Electing a Leader

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The following scenario was examined in [1]: we toss n ideal coins, then toss those which show tails after the first toss, then toss those which show tails after the second toss, etc. Observe that if, at a given toss, only heads appear, then the process immediately terminates. Suppose instead that we require the coins (all of which showed heads) to be tossed again? Under such a change of rules, it is clear that the final toss will always involve exactly one coin. This solitary coin is called the **leader** and the process of selecting such is called an **election**.

Certain parameters governing the election (a random incomplete trie) are of interest. In Figure 1, the **size** $v_7 = 10$ is the number of vertices in the tree. The **height** $h_7 = 6$ is the length of the longest root-to-leaf path, that is, the time duration to choose a leader. Finally, $c_7 = 21$ is the total number of coin tosses. Let C_n denote likewise, given arbitrary n and a random election. It is surprising that $\mathrm{E}(C_n) = 2n$ for $n \geq 2$; the random variables V_n and H_n are more complicated [2].

The following sums involving Bernoulli numbers [3] and binomial coefficients are relevant and interesting [2, 4]:

$$\frac{1}{n}\sum_{k=2}^{n-1}\binom{n}{k}\frac{B_k}{2^{k-1}-1} \sim \frac{\ln(n)}{2\ln(2)} - \left(\frac{\ln(\pi)}{2\ln(2)} - \frac{\gamma}{2\ln(2)} + \frac{3}{4}\right) + \delta_1\left(\frac{\ln(n)}{\ln(2)}\right),$$

$$\sum_{k=1}^{n-1}\binom{n}{k}\frac{B_k}{2^{k}-1} \sim -\frac{\ln(n)}{\ln(2)} + \frac{1}{2} + \delta_2\left(\frac{\ln(n)}{\ln(2)}\right),$$

$$n\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_k}{2^{k+1}-1} \sim \frac{\pi^2}{6\ln(2)} + \delta_3\left(\frac{\ln(n)}{\ln(2)}\right)$$

where, for $m = 1, 2, 3$,

$$\delta_m(x) = \frac{1}{\ln(2)}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\zeta\left(m-1-\frac{2\pi i k}{\ln(2)}\right)\Gamma\left(m-1-\frac{2\pi i k}{\ln(2)}\right)\exp\left(2\pi i k x\right)$$

