Lane-Ritter-Emden Constants

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The Lane-Emden equation of index p:

$$y''(x) + \frac{2}{x}y'(x) + y(x)^p = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

is useful in astrophysics for computing the structure of interiors of polytropic stars [1, 2, 3]. A well-known series for $y(x)$ is [4]

$$y(x) = \sum_{k=0}^{\infty} a_k x^{2k}, \qquad x \approx 0$$

where $a_0 = 1$, $a_1 = -1/6$ and

$$a_k = \frac{1}{(k-1)k(2k+1)} \sum_{j=1}^{k-1} (j\,p + j - k + 1)(k-j)(2k-2j+1)a_j a_{k-j}, \qquad k \geq 2.$$

This series has radius of convergence [5, 6, 7, 8, 9, 10]

$$\gamma = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|^{1/2} = \begin{cases} \infty & \text{if } p = 0 \text{ or } p = 1, \\ 3.6537537362... = \sqrt{13.3499163649...} & \text{if } p = 3/2, \\ 3.9645856345... = \sqrt{15.7179392534...} & \text{if } p = 2, \\ 2.5748367419... = \sqrt{6.6297842476...} & \text{if } p = 3, \\ 2.0348941557... = \sqrt{4.1407942251...} & \text{if } p = 4, \\ 1.7320508075... = \sqrt{3} & \text{if } p = 5 \end{cases}$$

and details on relevant calculations will appear momentarily. The *dimensionless radius* of a polytropic star is the smallest positive x_0 for which $y(x_0) = 0$: [1, 11, 12, 13, 14, 15]

$$x_0 = \begin{cases} 2.4494897427... = \sqrt{6} & \text{if } p = 0, \\ 3.1415926535... = \pi & \text{if } p = 1, \\ 3.6537537362... = \sqrt{13.3499163649...} & \text{if } p = 3/2, \\ 4.3528745959... = \sqrt{18.9475172480...} & \text{if } p = 2, \\ 6.8968486193... = \sqrt{47.5665208786...} & \text{if } p = 3, \\ 14.9715463488... = \sqrt{224.1472000754...} & \text{if } p = 4, \\ \infty & \text{if } p = 5 \end{cases}$$

and the *dimensionless mass* of a polytropic star is x_0^2 multiplied by $-y'(x_0)$:

$$\mu = -x_0^2 y'(x_0) = \begin{cases} 4.8989794855... = 2\sqrt{6} & \text{if } p = 0, \\ 3.1415926535... = \pi & \text{if } p = 1, \\ 2.7140551201... & \text{if } p = 3/2, \\ 2.4110460120... & \text{if } p = 2, \\ 2.0182359509... & \text{if } p = 3, \\ 1.7972299144... & \text{if } p = 4, \\ 1.7320508075... = \sqrt{3} & \text{if } p = 5. \end{cases}$$

No closed-form expressions for constants associated with the range $1 < p < 5$ are known. The functions $\gamma(p)$ and $x_0(p)$ are initially equal for $p > 1$, but they separate at $p \approx 1.9121$ [7]. The function $\mu(p)$ initially decreases, but encounters a minimum at $p \approx 4.823$ and increases henceforth [16, 17].

A simpler formula for the coefficients $\{a_k\}$ is valid for $p = 2$: [18]

$$a_k = \frac{-1}{(2k)(2k+1)} \sum_{j=0}^{k-1} a_j a_{k-j}, \qquad k \geq 1$$

which makes the alternating character of the series obvious. Is there an analogous formula for $p = 3$ or $p = 4$?

Let us explain how $\gamma(p)$ is computed for $2 \leq p \leq 5$. Write $t = -x^2$ and $u = y^{-1/p}$, then

$$-6p\,u\frac{du}{dt} + 4p(p+1)t\left(\frac{du}{dt}\right)^2 - 4p\,t\,u\frac{d^2u}{dt^2} = u^{-p^2+p+2}$$

$$u(0) = 1, \qquad \frac{du}{dt}(0) = -\frac{1}{6p}.$$

For example, supposing $p = 2$, we find $u(10) = 0.312...$ and $\frac{du}{dt}(10) = -0.058....$ By the Inverse Function Theorem,

$$-6p\,u\left(\frac{dt}{du}\right)^2 + 4p(p+1)t\frac{dt}{du} + 4p\,t\,u\frac{d^2t}{du^2} = u^{-p^2+p+2}\left(\frac{dt}{du}\right)^3.$$

In the case $p = 2$, initial conditions $t(0.312...) = 10$ and $\frac{dt}{du}(0.312...) = \frac{1}{-0.058...}$ clearly hold. We find $t(0) = 15.717...$, thus $x = (\pm 3.964...)i$ correspond to where $y = u^{-p}$ explodes [19]. This technique works because 10 is large enough that $u(10)$ is small, making the computation of $t(0)$ feasible.

