pmatrix}$$

be the initial output of Schmidt's algorithm, starting with input z, and let

$$\begin{aligned} \tilde{\varphi}(z) &= -\ln |\tilde{c}_1(z)z - \tilde{a}_1(z)| \\ &= \begin{cases} 0 & \text{if } z \in F(V_1) \cup F(E_2) \cup F(E_3) \cup F^*(V_1) \\ -\frac{1}{2}\ln\left(2\left((x-\frac{1}{2})^2 + (y-\frac{1}{2})^2\right)\right) & \text{if } z \in F(E_1) \cup F(C) \cup F^*(C) \\ -\frac{1}{2}\ln\left(x^2 + (y-1)^2\right) & \text{if } z \in F(V_2) \cup F^*(V_2) \\ -\frac{1}{2}\ln\left((x-1)^2 + (y-1)^2\right) & \text{if } z \in F(V_3) \cup F^*(V_3). \end{cases} \end{aligned}$$

Then $(1/n)\ln\left|q_n^{(\ell)}\right|$ converges to [10]

$$\int\limits_{F(I)\cup F^*(I)} \tilde{\varphi}(z)\tilde{f}(z)\,dz = 0.29156...$$

via numerical calculation of each component of the integral. Closed-form expressions for the components appear to be impossible. Nakada [14, 15, 16], however, proved by a different approach that the powerless Schmidt analog of Lévy's constant is

$$\frac{G}{\pi} = 0.2915609040... = \ln(1.3385151519...)$$

where G is Catalan's constant [17, 18].

For the powerful scenario, let

$$T_1(z) = \begin{pmatrix} a_1(z) & b_1(z) \\ c_1(z) & d_1(z) \end{pmatrix}$$

be the initial full term in the complex continued fraction expansion of z, and let

$$\begin{aligned}\varphi(z) &= -\ln|c_1(z)z-a_1(z)| \\ &= \begin{cases} 0 & \text{if } z\in F(V_1)\cup F(E_2)\cup F(E_3)\cup F^*(V_1) \\ -\frac{1}{2}\ln\left(2\left((x-\frac{1}{2})^2+(y-\frac{1}{2})^2\right)\right) & \text{if } z\in F(E_1)\cup F(C)\cup F^*(C) \\ -\frac{1}{2}\ln\left(m^2x^2+(m\,y-1)^2\right) & \text{if } z\in\left(F(V_2^m)-F(V_2^{m+1})\right)\cup\left(F^*(V_2^m)-F^*(V_2^{m+1})\right) \\ -\frac{1}{2}\ln\left(m^2(x-1)^2+(m\,y-1)^2\right) & \text{if } z\in\left(F(V_3^m)-F(V_3^{m+1})\right)\cup\left(F^*(V_3^m)-F^*(V_3^{m+1})\right). \end{cases}\end{aligned}$$

Then $(1/n)\ln\left|q_n^{(\ell)}\right|$ converges to [10]

$$\int\limits_{F(I)\cup F^*(I)}\varphi(z)f(z)\,dz=0.4859...$$

via numerical calculation of each component of the integral and summation over $m\geq 1$. Closed-form expressions for the components again appear to be impossible. Nakada [15] proved, as a corollary of his aforementioned result, that the powerful Schmidt analog of Lévy's constant is

$$\frac{G}{\kappa}=0.4859540077...=\ln(1.6257252237...).$$

These are magnificient formulas, needless to say!

Complex continued fractions built upon the Eisenstein-Jacobi integers (rather than the Gaussian integers) were introduced in [19], but no comparable ergodic theory has been published, as far as is known.

We merely mention the **Jacobi-Perron algorithm** [20, 21, 22, 23, 24]

$$T_{\mathrm{JPA}}(x,y)=\left(\frac{y}{x}-\left\lfloor\frac{y}{x}\right\rfloor,\frac{1}{x}-\left\lfloor\frac{1}{x}\right\rfloor\right)$$

and the **Podsypanin algorithm** [25, 26, 27, 28]

$$T_{\mathrm{MJPA}}(x,y)=\begin{cases}\left(\dfrac{y}{x},\dfrac{1}{x}-\left\lfloor\dfrac{1}{x}\right\rfloor\right) & \text{if } x\geq y\wedge x\neq 0, \\ \left(\dfrac{1}{y}-\left\lfloor\dfrac{1}{y}\right\rfloor,\dfrac{x}{y}\right) & \text{if } x<y\wedge y\neq 0, \\ 0 & \text{if } x=y=0\end{cases}$$

for $(x,y)\in[0,1]\times[0,1]$. Both possess unique invariant densities but only the latter has a closed-form expression:

$$f_{\mathrm{MJPA}}(x,y)=\frac{1}{2c}\frac{2+x+y}{(1+x)(1+y)(1+x+y)}$$

where

$$c = \frac{\pi^2}{12} + \mathrm{Li}_2\left(-\frac{1}{2}\right) = 0.3740528265....$$

If q_n denotes the common denominator in the n^{th} partial convergent to (x, y), then

$$\lim_{n\to\infty} \frac{1}{n} \ln(q_n) = -\int_0^1 \int_0^1 \ln\left(\max\{x, y\}\right) f_{\mathrm{MJPA}}(x, y)\, dx\, dy = 0.6695004121...$$

almost certainly (we omit the complicated exact formula involving dilogarithms and $\zeta(3)$). A precise estimate of the entropy associated with T_{JPA} would be good to see someday.

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