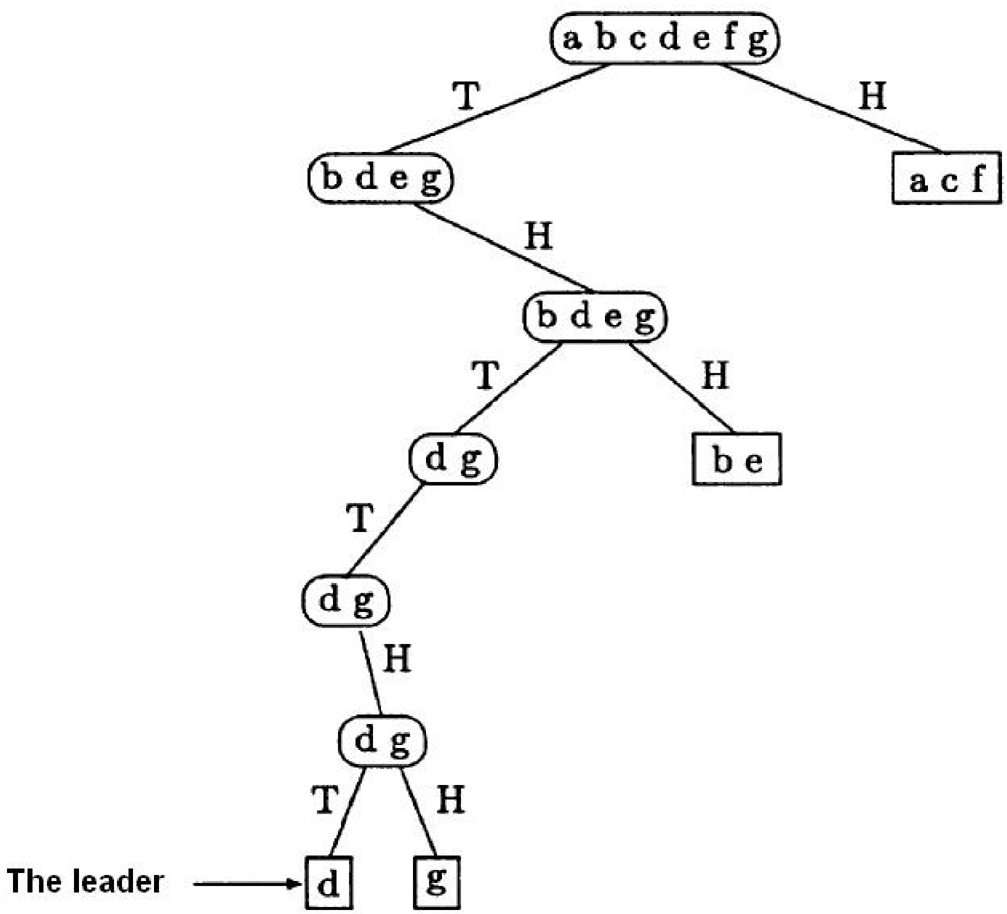

Figure 1: A typical election, starting with $n = 7$ candidates.

are periodic functions of period 1 and very small amplitude. For example, $|\delta_2(x)| < 1.927 \times 10^{-5}$ for all x. Each fluctuates symmetrically about 0. Define also [5]

$$\varepsilon(x) = \frac{2}{\ln(2)^2} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \nu\left(\frac{2\pi i k}{\ln(2)}\right) \exp(2\pi i k x) - \delta_2^2(x)$$

where

$$\nu(s) = \zeta(1-s)\Gamma(-s) - s\,\zeta'(1-s)\Gamma(-s) - s\,\zeta(1-s)\Gamma'(-s).$$

This again has period 1 and small amplitude – we have $|\varepsilon(x)| < 1.398 \times 10^{-4}$ always – fluctuations are symmetrical not about 0, but instead about

$$\int_0^1 \varepsilon(x)dx = -\frac{1}{\ln(2)^2} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left|\zeta\left(1 - \frac{2\pi i k}{\ln(2)}\right)\Gamma\left(1 - \frac{2\pi i k}{\ln(2)}\right)\right|^2 \approx -1.856 \times 10^{-10}.$$

Let us return to coin tossing. Prodinger [2] showed that

$$\mathrm{E}(V_n) \sim \frac{2\ln(n)}{\ln(2)} + \left(2 - \frac{\ln(\pi) - \gamma}{\ln(2)}\right) + 2\delta_1\left(\frac{\ln(n)}{\ln(2)}\right) - \delta_2\left(\frac{\ln(n)}{\ln(2)}\right),$$

$$\mathrm{E}(H_n) = -\sum_{k=1}^{n-1} \binom{n}{k} \frac{B_k}{1 - 2^{-k}} \sim \frac{\ln(n)}{\ln(2)} + \frac{1}{2} - \delta_2\left(\frac{\ln(n)}{\ln(2)}\right)$$

asymptotically as $n \to \infty$, assuming that the election is conducted exactly as described earlier. If we alter the rules so that a draw between two coins is allowed (if precisely two coins are left, they *both* are declared leaders), then

$$\mathrm{E}(\tilde{V}_n) \sim \frac{2\ln(n)}{\ln(2)} + \left(2 - \frac{\ln(\pi) - \gamma + \frac{\pi^2}{16}}{\ln(2)}\right) + 2\delta_1\left(\frac{\ln(n)}{\ln(2)}\right) - \delta_2\left(\frac{\ln(n)}{\ln(2)}\right) - \frac{3}{8}\delta_3\left(\frac{\ln(n)}{\ln(2)}\right),$$

$$\mathrm{E}(\tilde{H}_n) \sim \frac{\ln(n)}{\ln(2)} + \frac{1}{2} - \frac{\pi^2}{12\ln(2)} - \delta_2\left(\frac{\ln(n)}{\ln(2)}\right) - \frac{1}{2}\delta_3\left(\frac{\ln(n)}{\ln(2)}\right),$$

$$\mathrm{E}(\tilde{C}_n) \sim 2n - \frac{\pi^2}{6\ln(2)} - \delta_3\left(\frac{\ln(n)}{\ln(2)}\right).$$

For example, the constant for $\mathrm{E}(V_n)$ is 1.1812500478...; the difference $\pi^2/(16\ln(2)) = 0.8899268328...$ quantifies how much is saved by stopping earlier to give $\mathrm{E}(\tilde{V}_n)$. For $\mathrm{E}(H_n)$ versus $\mathrm{E}(\tilde{H}_n)$, the difference $\pi^2/(12\ln(2)) = 1.1865691104...$ is slightly greater.