See also [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62]; we hope to report on [63, 64, 65, 66, 67, 68] later.

0.1. Polytropic and Isothermal Spheres. A generalization of the Lane-Emden equation is [17, 20, 21, 22, 23, 24]

$$y''(x)+\frac{N}{x}y'(x)+y(x)^{p}=0,\qquad y(0)=1,\qquad y'(0)=0$$

corresponding to N-dimensional polytropic spheres in $\mathbb{R}^{N+1}$. The case $N=2$ was discussed earlier. For $N=1$ (polytropic cylinders), we have

$$x_0=\begin{cases}2 & \text{if } p=0,\\ 2.4048255576...=z & \text{if } p=1,\\ 2.6477767662... & \text{if } p=3/2,\\ 2.9213207237... & \text{if } p=2,\\ 3.5739009819... & \text{if } p=3\end{cases}$$

where z is the smallest positive zero of the Bessel function J_0 and [25]

$$\mu=-x_0y'(x_0)=\begin{cases}2 & \text{if } p=0,\\ 1.2484591696...=z\,J_1(z) & \text{if } p=1,\\ 1.0611147888... & \text{if } p=3/2,\\ 0.9253532703... & \text{if } p=2,\\ 0.7401221205... & \text{if } p=3.\end{cases}$$

No closed-form expressions for constants associated with $p>1$ are known. In contrast, for $N=0$ (polytropic slabs),

$$x_0=\left(\frac{\pi}{2(p+1)}\right)^{1/2}\frac{\Gamma\left(\frac{1}{p+1}\right)}{\Gamma\left(\frac{p+3}{2(p+1)}\right)}=\begin{cases}1.4142135623...=\sqrt{2} & \text{if } p=0,\\ 1.5707963267...=\pi/2 & \text{if } p=1,\\ 1.6453408471... & \text{if } p=3/2,\\ 1.7173153422... & \text{if } p=2,\\ 1.8540746773... & \text{if } p=3\end{cases}$$

and

$$\mu=-y'(x_0)=\left(\frac{2}{p+1}\right)^{1/2}=\begin{cases}1.4142135623...=\sqrt{2} & \text{if } p=0,\\ 1 & \text{if } p=1,\\ 0.8944271909... & \text{if } p=3/2,\\ 0.8164965809... & \text{if } p=2,\\ 0.7071067811...=1/\sqrt{2} & \text{if } p=3.\end{cases}$$

A different generalization involves the limit as $p\to\infty$:

$$y''(x)+\frac{2}{x}y'(x)=e^{-y(x)},\qquad y(0)=y'(0)=0.$$

This corresponds to 2-dimensional isothermal spheres in $\mathbb{R}^3$ and has the following series expansion:

$$y(x) = \sum_{k=1}^{\infty} b_k x^{2k}, \qquad x \approx 0$$

where $b_1 = 1/6$ and

$$b_k = \frac{-1}{(k-1)k(2k+1)} \sum_{j=1}^{k-1} j(k-j)(2k-2j+1) b_j b_{k-j}, \qquad k \geq 2.$$

The radius of convergence, squared, is [7, 8, 69]

$$\lim_{k\to\infty} \left| \frac{b_k}{b_{k+1}} \right| = 10.7170288238... = 2(5.3585144119...).$$

This is computed by writing $t = -x^2$ and $u = e^{y/2}$, then applying the Inverse Function Theorem to

$$-12u\frac{du}{dt} + 8t\left(\frac{du}{dt}\right)^2 - 8t\,u\frac{d^2u}{dt^2} = 1,$$

$$u(0) = 1, \qquad \frac{du}{dt}(0) = -\frac{1}{12}.$$

It is also known that

$$-y(x) \sim \ln\left(\frac{2}{x^2}\right) + \frac{C}{\sqrt{x}}\cos\left(\frac{\sqrt{7}}{2}\ln(x) - c\right)$$

as $x \to \infty$ for certain unspecified constants C and c. A more precise statement of this asymptotic formula, with expressions for C and c, would be good to see. Related materials include [70, 71, 72, 73, 74, 75].

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