Fill, Mahmoud & Szpankowski [5] proved that

$$\mathrm{Var}(H_n) \sim \frac{1}{12} + \frac{\pi^2}{6\ln(2)^2} - \frac{\gamma^2 + 2\gamma_1}{\ln(2)^2} + \varepsilon\left(\frac{\ln(n)}{\ln(2)}\right)$$

asymptotically as $n \to \infty$, where γ_1 is the first Stieltjes constant [6]. As predicted in [2], this is a nontrivial result. The constant 3.1166951643... also appears in [7]; another treatment is given by [8]. As far as is known, evaluating $\mathrm{Var}(\tilde{H}_n)$ remains open. The parameters V_n, C_n, $\tilde{V}_n$, $\tilde{C}_n$ deserve more attention. Random elections yielding a predetermined number > 1 of leaders are examined in [9].

0.1. Non-Ideal Coins. Instead of assuming that coins are ideal (independent probability of tails $= 1/2$), let us suppose that coins "know" their count just before each toss. More precisely, if $n_1 = n$ is the count before the first toss and n_j is the count before the j^{th} toss, $j \geq 1$, then at time j, each coin enjoys independent probability of tails $= 1/n_j$. Since $n_{j+1} \leq n_j$, the odds that any active candidate becomes the leader improve with time. If $n_{j+1} = 1$, the election is over. If $n_{j+1} = 0$, then n_{j+1} is overwritten with n_j and the coins are tossed again.

Clearly $\mathrm{E}(H_1) = 0$. From the recursion

$$\left[1 - \left(1 - \frac{1}{n}\right)^n - \left(\frac{1}{n}\right)^n\right] \mathrm{E}(H_n) = 1 + \sum_{k=2}^{n-1} \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k} \mathrm{E}(H_k)$$

for $n \geq 2$, we obtain $\mathrm{E}(H_2) = 2$, $\mathrm{E}(H_3) = 13/6$, $\mathrm{E}(H_4) = 65/29$ and [10, 11]

$$\lim_{n\to\infty} \mathrm{E}(H_n) = 2.4417158788....$$

A more complicated recursion gives $\lim_{n\to\infty} \mathrm{Var}(H_n) = 2.832554383....$

Is $1/n$ the optimal probability? Replacing $1/n$ everywhere by t/n for $0 < t < 2$ in the preceding, we obtain

$$\mathrm{E}(H_2) = \frac{2}{(2-t)t}, \qquad \mathrm{E}(H_3) = \frac{18 - 3t - 2t^2}{3(3-t)(2-t)t}.$$

Differentiating the recursion with respect to t allows us to find a minimum point $t^* = 1.0654388051...$ and thus [11]

$$\lim_{n\to\infty} \mathrm{E}(H_n^*) = 2.4348109638....$$

No one has evaluated $\lim_{n\to\infty} \mathrm{Var}(H_n^*)$, as far as is known. Related topics in random elections are found in [12].

0.2. Number Games. The following game, proposed by Gilbert [13], was revisited by Fokkink [14]. A player A chooses a secret integer from 1 to n. Another player B attempts to guess A's integer. After each guess, A tells B whether the guess is too high, too low or correct. If B has guessed A's integer, the game ends. If not, then A may change the secret integer, but the new integer must be consistent with all the information so far provided. Assuming both players adopt optimal, equilibrium, randomized strategies, the expected number ξ_n of guesses is conjectured to satisfy [14]

$$\xi_n \sim \frac{\ln(n)}{\ln(2)} - (0.487...)$$

asymptotically as $n \to \infty$. It is further acknowledged in [14] that this formula may require a small amplitude oscillation and [9] is cited. A verification of either claim would be good to see.

Here is a comparatively simple game, proposed by Häggström [15] as a model for the Swedish National Lottery. Every contestant chooses a positive integer. The person who submits the smallest integer not chosen by anybody else is the winner. (If no integer is chosen by exactly one person, then there is no winner.) Let us focus on the case where there are exactly three contestants. Assuming all three adopt optimal, equilibrium, randomized strategies, each of them independently draws an integer according to a shifted geometric distribution:

$$\mathrm{P}\left(\ell \text{ is selected}\right) = (1-r)r^{\ell-1}$$

where $\ell = 1, 2, 3, \ldots$ and $r = 0.5436890126...$ satisfies the cubic equation

$$\frac{1}{r^3} - \frac{1}{r^2} - \frac{1}{r} - 1 = 0.$$

This constant is the reciprocal growth rate for the so-called Tribonacci sequence [16]. What can be said if instead there are exactly four contestants? The only other reference found on this subject, [17], contains more elaborate analyses (assuming a Poisson random count of players or an upper bound on playable numbers, if not both).

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